IS THERE A PREDICTABLE CRITERION FOR MUTUAL SINGULARITY OF TWO PROBABILITY MEASURES ON A FILTERED SPACE?

W. Schachermayer and W. Schachinger

Department of Statistics, University of Vienna

Abstract. The theme of providing predictable criteria for absolute continuity and for mutual singularity of two density processes on a filtered probability space is extensively studied, e.g., in the monograph by J. Jacod and A. N. Shiryaev [JS]. While the issue of absolute continuity is settled there in full generality, for the issue of mutual singularity one technical difficulty remained open ([JS], p210): "We do not know whether it is possible to derive a *predictable* criterion (necessary and sufficient condition) for $P'_T \perp P_T, \ldots$ ". It turns out that to this question raised in [JS] which we also chose as the title of this note, there are two answers: on the negative side we give an easy example, showing that in general the answer is no, even when we use a rather wide interpretation of the concept of "predictable criterion". The difficulty comes from the fact that the density process of a probability measure P with respect to another measure P' may suddenly jump to zero.

On the positive side we can characterize the set, where P' becomes singular with respect to P — provided this does not happen in a sudden but rather in a continuous way — as the set where the Hellinger process diverges, which certainly is a "predictable criterion". This theorem extends results in the book of J. Jacod and A. N. Shiryaev [JS].

1. Introduction

We adopt the notation of [JS], which means that we are given two fixed probability measures P, P' on a filtered space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathcal{F})$ with right continuous filtration and $\mathcal{F} = \bigvee_t \mathcal{F}_t$, $Q = \frac{P+P'}{2}$, z and z' denote the density processes of P and P', relative to Q. We define the process $(Y_t)_{t\geq 0}$ by $Y_t = \sqrt{z_t z'_t}$ and let $h = h(\frac{1}{2})$ denote the Hellinger process of order $\frac{1}{2}$, i. e., the predictable increasing process h such that $h_0 = 0$ and

$$M = Y + Y_- \bullet h \tag{1}$$

is a Q-martingale. For a stopping time T we denote by P_T and P'_T the restrictions of P and P' to $\mathcal{F}_T$. We shall investigate the following question:

1.1 Problem. *Under which conditions can we assert that $P'_T \ll P_T$ or $P'_T \perp P_T$? More generally, can we find a Hahn-decomposition of Ω into two sets, such that P'_T is absolutely continuous (resp. singular) with respect to P_T on these sets?*

The answer to this question should be in terms of a "predictable criterion". By this concept we mean that the answer should be in terms of the values of a predictable process, such as the Hellinger process h, evaluated at time T.

Before proceeding to answering this question we pause for some remarks: In order to avoid irrelevant complications at $t = 0$ we suppose that $P_0 \sim P'_0$. We define the stopping time S as the first moment when either z or z' vanishes,

$$S = \inf\{t : z_t = 0 \text{ or } z'_t = 0\}. \tag{2}$$

Noting that (1) determines the Hellinger process h only up to time S, we define h to be constant after S, i. e., we consider the "Hellinger process in the strict sense" in the terminology of [JS]. We also introduce the Hellinger process $h(0)_t$ of order 0 as the compensator of the process $\mathbb{1}_{[\![S,\infty[\![}$.

Now we review the known results: there is a very satisfactory answer to our problem as regards the question of absolute continuity:

Key words and phrases. continuity and singularity of probability measures, Hellinger processes, stochastic integrals, stopping times.

1.2 Theorem. ([JS], Thm IV 2.6) *In the above setting we have, for every stopping time T,*

$$P'_T \ll P_T \Leftrightarrow P'(G_T) = 1,$$

where $G_T = \{h_T < \infty\} \cap \{h(0)_T = 0\}$.

