A BIPOLAR THEOREM FOR $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$

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Abstract. A consequence of the Hahn-Banach theorem is the classical bipolar theorem which states that the bipolar of a subset of a locally convex vector space equals its closed convex hull.

The space $L^0(\Omega, \mathcal{F}, \mathbb{P})$ of real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the topology of convergence in measure fails to be locally convex so that — a priori — the classical bipolar theorem does not apply. In this note we show an analogue of the bipolar theorem for subsets of the positive orthant $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$, if we place $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ in duality with itself, the scalar product now taking values in $[0, \infty]$. In this setting the order structure of $L^0(\Omega, \mathcal{F}, \mathbb{P})$ plays an important role and we obtain that the bipolar of a subset of $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ equals its closed, convex and solid hull.

In the course of the proof we show a decomposition lemma for convex subsets of $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ into a "bounded" and a "hereditarily unbounded" part, which seems interesting in its own right.

1. The Bipolar Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and denote by $L^0(\Omega, \mathcal{F}, \mathbb{P})$ the vector space of (equivalence classes of) real-valued measurable functions defined on $(\Omega, \mathcal{F}, \mathbb{P})$ which we equip with the topology of convergence in measure (see [KPR 84], chapter II, section 2). Recall the wellknown fact (see, e.g.,[KPR 84], theorem 2.2) that, for a diffuse measure $\mathbb{P}$, the topological dual of $L^0(\Omega, \mathcal{F}, \mathbb{P})$ is reduced to $\{0\}$ so that there is no counterpart to the duality theory, which works so nicely in the context of locally convex spaces (compare [Sch 67], chapter IV).

By $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ we denote the positive orthant of $L^0(\Omega, \mathcal{F}, \mathbb{P})$, i.e.,

$$L^0_+(\Omega, \mathcal{F}, \mathbb{P}) = \{f \in L^0(\Omega, \mathcal{F}, \mathbb{P}), f \geq 0\}.$$

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We may consider the dual pair of convex cones $\langle L^0_+(\Omega,\mathcal{F},\mathbb{P}), L^0_+(\Omega,\mathcal{F},\mathbb{P})\rangle$ where we define the scalar product $\langle f,g\rangle$ by

$$\langle f,g\rangle = \mathbb{E}[fg], \qquad f,g \in L^0_+(\Omega,\mathcal{F},\mathbb{P}).$$

Of course, this is not a scalar product in the usual sense of the word as it may assume the value $+\infty$. But the expression $\langle f,g\rangle$ is a welldefined element of $[0,\infty]$ and the application $(f,g) \longrightarrow \langle f,g\rangle$ has — mutatis mutandis — the obvious properties of a bilinear function.

The situation is similar to the one encountered at the very foundation of measure theory: to overcome the difficulty that $\mathbb{E}[f]$ does not make sense for a general element $f \in L^0(\Omega,\mathcal{F},\mathbb{P})$ one may either restrict to elements $f \in L^1(\Omega,\mathcal{F},\mathbb{P})$ or to elements $f \in L^0_+(\Omega,\mathcal{F},\mathbb{P})$, admitting in the latter case the possibility $\mathbb{E}[f] = +\infty$. In the present note we adopt this second point of view.

1.1 Definition. We call a subset $C \subseteq L^0_+$ *solid*, if $f \in C$ and $0 \le g \le f$ implies that $g \in C$. The set C is said to be *closed in probability* or simply *closed*, if it is closed with respect to the topology of convergence in probability.

1.2 Definition. For $C \subseteq L^0_+$ we define the *polar* C^0 of C by

$$C^0 = \{g \in L^0_+ : \mathbb{E}[fg] \le 1, \quad \text{for each } f \in C\}$$

1.3 Bipolar Theorem. *For a set $C \subseteq L^0_+(\Omega,\mathcal{F},\mathbb{P})$ the polar C^0 is a closed, convex, solid subset of $L^0_+(\Omega,\mathcal{F},\mathbb{P})$.*

The bipolar

$$C^{00} = \{f \in L^0_+ : \mathbb{E}[fg] \le 1, \quad \textit{for each } g \in C^0\}$$

is the smallest closed, convex, solid set in $L^0_+(\Omega,\mathcal{F},\mathbb{P})$ containing C.

