Erratum to "Utility Maximization in Incomplete Markets with Random Endowment"

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Abstract

K. Larsen, M. Soner, and G. Zitkovic kindly pointed out to us an error in our paper [1] which appeared in 2001 in this journal. They also provide an explicit counter-example in [4].

In Theorem 3.1 of [1] it was incorrectly claimed (among several other correct assertions) that the value function $u(x)$ is continuously differentiable. The erroneous argument for this assertion is contained in Remark 4.2 of [1] where it was claimed that the dual value function $v(y)$ is strictly concave. As the functions u and v are mutually conjugate the continuous differentiability of u is equivalent to the strict convexity of v. By the same token, in Remark 4.3 the assertion on the uniqueness of the element $\hat{y}$ in the supergradient of $u(x)$ is also incorrect.

Similarly, the assertion in Theorem 3.1 (ii) that $\hat{y}$ and x are related via $\hat{y} = u'(x)$ is incorrect. It should be replaced by the relation $x = -v'(\hat{y})$ or, equivalently, by requiring that $\hat{y}$ is in the supergradient of $u(x)$.

To the best of our knowledge all the other statements in [1] are correct.

As we believe that the counter-example in [4] is beautiful and instructive in its own right we take the opportunity to present it in some detail.

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1 Discussion

We sketch the counter-example in [4] in a slightly modified and self-contained way. In the sequel we suppose that the reader is familiar with the paper [1] as well as with ([3], Example 5.1 bis). We briefly recall the notation of this example which is the basis of the counter-example in [4].

The stock price process $S = (S_0, S_1)$ is defined by $S_0 = 1$ and by letting S_1 assume the value $x_0 = 2$ with probability $p_0 = 1 - \alpha$ and, for $n \geq 1$, the value $x_n = \frac{1}{n}$ with probability $p_n = \alpha 2^{-n}$, for $0 < \alpha < 1$ sufficiently small.

For logarithmic utility $U(x) = \ln(x)$ we obtain, for given endowment $x > 0$, that it is optimal to invest the entire endowment into the stock S so that we end up at time $t = 1$ with the random wealth $\hat{X}(x) = xS_1$. For the value function $u(x)$ we thus obtain (see [3], Example 5.1 bis for details) the expected utility of $U(\hat{X}(x))$, i.e.

$$u(x) = \mathbb{E}[U(\hat{X}(x))] = \log(x) + \sum_{n=0}^{\infty} p_n \ln(x_n). \tag{1}$$

Using the notation from [1], the point of this example is that the dual optimizer $\hat{Q}(y)$ is not an element of L^1, but only of its bidual $(L^\infty)^*$. In other words, $\hat{Q}(y)$ defines a finitely additive probability measure on $(\Omega, \mathcal{F})$ which fails to be sigma-additive. We write $\hat{Q}(y) = \hat{Q}^r(y) + \hat{Q}^s(y)$ for the decomposition of $\hat{Q}(y)$ into its regular and its singular part.

To verify that $\|\hat{Q}^r(y)\| < \|\hat{Q}(y)\|$, fix $x = y = 1$. Writing $\hat{X}$ for $\hat{X}(1)$, $\hat{Q}$ for $\hat{Q}(1)$, and q_n for $\hat{Q}[S_1 = x_n]$ we have the relation ([3], Th. 2.2)

$$U'(\hat{X}) = \frac{d\hat{Q}^r}{d\mathbb{P}}, \tag{2}$$

so that

$$(x_n)^{-1} = \frac{q_n}{p_n}, \quad n \geq 0, \tag{3}$$

which yields

$$\sum_{n=0}^{\infty} q_n = \frac{1-\alpha}{2} + \alpha \sum_{n=1}^{\infty} n p_n. \tag{4}$$

This term is smaller than 1 (recall that $0 < \alpha < 1$ is small), which readily shows that the regular part $\hat{Q}^r$ has a smaller mass than $\hat{Q}$.

So far we just recalled Example 5.1 bis from [3]. For the next step we follow [2] and distinguish between the odd and the even numbers $n \in \mathbb{N}_0$:

$$A = \{S_1 = x_n, \quad \text{for odd}\, n \geq 1\}, \qquad B = \{S_1 = x_n, \quad \text{for even}\, n \geq 0\}.$$

Now comes the beautiful idea from [4]. Define the process $\tilde{S} = (\tilde{S}_0, \tilde{S}_1)$ by $\tilde{S}_0 = S_0 = 1$ and

$$\tilde{S}_1 = S_1 + \frac{1}{2}\mathbb{1}_A. \tag{5}$$

We also consider the random variable e_T which we define as

$$e_T = -\frac{1}{2}\mathbb{1}_A. \tag{6}$$

For the utility maximization problem, subject to the additional random endowment e_T, we define the value function $\tilde{u}(x)$ as in [1]. This boils down to the formula

$$\tilde{u}(x) = \sup_{\lambda \in \mathbb{R}} \mathbb{E}[\ln(x + \lambda(\tilde{S}_1 - \tilde{S}_0) + e_T)]. \tag{7}$$