Note that the set G_T is defined in terms of the values of the two predictable processes h and $h(0)$ at time T. To compare this result with the situation of mutual singularity below we make the following (trivial) reformulation: denote by H the predictable, increasing, $[0,\infty]$-valued process $H_t = h_t + \infty \cdot h(0)_t$ (where $\infty \cdot 0 = 0$). Then

$$P'_T \ll P_T \Leftrightarrow P'(H_T < \infty) = 1,$$

For the question of mutual singularity (or, more generally, for the question of the Hahn-decomposition) the situation is more subtle, the difficulty arising from the fact that z_t may jump to zero as we shall presently see. Of course, we can always get rid of this difficulty by shifting it into the assumptions of a theorem: we thus obtain the subsequent result which directly follows from ([JS], lemma IV 2.12 a), b) and d)):

1.3 Theorem. *In the above setting suppose in addition that*

$$P'_T \left[S \le T \text{ and } z_{S_-} > z_S = 0 \right] = 0.$$

Then the restriction of P'_T to $\{h_T < \infty\}$ is absolutely continuous with respect to P_T while the restriction of P'_T to $\{h_T = \infty\}$ is singular with respect to P_T.

In particular

$$P'_T \perp P_T \Leftrightarrow P'(h_T = \infty) = 1.$$

Our main task in this note is to analyze what we still can say when we don't use the simplifying assumption that z_t is not allowed to jump to zero; unfortunately there is no hope for a complete analogue to theorem 1.2, as the following elementary example, which will be constructed in section 3 below, shows.

1.4 Example. There is a filtered space $(\Omega, (\mathcal{F}_t)_{t=0}^2, \mathcal{F})$ equipped with two probability measures P and P' with the following property:

There is no $[0,\infty]$-valued predictable process H such that, for every stopping time T,

$$P'_T \perp P_T \Leftrightarrow P'(H_T = \infty) = 1.$$

In fact Ω may be chosen to consist only of 4 elements.

Despite this discouraging example we can formulate an interesting positive result, where we shift the problem, that z_t may jump to zero from the assumption (as in theorem 1.3 above) into the conclusion of the theorem:

1.5 Theorem. *Under the assumption that $P_0 \sim P'_0$ we have, for every stopping time T,*

$$\{S \le T, z_{S_-} = 0\} = \{h_T = \infty\} \quad P'\text{-a.s.} \tag{3}$$

The set on the left hand side may be interpreted as the set where P'_T is singular with respect to P_T, but in such a way that, as $t \nearrow S \le T$, this singularity was not obtained by a "sudden jump", but rather in a continuous way. The assertion of the theorem is that — even in the presence of jumps of z_t to zero — it is precisely this set which is characterized by the divergence of the Hellinger process.

Remark. We tried to formulate Thm 1.5 in a manner that suits best for comparison with the results in [JS], but there are other ways to state it. Actually, the assertion of Thm 1.5 is equivalent to the assertion

$$\{z_{S_-} = 0\} = \{h_S = \infty\} \quad P'\text{-a.s.} \tag{3'}$$

To verify (3) $\Rightarrow$ (3′), we take $T := S$, and (3′) $\Rightarrow$ (3) follows from

$$\{S \le T, z_{S-} = 0\} = \{S \le T, h_S = \infty\} = \{S \le T, h_T = \infty\} = \{h_T = \infty\} \quad P'-\text{a.s.}$$

where we used the constancy of h from time S on and the fact that $h_T < \infty$ for $T < S$, $P'-$a.s., which follows e.g. from ([JS], Thm IV 1.18).

Theorem 1.5 and example 1.4 answer the question raised in [JS] right after corollary IV 2.8 and also sharpen the assertions of ([JS], lemma IV 2.12). Our proof is quite different from the methodology used in [JS]: it uses a close monitoring of those paths of z_t, for which $z_{S-} = 0$, and an extension of the Borel-Cantelli lemma due to P. Lévy. Although the proof is elementary it is somewhat labourious and technical.

The paper is organized as follows: In section 2 we give the proof of theorem 1.5 and in section 3 we construct example 1.4.

2. Proof of theorem 1.5

The inclusion $\{0 < S \le T, z_{S-} = 0\} \supseteq \{h_T = \infty\}$, P'-a.s. is proved in ([JS], lemma IV 2.12a). The reverse inclusion can be deduced from ([JS], lemma IV 2.12b+d) in the case, where z isn't allowed to jump to 0 up to time T, i. e., under the assumption of theorem 1.3

$$P'_T\left[0 < S \le T, z_{S-} > z_S = 0\right] = 0. \tag{4}$$

It remains to prove

$$\{0 < S \le T, z_{S-} = 0\} \subseteq \{h_T = \infty\} \quad P'-\text{a.s.} \tag{5}$$

without assuming (4). As our proof will rely heavily on the fact that the local martingale M given by (1) is a Q-martingale, we are aiming at Q-almost sure results. So the first thing to do is replace (5) by

$$\{0 < S \le T, z_{S-} = 0 \text{ or } z'_{S-} = 0\} \subseteq \{h_T = \infty\} \quad Q-\text{a.s.}, \tag{5′}$$

which is indeed equivalent to (5). We need the following lemma:

Lemma 2.1. *Let $(A_t)_{t\in[0,\infty[}$ be an adapted increasing process on a filtered space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$, $(W_n)_{n\in\mathbb{N}}$ an increasing sequence of stopping times, such that for all $n \in \mathbb{N}$*

$$\mathbb{P}\left[W_{n-1} < \infty, \mathbb{P}\left[A_{W_n} - A_{W_{n-1}} \ge \alpha | \mathcal{F}_{W_{n-1}}\right] < \alpha\right] \le 2^{-n}, \quad \text{for some } \alpha > 0. \tag{6}$$

Then

$$\bigcap_{n=1}^{\infty} \{W_n < \infty\} \subseteq \{A_\infty = \infty\} \quad \mathbb{P}-\text{a.s.} \tag{7}$$

Proof. Let $B_n = \{W_n < \infty\}$, $B = \bigcap_{n=1}^{\infty}\{W_n < \infty\}$, $E_n = \{A_{W_n} - A_{W_{n-1}} \ge \alpha\}$ and $\xi_n = \mathbb{P}\left[E_n | \mathcal{F}_{W_{n-1}}\right]$. If $\mathbb{P}[B] = 0$, (7) is trivially satisfied, so we assume $\mathbb{P}[B] = a > 0$. Since $B_n \searrow B$, we have

$$\mathbb{P}[\xi_n < \alpha | B] \le \frac{\mathbb{P}[\xi_n < \alpha, B_n]}{\mathbb{P}[B]} \le a^{-1}2^{-n}.$$

The Borel-Cantelli lemma yields $\mathbb{P}[\xi_n \ge \alpha \text{ i.o.} | B] = 1$, therefore

$$B \subseteq \left\{\sum_{k=1}^{n} \xi_k \to \infty\right\} \quad \mathbb{P}-\text{a.s.}$$

By Levy's extension of the Borel-Cantelli lemmas (cf. [S], p518 or [W], Thm 12.15)

$$\left\{\sum_{k=1}^{\infty} \xi_k = \infty\right\} = \left\{\sum_{k=1}^{\infty} \mathbb{I}_{E_k} = \infty\right\} \quad \mathbb{P}-\text{a.s.},$$

and the observation $A_{W_n} \geq \alpha \sum_{k=1}^{n} \mathbb{I}_{E_k}$ completes the proof. □

Remark. Given $\alpha > 0$ and a nonnegative random variable X, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we point out the following simple implications

$$\mathbb{E}[X \wedge 2] \geq 3\alpha \;\Rightarrow\; \mathbb{P}[X \wedge 2 \geq \alpha] \geq \alpha \;\Rightarrow\; \mathbb{E}[X \wedge 2] \geq \alpha^2,$$

which justify replacing hypothesis (6) by the equivalent hypothesis

$$(6') \qquad \mathbb{P}\left[W_{n-1} < \infty, \mathbb{E}\left[(A_{W_n} - A_{W_{n-1}}) \wedge 2 | \mathcal{F}_{W_{n-1}}\right] < \alpha'\right] \leq 2^{-n}, \quad \text{for some } \alpha' > 0.$$

We now proceed to the proof of theorem 1.5. First we define an increasing sequence of stopping times $(U_n)_{n\in\mathbb{N}}$ by

$$U_0 = 0, \quad U_n = \inf\left\{t : U_{n-1} \leq t \leq T, z_t \neq 0, z'_t \neq 0, \left\{\frac{z_t}{z_{U_{n-1}}}, \frac{z'_t}{z'_{U_{n-1}}}\right\} \not\subset \left]\tfrac{1}{2}, 2\right[\right\}, \text{ for } n \geq 1$$

where $\inf \emptyset := \infty$. This definition ensures, that

$$(8) \qquad 0 \leq Y_{\tau-} \leq 2Y_{U_{n-1}} \quad Q\text{-a.s. for stopping times } \tau \in]\!]U_{n-1}, U_n]\!]$$