To prove theorem 1.3 we need a decomposition result for convex subsets of L^0_+ we present in the next section. The proof of theorem 1.3 will be given in section 3.

We finish this introductory section by giving an easy extension of the bipolar theorem 1.3 to subsets of L^0 (as opposed to subsets of L^0_+). Recall that, with the usual definition of *solid* sets in vector lattices (see [Sch 67], chapter V, section 1), a set $D \subset L^0$ is defined to be *solid* in the following way.

1.4 Definition. A set $D \subset L^0$ is *solid*, if $f \in D$ and $h \in L^0$ with $|h| \le |f|$ implies $h \in D$.

Note that a set $D \subset L^0$ is solid if and only if the set of its absolut values $|D| = \{|h| : h \in D\} \subset L^0_+$ form a solid subset of L^0_+ as defined in 1.1 and $D = \{h \in L^0 : |h| \in |D|\}$. Hence the second part of theorem 1.3 implies:

1.5 Corollary. *Let $C \subset L^0$ and $|C| = \{|f| : f \in C\}$. Then the smallest closed, convex, solid set in L^0 containing C equals $\{f \in L^0 : |f| \in |C|^{00}\}$.*

Proof. Let D' be the smallest closed, convex, solid set in L^0_+ containing $|C|$ and $D = \{f : |f| \in D'\}$. One easily verifies that D is the smallest closed, convex and solid subset of L^0 containing C. Applying theorem 1.3 to $|C|$, we obtain that $D' = |C|^{00}$, which implies that $D = \{f \in L^0 : |f| \in |C|^{00}\}$. $\square$

For more detailed results in the line of corollary 1.5 concerning more general subsets of L^0 we refer to [B 97].

2. A Decomposition Lemma for Convex Subsets of $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$

Recall that a subset of a topological vector space X is *bounded* if it is absorbed by every zero-neighborhood of X ([Sch 67], Chapter I, Section 5). In the case of $L^0(\Omega, \mathcal{F}, \mathbb{P})$ this amounts to the following well-known concept.

2.1 Definition. A subset $C \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ is *bounded in probability* if, for $\varepsilon > 0$, there is $M > 0$ such that

$$\mathbb{P}[|f| > M] < \varepsilon, \qquad \text{for } f \in C.$$

We now introduce a concept which describes a strong form of unboundedness in $L^0(\Omega, \mathcal{F}, \mathbb{P})$.

2.2 Definition. A subset $C \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ is called *hereditarily unbounded in probability on a set* $A \in \mathcal{F}$, if, for every $B \in \mathcal{F}, B \subseteq A, \mathbb{P}[B] > 0$ we have that $C|_B = \{f\chi_B : f \in C\}$ fails to be a bounded subset of $L^0(\Omega, \mathcal{F}, \mathbb{P})$.

We now are ready to formulate the decomposition result:

2.3 Lemma. *Let C be a convex subset of $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$. There exists a partition of Ω into disjoint sets $\Omega_u, \Omega_b \in \mathcal{F}$ such that*

(1) *The restriction $C|_{\Omega_b}$ of C to Ω_b is bounded in probability.*
(2) *C is hereditarily unbounded in probability on Ω_u.*

The partition $\{\Omega_u, \Omega_b\}$ is the unique partition of Ω satisfying (1) and (2) (up to null sets). Moreover