For example, for $x = 1$, we again find that the optimizer $\hat{\lambda}$ in (7) equals 1, i.e. it again is optimal to invest the entire initial endowment $x = 1$ into the stock $\tilde{S}$: in this case the novel terms $\mathbb{1}_A$ in (5) and (6) cancel out perfectly. In fact, for all $x \geq 1$, we find that the optimal $\hat{\lambda}$ in (7) equals $\hat{\lambda}(x) = x$, just as in (1). Indeed, for $x \geq 1$, the crucial constraint is that we cannot invest more than the amount $\hat{\lambda} = x$ into the stock due to the definition of $\tilde{S}$ and e_T on the set B, which corresponds to the even numbers $n \in \mathbb{N}_0$.

On the other hand, for $x < 1$, the picture changes: now the binding constraint is given by the odd numbers $n \in \mathbb{N}_0$, i.e. the behaviour of $\tilde{S}$ and e_T on the set A. For $\frac{1}{2} < x \leq 1$, we obtain that the optimizer $\hat{\lambda}$ in (7) equals $\hat{\lambda}(x) = 1 - 2(1 - x) = 2x - 1$. The remaining amount $x - (2x - 1) = 1 - x$ of the initial wealth x is kept in the bond. Note that, for $x \leq \frac{1}{2}$ there is no admissible solution λ in (7), i.e. $\tilde{u}(x) = -\infty$ in (7).

We thus obtain for the optimal terminal wealth $\hat{X}(x) = x + \hat{\lambda}(\tilde{S}_1 - \tilde{S}_0) + e_T$, for $\frac{1}{2} < x \leq 1$,

$$\hat{X}(x) = \begin{cases} (2x-1)x_n + (1-x), & n \quad \text{even} \\ (2x-1)(x_n + \frac{1}{2}) + (\frac{1}{2} - x), & n \quad \text{odd} \end{cases} \tag{8}$$

and, for $1 \leq x < \infty$,

$$\hat{X}(x) = \begin{cases} x x_n, & n \quad \text{even} \\ x(x_n + \frac{1}{2}) - \frac{1}{2}, & n \quad \text{odd.} \end{cases} \tag{9}$$

Note that we always have that $\hat{X}(x)$ is an a.s. strictly positive random variable whose essential infimum is zero. The latter property is obtained by considering the sets $\{S_n = x_n\}$ with n tending to infinity, where we have to consider the odd $n's$, in the case $\frac{1}{2} < x \leq 1$, and the even $n's$, in the case $x \geq 1$.

Clearly, the definitions (8) and (9) coincide for $x = 1$ in which case we obtain $\hat{X}(1) = S_1$. In particular, the value function

$$\tilde{u}(x) = \mathbb{E}[\ln(\hat{X}(x))]$$

is continuous at $x = 1$, as must be the case.

We shall see that $\tilde{u}$ has a kink at $x = 1$. Indeed, we may calculate the derivative of $\tilde{u}(x)$, for $x \in]\frac{1}{2}, 1[$ as well as for $x \in]1, \infty[$ by using the formula

$$\frac{d}{dx}\tilde{u}(x) = \mathbb{E}[\frac{d}{dx}\ln(\hat{X}(x))], \quad x \in]\frac{1}{2}, 1[\cup]1, \infty[.$$

Hence the difference $\Delta\tilde{u}'(1) = \lim_{x\searrow 1}(\frac{d}{dx}\tilde{u}(x)) - \lim_{x\nearrow 1}(\frac{d}{dx}\tilde{u}(x))$ of the right and left derivative of $\tilde{u}$ at $x = 1$ can be explicitly computed as

$$\Delta\tilde{u}'(1) = \sum_{n=0}^{\infty} p_n(\frac{1}{x_n} - \frac{2}{x_n}) = -(\frac{p_0}{2} + \sum_{n=1}^{\infty} np_n),$$

which clearly shows that the function $\tilde{u}(x)$ fails to be differentiable at $x = 1$.

Summing up, following [4] we constructed an example where the value function $\tilde{u}(\cdot)$ fails to be differentiable.

We still want to have a closer look at the dual problem associated to the above example. In particular, we want to spot precisely where the erroneous argument in [1] has emerged.

Define

$$y_1 = \lim_{x\searrow 1}\Big(\frac{d}{dx}\tilde{u}(x)\Big) \quad \text{and} \quad y_2 = \lim_{x\nearrow 1}\Big(\frac{d}{dx}\tilde{u}(x)\Big).$$

As the dual value function $\tilde{v}$ (see [1] for the definition)

$$\tilde{v}(y) = \min_{Q\in\mathcal{D}}\Big\{\mathbb{E}\Big[V\Big(y\frac{dQ^r}{d\mathbb{P}}\Big)\Big] + y\langle Q, e_T\rangle\Big\} \tag{10}$$

is conjugate to $\tilde{u}$ (see [1]), we know from the fact that $\tilde{u}(x)$ has a kink at $x = 1$ that $\tilde{v}(y)$ is an affine function with slope -1 on the interval $[y_1, y_2]$, in view of the basic relation

$$\tilde{v}(y) = \sup_x\{\tilde{u}(x) - xy\}.$$

What are the dual optimizers $\hat{Q}_y$ for $y \in [y_1, y_2]$, given by ([1], Theorem 3.1 and Lemma 4.1)?