and that on the set $\{U_n < \infty\}$ we have $\frac{z_{U_n}}{z_{U_{n-1}}} \leq \frac{1}{2}$ or $\frac{z_{U_n}}{z_{U_{n-1}}} \geq 2$ or $\frac{2-z_{U_n}}{2-z_{U_{n-1}}} \leq \frac{1}{2}$ or $\frac{2-z_{U_n}}{2-z_{U_{n-1}}} \geq 2$. Therefore

$$z_{U_n} \in C(z_{U_{n-1}}) \quad Q\text{-a.s. on the set } \{U_n < \infty\},$$

where

$$(9) \qquad C(x) := \left]0, \tfrac{x}{2} \vee 2x - 2\right] \cup \left[2x \wedge 1 + \tfrac{x}{2}, 2\right[.$$

If $Q[\bigcap_{n=1}^{\infty}\{U_n < \infty\}] > 0$, then for any $\mu_0 > 0$

$$(10) \qquad Q[z_{U_n} \in]0, \mu_0[\cup]2 - \mu_0, 2[\,|\,U_n < \infty] \to 1, \quad \text{as } n \to \infty,$$

since on the set $\bigcap_{n=1}^{\infty}\{U_n < \infty\}$ the sequence $(z_{U_n})_{n\in\mathbb{N}}$ converges Q-a.s. to a random variable, which takes values in the two element set $\{0, 2\}$ (other values are not possible by the definition of $(U_n)_{n\in\mathbb{N}}$), and since $Q[U_n < \infty] \to Q[\bigcap_{n=1}^{\infty}\{U_n < \infty\}]$ as $n \to \infty$. In particular, we have

$$\left\{0 < S \leq T, z_{S-} = 0 \text{ or } z'_{S-} = 0\right\} = \bigcap_{n=1}^{\infty}\{U_n < \infty\} \quad Q\text{-a.s.},$$

so $(5')$ and thus theorem 1.5 will follow from lemma 2.1, if we can establish the truth of the subsequent lemma 2.2. □

Lemma 2.2. *There exists* $\gamma > 0$ *such that*

$$Q\left[U_{n-1} < \infty, \mathbb{E}\left[(h_{U_n} - h_{U_{n-1}}) \wedge 2 | \mathcal{F}_{U_{n-1}}\right] < \gamma^2\right] \to 0$$

as $n \to \infty$.

Proof. Suppose to the contrary that, for any $0 < \gamma < 1$, there is a constant $\beta = \beta(\gamma) > 0$ and for infinitely many n (depending on γ) an $\mathcal{F}_{U_{n-1}}$-measurable set $E_{n-1} = E_{n-1}(\gamma) \subset \{U_{n-1} < \infty\}$, such that $Q[E_{n-1}] \geq \beta$ and on E_{n-1} we have

$$(11) \qquad Q\left[h_{U_n} - h_{U_{n-1}} \geq \gamma | \mathcal{F}_{U_{n-1}}\right] < \gamma.$$

For given small $\mu_0 > 0$ (to be chosen later), we can and will assume by passing again to a subsequence and invoking (10), that

$$z_{U_{n-1}} \in]0, \mu_0[\cup]2-\mu_0, 2[\tag{12}$$

on an $\mathcal{F}_{U_{n-1}}$-measurable subset of E_{n-1} of Q-probability at least $\frac{\beta}{2}$, which we again denote by E_{n-1}.

We then define a sequence of stopping times $(V_n)_{n\in\mathbb{N}}$ by

$$V_n = \inf\left\{t : U_{n-1} \le t, h_t - h_{U_{n-1}} \ge \gamma\right\},$$

so that (11) implies

$$\mathbb{E}\left[\mathbb{1}_{\{V_n < U_n\}} | \mathcal{F}_{U_{n-1}}\right] < \gamma \quad Q\text{-a.s. on } E_{n-1}. \tag{13}$$

The definition of $(V_n)_{n\in\mathbb{N}}$ ensures, that

$$h_{V_n} - h_{U_{n-1}} < 2 \quad Q\text{-a.s. }, \tag{14}$$

since the jumps of h are bounded by 1 (see, e. g. [JS], IV 1.30).