(3) *If $\mathbb{P}[\Omega_b] > 0$ we may find a probability measure Q_b equivalent to the restriction $\mathbb{P}|_{\Omega_b}$ of $\mathbb{P}$ to Ω_b such that C is bounded in $L^1(\Omega, \mathcal{F}, Q_b)$. In fact, we may choose Q_b such that $\frac{dQ_b}{d\mathbb{P}}$ is uniformly bounded.*
(4) *For $\varepsilon > 0$ there is $f \in C$ s.t.*
$$\mathbb{P}[\Omega_u \cap \{f < \varepsilon^{-1}\}] < \varepsilon.$$
(5) *Denote by D the smallest closed, convex, solid set containing C. Then D has the form*
$$D = D|_{\Omega_b} \oplus L^0_+|_{\Omega_u},$$
where $D|_{\Omega_b} = \{\, u\,\chi_{\Omega_b} \ : \ u \in D\,\}$ and $L^0_+|_{\Omega_u} = \{\, v\,\chi_{\Omega_u} \ : \ v \in L^0_+(\Omega, \mathcal{F}, \mathbb{P})\,\}$.

Proof. Noting that the lemma holds true for C iff it holds true for the solid hull of C we may assume w.l.g. that C is solid and convex.

We now use a standard exhausting argument to obtain Ω_u. Denote by $\mathcal{B}$ the family of sets $B \in \mathcal{F}, \mathbb{P}[B] > 0$, verifying

$$\text{for } \varepsilon > 0 \text{ there is } f \in C, \text{ s.t. } \mathbb{P}[B \cap \{f < \varepsilon^{-1}\}] < \varepsilon.$$

Note that $\mathcal{B}$ is closed under countable unions: indeed, for $(B_n)_{n=1}^\infty$ is $\mathcal{B}$ and $\varepsilon > 0$, find elements $(f_n)_{n=1}^\infty$ in C such that

$$\mathbb{P}[B_n \cap \{f_n < 2^n\varepsilon^{-1}\}] < 2^{-n}\varepsilon.$$

Then, by the convexity and solidity of C

$$F_N = \sum_{n=1}^{N} 2^{-n} f_n$$

is in C and, for N large enough,

$$\mathbb{P}[B \cap \{F_N < \varepsilon^{-1}\}] < \varepsilon.$$

Hence there is a set of maximal measure in $\mathcal{B}$, which we denote by Ω_u and which is unique up to null-sets. Let $\Omega_b = \Omega \backslash \Omega_u$.

(1) and (3): If $\mathbb{P}[\Omega_b] = 0$ assertions (1) and (3) are trivially satisfied; hence we may assume that $\mathbb{P}[\Omega_b] > 0$. We want to verify (3). Note, since C is a solid subset of L^0_+, the convex set $C' = C \cap L^1(\Omega, \mathcal{F}, \mathbb{P}|_{\Omega_b})$ is dense in C with respect to the convergence in probability $\mathbb{P}|_{\Omega_b}$; hence, by Fatou's Lemma, it is enough to find a probability measure $Q_b \sim \mathbb{P}|_{\Omega_b}$ such that C' is bounded in $L^1(\mathbb{Q}_b)$. To this end we apply Yan's theorem ([Y 80], theorem 2) to C'. For convex, solid subsets C' of $L^1_+(\mathbb{P}|_{\Omega_b})$, this theorem states, that the following two assertions are equivalent:

- (i) for each $A \in \mathcal{F}$ with $\mathbb{P}|_{\Omega_b}[A] = \mathbb{P}[\Omega_b \cap A] > 0$, there is $M > 0$ such that $M\chi_A$ is not in the $L^1(\Omega, \mathcal{F}, \mathbb{P}|_{\Omega_b})$-closure of C';
- (ii) there exists a probability measure Q_b equivalent to $\mathbb{P}|_{\Omega_b}$ such that C' is a bounded subset of $L^1_+(\Omega, \mathcal{F}, Q_b)$. In addition, we may choose Q_b such that $\frac{dQ_b}{d\mathbb{P}}$ is uniformly bounded.

Assertion (i) is satisfied because otherwise we could find a subset $A \in \mathcal{F}, A \subset \Omega_b, \mathbb{P}[A] > 0$ belonging to the family $\mathcal{B}$, in contradiction to the construction of Ω_u above.