We know from [1] that the regular parts $\hat{Q}_y^r$ are unique and are given by the formula

$$U'(\hat{X}(x)) = y\frac{d\hat{Q}_y^r}{d\mathbb{P}}, \tag{11}$$

as in (2) above. The number x is associated to y via the relation $-\tilde{v}'(y) = x$ which yields $x = 1$, for $y \in [y_1, y_2]$.

This implies the amazing fact that the regular parts $y\hat{Q}_y^r$ of the dual optimizers $y\hat{Q}_y$ are identical, for all $y \in [y_1, y_2]$. Note that the total mass of the elements $y\hat{Q}_y \in (L^\infty)^*$ equals $\|y\hat{Q}_y\| = y$. If we pass, as usual, to the normalized finitely additive probability measures $\hat{Q}_y$, their regular parts $\hat{Q}_y^r$ scale by the factor y^{-1}.

As regards the singular part $\hat{Q}_y^s$ of $\hat{Q}_y$ it is clear that $\hat{Q}_y^s$ is supported by each of the sets

$$C_N = U_{n=N}^{\infty}\{S_n = x_n\}.$$

Indeed, for each $\epsilon > 0$, the singular measure $\hat{Q}_y^s$ is supported by the set $\{\hat{X}(x) < \epsilon\}$, where $-\tilde{v}'(y) = x$. This follows from the analysis in ([3], Example 5.1 bis). But now the additional aspect of the odd and even $n's$ arises: how much of this singular mass sits on $C_N \cap A$ and how much on $C_N \cap B$?

It follows from (8), (9) and the subsequent discussion that, for $\frac{1}{2} < x < 1$ and $\tilde{u}'(x) = y$, the singular measure $\hat{Q}_y^s$ is supported by A, while for $1 < x < \infty$ and $\tilde{u}'(x) = y$ the singular measure $\hat{Q}_y^s$ is supported by B. One may also pass to the limits $x \nearrow 1$ and $x \searrow 1$ to show that $\hat{Q}_{y_1}^s$ is supported by A, while $\hat{Q}_{y_2}^s$ is supported by B. It turns out that, for general $y \in [y_1, y_2]$ of the form $y = \mu y_1 + (1-\mu)y_2$, we have the affine relations

$$y\hat{Q}_y^s[A] = \mu y_1 \hat{Q}_{y_1}^s[A] \tag{12}$$

and

$$y\hat{Q}_y^s[B] = (1-\mu)y_2\hat{Q}_{y_2}^s[B]. \tag{13}$$

Indeed, as for an equivalent martingale measure Q for $\tilde{S}$ we have that $\mathbb{E}_Q[\tilde{S}_1 - \tilde{S}_0] = 0$, we also obtain $\langle \hat{Q}_y, \tilde{S}_1 - \tilde{S}_0 \rangle = 0$ by weak-star continuity. As $y\hat{Q}_y^r$ does not depend on y we obtain that $\langle y\hat{Q}_y^s, \tilde{S}_1 - \tilde{S}_0 \rangle$ does not depend on y either, for $y \in [y_1, y_2]$. On the set $C_N \cap A$ (resp. $C_N \cap B$) the random variable $\tilde{S}_1 - \tilde{S}_0$ equals $-\frac{1}{2}$ (resp. -1), up to an error of at most $\frac{1}{N}$, which disappears in the limit $N \mapsto \infty$. This implies that $-\frac{1}{2}y\hat{Q}_y^s[A] - y\hat{Q}_y^s[B]$ is constant when y varies in $[y_1, y_2]$ and readily yields the affine relations (12) and (13).

Finally let us have a closer look where the mistake in Remark 4.2 of [1] occurred. In this argument we have fixed numbers $0 < y_1 < y_2$ (which may or may not coincide with the y_1, y_2 considered above) and considered the value function $\tilde{v}(y)$ as in (10). For $y_1 \neq y_2$ we have that $y_1\hat{Q}_{y_1}$ is different from $y_2\hat{Q}_{y_2}$ as shown in [1]. Up to this point the reasoning was correct. We then tacitly (and incorrectly) assumed that this implies that their regular parts $y_1\hat{Q}^r_{y_1}$ and $y_2\hat{Q}^r_{y_2}$ must be different too! This would imply the strict inequality claimed in Remark 4.2 of [1]. But as we just have seen, it may happen that these two measures coincide. In addition, the singular parts $y\hat{Q}^s_y$ satisfy the affine relations (12) and (13) which also prevent the inequality in Remark 4.2 of becoming strict.

We finish this erratum by thanking again K. Larsen, M. Soner, and G. Zitkovic for providing this illuminating counter-example which we expect to have applications and allow for additional insight also beyond the present context.

References

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