In order to get grip of h, we employ the Doob-Meyer decomposition of the supermartingale Y given by (1)

$$Y = M - Y_- \bullet h$$

where h is the Hellinger process of order $\frac{1}{2}$ in the strict sense and M is a uniformly integrable martingale (cf. [JS], IV 1.18). Taking the difference $Y_t - Y_{U_{n-1}}$, and dividing by $Y_{U_{n-1}}$, we formally arrive at

$$\frac{Y_t}{Y_{U_{n-1}}} = 1 + \frac{1}{Y_{U_{n-1}}}\left(M_t - M_{U_{n-1}}\right) - \left[\left(\frac{Y_-}{Y_{U_{n-1}}} \bullet h\right)_t - \left(\frac{Y_-}{Y_{U_{n-1}}} \bullet h\right)_{U_{n-1}}\right].$$

This can be looked upon as the Doob-Meyer decomposition of the supermartingale

$$Y_t^{(n)} := \mathbb{1}_{\{t \le U_{n-1}\}} + \mathbb{1}_{\{t > U_{n-1}\}} \frac{Y_t}{Y_{U_{n-1}}}$$

on the set $\{t > U_{n-1}\}$, which we rewrite as

$$Y^{(n)} = M^{(n)} - Y_-^{(n)} \bullet h^{(n)},$$

with $M_t^{(n)} := 1 + \frac{1}{Y_{U_{n-1}}}\left(M_t - M_{U_{n-1}}\right)\mathbb{1}_{\{t > U_{n-1}\}}$ and $h_t^{(n)} := \left(h_t - h_{U_{n-1}}\right)\mathbb{1}_{\{t > U_{n-1}\}}$. The martingale $M^{(n)}$ is again uniformly integrable and starts at 1. In the sequel all expectations are taken with respect to Q. We are going to derive a contradiction in computing expectations at time $U_n \wedge V_n$ conditional on $\mathcal{F}_{U_{n-1}}$:

$$\mathbb{E}\left[Y^{(n)}_{U_n \wedge V_n} | \mathcal{F}_{U_{n-1}}\right] = \mathbb{E}\left[M^{(n)}_{U_n \wedge V_n} | \mathcal{F}_{U_{n-1}}\right] - \mathbb{E}\left[Y_-^{(n)} \bullet h^{(n)}_{U_n \wedge V_n} | \mathcal{F}_{U_{n-1}}\right].$$

From corollary 2.1 below (which is a kind of "uniformly strict" Jensen inequality for the concave function $f(x) = \sqrt{x(2-x)}$) it follows that for properly chosen μ_0 (cf. (12)) there exists $\epsilon > 0$ such that on the set E_{n-1} we have Q-a.s.

$$\mathbb{E}\left[Y^{(n)}_{U_n} | \mathcal{F}_{U_{n-1}}\right] \le 1 - \epsilon + \epsilon Q\left[U_n = \infty | \mathcal{F}_{U_{n-1}}\right].$$

This yields

$$\begin{aligned}
\mathbb{E}\left[Y^{(n)}_{U_n \wedge V_n} | \mathcal{F}_{U_{n-1}}\right] &= \mathbb{E}\left[Y^{(n)}_{U_n} | \mathcal{F}_{U_{n-1}}\right] - \mathbb{E}\left[Y^{(n)}_{U_n}\mathbb{1}_{\{V_n < U_n\}} | \mathcal{F}_{U_{n-1}}\right] + \mathbb{E}\left[Y^{(n)}_{V_n}\mathbb{1}_{\{V_n < U_n\}} | \mathcal{F}_{U_{n-1}}\right] \\
&\le \left[1 - \epsilon + \epsilon Q\left[U_n = \infty | \mathcal{F}_{U_{n-1}}\right]\right] + 2\gamma,
\end{aligned}$$

where the estimate of the third term employs (8) and (13).