Hence assertion (ii) holds true which implies assertion (3) of the lemma. Obviously (3) implies assertion (1).

(2) and (4): As Ω_u is an element of $\mathcal{B}$ we infer that (4) holds true which in turn implies (2).

(5): Obviously $D \subset D|_{\Omega_b} \oplus L^0_+|_{\Omega_u}$. To show the reverse inclusion let $f = v + w$ with $v \in D|_{\Omega_b}$ and $w \in L^0_+|_{\Omega_u}$. We have to show that $f \in D$. Property (2) implies that, for every $n \in \mathbb{N}$, we find an $f_n \in C$ such that $\mathbb{P}[\{f_n \leq n^2\} \cap \Omega_u] \leq (1/n)$. Since $h_n = (1 - (1/n))\, v + (1/n)\, (f_n \wedge (n\, w)) \in D$ and $h_n \to v + w$ in probability, it follows that $f \in D$.

According to (2), C is unbounded in probability in $L^0(\Omega, \mathcal{F}, \mathbb{P}|_B)$ for each $B \subseteq \Omega_u$ with $\mathbb{P}[B] > 0$; the uniqueness of the decomposition $\Omega = \Omega_u \cup \Omega_b$ (up to null sets) with respect to the assertions (1) and (2) immediately follows from this. $\square$

3. The Proof of the Bipolar Theorem 1.3

To prove the first assertion of theorem 1.3 fix a set $C \subset L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ and note that the convexity and solidity of C^0 are obvious and the closedness of C^0 follows from Fatou's lemma.

To prove the second assertion of the theorem denote by D the intersection of all closed, convex and solid sets in L^0_+ containing C. Clearly D is closed, convex and solid, which implies the inclusion $D \subset C^{00}$. We have to show that $C^{00} \subseteq D$.

Using assertion (5) of lemma 2.3 we may decompose Ω into $\Omega = \Omega_b \cup \Omega_u$ such that $D = D|_{\Omega_b} \oplus L^0_+|_{\Omega_u}$ and (if $\mathbb{P}[\Omega_b] > 0$) we find a probability measure Q_b supported by Ω_b and equivalent to the restriction $\mathbb{P}|_{\Omega_b}$ of $\mathbb{P}$ to Ω_b such that D is bounded in $L^1(\Omega, \mathcal{F}, Q_b)$ (assertion (2)).

Now suppose that there is $f_0 \in C^{00} \backslash D$ and let us work towards a contradiction. Let $f_b = f_0 \chi_{\Omega_b}$ denote the restriction of f_0 to Ω_b. It is enough to show that f_b is in D. Let us denote by $D_b = \{f\chi_{\Omega_b} : f \in D\}$ the restriction of D to Ω_b and by

$$\tilde{D}_b = D_b - L^1_+(\Omega, \mathcal{F}, Q_b) = \{h \in L^1(\Omega, \mathcal{F}, Q_b)\} : \exists\, f \in D_b \text{ s.t. } h \leq f, Q_b - \text{a.s.}\}$$

the set of elements of $L^1(Q_b)$ dominated by an element of D_b. It is straightforward to verify that D_b and $\tilde{D}_b$ are $L^1(Q_b)$-closed, convex subsets of $L^1_+(Q_b)$ and $L^1(Q_b)$ respectively, and that D_b is bounded in $L^1_+(Q_b)$.

To show that f_b is contained in D (equivalently in D_b or in $\tilde{D}_b$) it suffices to show that $f_b \wedge M$ is in D_b, for each $M \in \mathbb{R}_+$. Indeed, by the $L^1(Q)$-boundedness and $L^1(Q)$-closedness of D_b this will imply that $f_b = L^1(Q) - \lim_{M\to\infty} f_b \wedge M$ is in D.