Furthermore (8), (13) and (14) yield

$$\mathbb{E}\left[\left(Y_-^{(n)}\bullet h\right)_{U_n\wedge V_n}-\left(Y_-^{(n)}\bullet h\right)_{U_{n-1}}\Big|\mathcal{F}_{U_{n-1}}\right]\le 2\mathbb{E}\left[(h_{U_n\wedge V_n}-h_{U_{n-1}})|\mathcal{F}_{U_{n-1}}\right]$$

$$=2\mathbb{E}\left[(h_{V_n}-h_{U_{n-1}})1\!\!1_{\{V_n<U_n\}}|\mathcal{F}_{U_{n-1}}\right]+2\mathbb{E}\left[(h_{U_n}-h_{U_{n-1}})1\!\!1_{\{V_n\ge U_n\}}|\mathcal{F}_{U_{n-1}}\right]\le 4\gamma+2\gamma$$

also Q-a.s. on the set E_{n-1}. Thus we obtain Q-a.s. on the set E_{n-1}

$$1-\epsilon+\epsilon Q\left[U_n=\infty|\mathcal{F}_{U_{n-1}}\right]\ge 1-8\gamma,$$

which, if γ is small enough ($\gamma=\frac{\epsilon}{10}$ will do), can only be true for finitely many n, since $Q\left[U_n=\infty,U_{n-1}<\infty|\mathcal{F}_{U_{n-1}}\right]\to 0$ in Q-probability. This is the desired contradiction. $\square$

In order to have a means of discussing the problem of replacing h in Theorem 1.5 by certain Hellinger-like processes (we will address this problem in more detail in the remark after corollary 2.1), we introduce some classes of concave functions, and prove a lemma, which is a "uniformly strict" version of Jensen's inequality for one of these classes of functions.

Definition 2.1. *Let $\mathbb{F}_0$ be the class of concave functions $f:[0,2]\to\mathbb{R}^+$, satisfying $f(0)=0$ and*

$$1-\frac{a}{2}-2^{-b}>0,\qquad 1+b-2^a>0,\tag{15}$$

where we denote

$$a=\limsup_{x\to 0}\frac{xf'(x)}{f(x)},\qquad b=\liminf_{x\to 0}\frac{xf'(x)}{f(x)}.\tag{16}$$

Let moreover $\mathbb{F}_2=\{f(\cdot):\ f(2-\cdot)\in\mathbb{F}_0\}$ and $\mathbb{F}=\mathbb{F}_0\cap\mathbb{F}_2$.

Note, that this classes are not empty. The set of admissible values of (a,b) in (15) is in fact a subset of $]0,1[^2$ containing the set $\{(a,b):\ 0<a=b<1\}$. Practical members of $\mathbb{F}$ are the functions $f(x)=x^\alpha(2-x)^\beta$, where $0<\alpha,\beta<1$.

Lemma 2.3. *Let $f\in\mathbb{F}_0$. Then there exist $\epsilon_f>0$ and $\mu_f>0$ such that for all $\mu\in]0,\mu_f]$ the inequality*

$$\frac{\mathbb{E}\left[f(X)\right]}{f(\mu)}\le 1-\epsilon_f p$$

holds for any random variable X ranging in $[0,2]$ and satisfying

$$\mathbb{E}X=\mu,\quad \mathbb{P}\left[X\in D(\mu)\right]=p>0,$$

where $D(\mu)=]0,2[\setminus]\frac{\mu}{2},2\mu[$.

Proof. Since f is concave, the affine function $\ell(x)=f(\mu)+f'(\mu)(x-\mu)$ satisfies $f(x)\le\ell(x)$ and $f(\mu)=\ell(\mu)$. On the set $D(\mu)$ we even have $f(x)\le\ell(x)-m(\mu)$ with

$$m(\mu)=\min\left(\ell(\tfrac{\mu}{2})-f(\tfrac{\mu}{2}),\ell(2\mu)-f(2\mu)\right).$$

Therefore

$$\mathbb{E}\left[f(X)\right]\le\mathbb{E}\left[\ell(X)-m(\mu)1\!\!1_{\{X\in D(\mu)\}}\right]=\ell(\mu)-m(\mu)\mathbb{P}\left[X\in D(\mu)\right]=f(\mu)-m(\mu)p.$$

It remains to show that $\frac{m(\mu)}{f(\mu)}$ is bounded away from 0 uniformly as $\mu\to 0$: By (16) we have $\mu f'(\mu)\le af(\mu)\,(1+o(1))$ and $\frac{f'(\mu)}{f(\mu)}\ge\frac{b}{\mu}\,(1+o(1))$. Integrating the latter inequality from $\frac{\mu}{2}$ to μ yields $\ln\frac{f(\mu)}{f(\frac{\mu}{2})}\ge b\ln 2\,(1+o(1))$ and thus $f(\frac{\mu}{2})\le 2^{-b}f(\mu)\,(1+o(1))$. Therefore, as $\mu\to 0$,

$$\ell(\frac{\mu}{2})-f(\frac{\mu}{2})=f(\mu)-\frac{\mu}{2}f'(\mu)-f(\frac{\mu}{2})\ge\left(1-\frac{a}{2}-2^{-b}+o(1)\right)f(\mu),$$

and similarely $\ell(2\mu)-f(2\mu)\ge(1+b-2^a+o(1))\,f(\mu)$. We have derived

$$\frac{m(\mu)}{f(\mu)}\ge\min(1-\frac{a}{2}-2^{-b},\ 1+b-2^a)+o(1).$$

Choosing now $\epsilon_f=\frac{1}{2}\min(1-\frac{a}{2}-2^{-b},\ 1+b-2^a)$ and μ_f such, that the function $g(x)$ implied by the symbol $o(1)$ in the last inequality satisfies $|g(x)|\le\epsilon_f$ for $x\in]0,\mu_f]$ makes the proof complete. $\square$

Corollary 2.1. *There exist $\epsilon > 0$ and $\mu_0 > 0$ such that for all $\mu \in]0, \mu_0] \cup [2-\mu_0, 2[$ and for any random variable X ranging in $[0,2]$ and satisfying*

$$\mathbb{E}X = \mu, \quad \mathbb{P}[X \in C(\mu)] = p > 0,$$

where $C(\mu)$ is given by (9), we have

$$\mathbb{E}\left[\sqrt{\frac{X(2-X)}{\mu(2-\mu)}}\right] \leq 1 - \epsilon p.$$

Proof. This follows from Lemma 2.3 and the fact that $f(2-x) = f(x)$.

Remark. For given concave function f we can define a supermartingale $Y_t = f(z_t) = M_t + A_t$, and ask, if the increasing process H, defined via $Y = M - Y_- \bullet H$ and required to be 0 at 0 and constant after S (cf.(2)), can replace the Hellinger process h of order $\frac{1}{2}$ in Theorem 1.5, i.e., satisfies

$$\{Y_{S-} = 0\} = \{H_S = \infty\} \quad Q\text{−a.s.} \tag{17}$$

for any Q–martingale $(z_t)_{t=0}^{\infty}$ with $z_0 = 1$ and $0 \leq z_t \leq 2$. Note that the Hellinger process $h(\frac{1}{2})$ (resp. $h(\alpha)$, for $0 < \alpha < 1$) corresponds to the choice $f(z) = z^\alpha(2-z)^{1-\alpha}$.

One obvious condition, f must satisfy, is $f(0) = f(2) = 0$. Otherwise H_t, given by $\int_0^t \frac{dA_s}{Y_{s-}}$, would be finite Q−a.s. on at least one of the sets $\{z_{S-} = 0\}$ resp. $\{z_{S-} = 2\}$.

Another necessary condition for (17) is $f'(0) = f'(2) = \infty$. To see this, take $z_t = 1 + B_{t\wedge S}$, where (B_t) is a standard Brownian motion with respect to Q and $S = \inf\{t \geq 0 : |B_t| = 1\}$, and take $f(x) = 1 - |x-1|$, which satisfies $f'(0) = 1$. The Tanaka formula then reveals that $H_t = L_{t\wedge S}$, where L denotes the local time at 0 of (B_t). Now $S < \infty$ Q-a.s. and therefore also $H_S < \infty$ Q-a.s., but $Y_{S-} = f(z_{S-}) = 0$ Q-a.s.

The methods of our paper suffice to prove the $\subseteq$-part of (17) for concave functions f belonging to the class $\mathbb{F}$. In particular equations (8) and (14) remain true for $f \in \mathbb{F}$. The $\supseteq$-part of (17) for $f \in \mathbb{F}$ can be proved as in ([JS], lemma IV 2.12 a). Thus, in (17), H can in particular be one of the Hellinger processes of order α, for $0 < \alpha < 1$. However, there are concave functions satisfying $f(0) = f(2) = 0$ and $f'(0) = f'(2) = \infty$, but not contained in $\mathbb{F}$, such as $f(x) = x(2-x)\ln(\frac{x}{e})\ln(\frac{2-x}{e})$ and $f(x) = \left(\ln(\frac{x}{2e})\ln(\frac{2-x}{2e})\right)^{-1}$.