So we are reduced to assuming that f_b is an element of $L^1(Q_b)$ which is not an element of $\tilde{D}_b$. Now we may apply a version of the Hahn-Banach theorem (the separation theorem [Sch 67], theorem 9.2) to the Banach space $L^1(Q_b)$ to find an element $g \in L^\infty(Q_b)$ such that

$$\mathbb{E}[f_b g] > 1 \text{ while } \mathbb{E}[fg] \leq 1, \text{ for } f \in \tilde{D}_b.$$

As $\tilde{D}_b$ contains the negative orthant of $L^1(Q_b)$ we conclude that $g \geq 0$. Considering g as an element of $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ by letting g equal zero on Ω_u we therefore have that $g \in C^0$ and the first inequality above implies that $f_b \notin C^{00}$ and so that $f \notin C^{00}$, a contradiction finishing the proof. □

4. Notes and Comments

4.1 Note: Our motivation for the formulation of the bipolar theorem 1.3 above comes from Mathematical Finance: in the language of this theory there often comes up a duality relation between a set of contingent claims and a set of state price densities, i.e., Radon-Nikodym derivatives of absolutely continuous martingale measures. In this setting it turns out that $L^0(\Omega, \mathcal{F}, \mathbb{P})$ often is the natural space to work in (as opposed to $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p > 0$), as it remains unchanged under the passage from $\mathbb{P}$ to an equivalent measure Q (while $L^p(\Omega, \mathcal{F}, \mathbb{P})$ does change, for $0 < p < \infty$). We refer, e.g., [DS 94] for a general exposition of the above described duality relations and to [KS 97] for an applications of the bipolar theorem 1.3.

4.2 Note: Lemma 2.3 may be viewed as a variation of theorem 1 in [Y 80], which is a result based on previous work of Mokobodzki (as an essential step in Dellacherie's proof of the semimartingale characterization theorem due to Bichteler and Dellacherie; see [Me 79] and [Y 80]). The proof of Yan's theorem is a blend of a Hahn-Banach and an exhaustion argument (see, e.g., [S 94] for a presentation of this proof and [Str 90], [S 94] for applications of Yan's theorem to Mathematical

Finance) In fact, these arguments have their roots in the proof of the Halmos-Savage theorem [HS 49] and the theorems of Nikishin and Maurey [N 70], [M 74].

4.3 Note: In the course of the proof of lemma 2.3 we have shown that a convex subset C of $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ is hereditarily unbounded in probability on a set $A \in \mathcal{F}$ iff, for $\varepsilon > 0$, there is $f \in C$ with

$$\mathbb{P}[A \cap \{f < \varepsilon^{-1}\}] < \varepsilon,$$

which seems a fact worth noting in its own right.

4.4 Note: Notice that by theorem 1.3 the bipolar C^{00} of a given set $C \subset L^0_+$, although originally defined with respect to $\mathbb{P}$, does not change if we replace $\mathbb{P}$ by an equivalent measure $\mathbb{Q}$. This may also be seen directly (without applying theorem 1.3) in the following way: If $\mathbb{Q} \sim \mathbb{P}$ are equivalent probability measures and $h = d\mathbb{Q}/d\mathbb{P}$ is the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, then the polar $C^0(\mathbb{Q})$ of a given convex set $C \subset L^0_+$ with respect to $\mathbb{Q}$ equals $C^0(\mathbb{Q}) = h^{-1} \cdot C^0(\mathbb{P})$, where $C^0(\mathbb{P})$ is the dual of C with respect to $\mathbb{P}$. On the other hand $\mathbb{E}_{\mathbb{P}}[f\,g] = \mathbb{E}_{\mathbb{P}}[f\,h\,h^{-1}\,g] = E_{\mathbb{Q}}[f\,h^{-1}\,g]$ for all $g \in L^0_+$ and therefore the polar $C^{00}(\mathbb{Q})$ of $C^0(\mathbb{Q})$ (defined with respect to $\mathbb{Q}$) coincides with the polar $C^{00}(\mathbb{P})$ of $C^0(\mathbb{P})$ (defined with respect to $\mathbb{P}$).

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