We do not know whether for general concave functions f the conditions

$$f(0) = f(2) = 0, \quad f'(0) = f'(2) = \infty$$

are also sufficient for (17) to hold, and leave this question for future research.

3. Example 1.4

Here we write down the example refered to in 1.4. Let $(\Omega, (\mathcal{F}_t)_{t=0}^2, \mathcal{F})$ and two probability measures P, P' be given by

$$\begin{aligned}
\Omega =& \{\omega_1, \omega_2, \omega_3, \omega_4\},\\
\mathcal{F}_0 =& \{\emptyset, \Omega\},\\
\mathcal{F}_1 =& \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}),\\
\mathcal{F}_2 =& \mathcal{F} = \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}),\\
P(\omega_1) =& 2P(\omega_3) = \tfrac{2}{3},\\
P'(\omega_2) =& 2P'(\omega_4) = \tfrac{2}{3}.
\end{aligned}$$

In this case we have $S = \mathbb{1}_{\{\omega_1, \omega_2\}} + 2 \cdot \mathbb{1}_{\{\omega_3, \omega_4\}}$.

Claim. *There is no $[0,\infty]$-valued predictable process H such that, for every stopping time T,*

$$P'_T \perp P_T \Leftrightarrow P'(H_T = \infty) = 1.$$

Proof. Assume that there is. Then $H_1 \equiv \text{const} < \infty$, P'-a.s., since $\mathcal{F}_0$ is trivial and $P'_1 \not\perp P_1$. On the other hand $H_S \equiv \infty$, P'-a.s., since $P_S \perp P'_S$. Now on the set $\{S = 1\}$, which has positive P'-measure, we have conflicting definitions of H_1. This contradicts our assumption. $\square$

Acknowledgement.

The authors thank an anonymous referee for suggestions which led to considerable improvement of the presentation.

References

[D] [redacted]. Durrett, *Probability: Theory and Examples*, Wadsworth, Inc., Belmont, California, 1991.

[DPV] K. [redacted]zhaparidze, P. Spreij, E. Valkeila, *On Hellinger Processes for Parametric Families of Experiments*, preprint (1996).

[HW] S. [redacted]V. He, J. G. Wang, *Remarks on Continuity, Contiguity and Convergence in Variation of Probability Measures*, Séminaire de Probabilités XXII, Lecture Notes in Mathematics 1321 (1988), Springer, Berlin Heidelberg.

[J1] [redacted] Jacod, *Processus de Hellinger, absolue continuité, contiguité*, Séminaire de Proba. de Rennes (1983), Rennes.

[J2] [redacted] Jacod, *Théorèmes limite pour les processus*, Ecole d'été de St-Flour XIII,1983, Lecture Notes in Mathematics 1117 (1985), Springer, Berlin Heidelberg.

[J3] [redacted] Jacod, *Filtered statistical models and Hellinger processes*, Stochastic Processes and their Appl. **32** (1989), 3-45.

[J4] [redacted] Jacod, *Convergence of filtered statistical models and Hellinger processes*, Stochastic Processes and their Appl. **32** (1989), 47-68.

[JM] J. [redacted]Jacod, J. Mémin, *Caractéristiques locales et conditions de continuité absolue pour les semimartingales*, Z. Wahrsch. Verw. Geb. **35** (1976), 1-37.

[JS] [redacted] Jacod, A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin, 1987.

[KLS] Ju[redacted]Kabanov, R. S. Liptser, A. N. Shiryaev, *Absolute continuity and singularity of locally absolutely continuous probability distributions (Part I)*, Math. USSR Sbornik **35** (1979), 631-680; *(Part II)* **36** (1980), 31-58.

[LS] R[redacted] S. Liptser, A. N. Shiryaev, *On the problem of "predictable" criteria of contiguity*, Proc. 5th Japan-USSR Symp. Lecture Notes in Mathematics 1021 (1983), Springer, Berlin Heidelberg, 384-418.

[S] A. N. Shiryaev, *Probability*, Springer, New York, 1996.

[W] D[redacted] Williams, *Probability with Martingales*, Cambridge University Press, Cambridge, 1991.