The fundamental theorem of asset pricing for continuous processes under small transaction costs *

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Abstract

A version of the fundamental theorem of asset pricing is proved for continuous asset prices with small proportional transaction costs.

Equivalence is established between: (a) the absence of arbitrage with general strategies for arbitrarily small transaction costs $\varepsilon > 0$, (b) the absence of free lunches with bounded risk for arbitrarily small transaction costs $\varepsilon > 0$, and (c) the existence of ε-consistent price systems – the analogue of martingale measures under transaction costs – for arbitrarily small $\varepsilon > 0$.

The proof proceeds through an explicit construction, as opposed to the usual separation arguments. The paper concludes comparing

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numéraire-free and numéraire-based notions of admissibility, and the corresponding martingale and local martingale properties for consistent price systems.

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1 Introduction

The equivalence between (a suitable notion of) absence of arbitrage and the existence of pricing functionals (e.g. equivalent martingale measures) is at the cornerstone of asset pricing. The theory of frictionless markets goes back to the seminal papers of Harrison, Kreps and Pliska ([11], [12], [19]), and after 25 years of research the situation is now well-understood, see e.g. [8] for an overview.

For transaction costs, the theory was initiated by E. Jouini and H. Kallal ([14], compare also to [5], [4]). A satisfactory analogue to the "fundamental theorem of asset pricing" was obtained in [18] and [23] for discrete-time models. The first treatment involving continuous-time models was [14]. This pioneering work used an L^2 setting and a strong concept of no free lunch to obtain the equivalence between absence of arbitrage and the existence of dual variables (*consistent price systems* in the terminology of [23]). Another equivalence using limits has been established in [3].

The present paper relates, in the continuous-time setting, a notion of absence of arbitrage admitting a clear-cut economic interpretation to the existence of *consistent price systems*, which correspond to equivalent martingale measures in the frictionless case.

To motivate our approach, consider the explicit example of geometric fractional Brownian motion with Hurst parameter $H \neq 1/2$. In [9] it was shown that this model is arbitrage-free under arbitrarily small (proportional) transaction costs. Contrast this result to the fact that — except in the Brownian case — fractional Brownian motion is not a semi-martingale. Thus, [6, Theorem 7.2] implies the existence of arbitrage opportunities in the absence of transaction costs (more precisely: a free lunch with vanishing risk with respect to simple integrands; compare also to [21], [2]).

It was made clear in [10] that the only feature of geometric fractional Brownian motion $(S_t)_{0\leq t\leq T}$ relevant to this result is its conditional full support in the space of trajectories $C^+_{S_0}[0,T]$, i.e. in the set of continuous positive functions on $[0,T]$ starting at S_0. For the class of processes with conditional full support it is shown in [10] that, for arbitrarily small transaction costs, there exist consistent price systems (see Definition 1.3 below), which implies, in particular, absence of arbitrage.

Hence, for continuous processes, the property of conditional full support is a *sufficient* condition for the existence of consistent price systems for arbitrarily small transaction costs (but it is not necessary, see the example in the Appendix). In this paper we address the question to identify *necessary and sufficient* conditions, i.e., to prove a version of the fundamental theorem of asset pricing for arbitrarily small transaction costs. For the present setting of continuous price processes we obtain precise characterisations in Theorems 1.11 and 3.17 below.

The financial market we consider consists of a risk-free asset B, normalized to $B_t \equiv 1$, and a risky asset S, based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbf{P})$ satisfying the usual conditions of right continuity and saturatedness. $\mathcal{F}_0$ is also assumed trivial. Throughout the paper we make the following:

Assumption 1.1 *$(S_t)_{0 \le t \le T}$ is adapted to $(\mathcal{F}_t)_{0 \le t \le T}$, with continuous and strictly positive paths.*

We begin with the precise notions of arbitrage, free lunch, and pricing systems in the present context. First we introduce the very simple concept of obvious arbitrage:

Definition 1.2 *S satisfying Assumption 1.1 allows for an* obvious arbitrage *if there are $\alpha > 0$ and $[0, T] \cup \{\infty\}$-valued stopping times $\sigma \le \tau$ such that $\{\sigma < \infty\} = \{\tau < \infty\}$, $\mathbf{P}[\sigma < \infty] > 0$ and*

$$\begin{aligned} & S_\tau / S_\sigma \ge \ 1 + \alpha \qquad \text{on } \{\sigma < \infty\}, && (1) \\ \text{or} \quad & S_\tau / S_\sigma \le (1 + \alpha)^{-1} \quad \text{on } \{\sigma < \infty\}. && (2) \end{aligned}$$

If (1) is satisfied then it is indeed rather obvious how to make an arbitrage profit using a buy and hold strategy: at time σ one goes long in the stock S and waits until time τ to clear the position again. In case of (2) one goes short rather than long.

The crucial observation is that such an arbitrage opportunity persists, even with sufficiently small (i.e. $\varepsilon < \alpha/(2 + \alpha)$) proportional transaction costs.

We now formulate the notion of an ε-consistent price system, in the spirit of the pioneering paper of E. Jouini and H. Kallal [14] (compare also to section 5 below).

Definition 1.3 *Let S satisfy Assumption 1.1. An ε-*consistent price system *is a pair $(\widetilde{S}, \mathbf{Q})$ of a probability $\mathbf{Q}$ equivalent to $\mathbf{P}$, and a $\mathbf{Q}$-martingale $\widetilde{S} = (\widetilde{S}_t)_{0\le t\le T}$, such that*

$$1-\varepsilon \le \frac{\widetilde{S}_t}{S_t} \le 1+\varepsilon, \ \ a.s. \quad 0 \le t \le T,$$

*If both inequalities are strict, then $\widetilde{S}$ is an ε-*strictly consistent price system*.*

The weak assumption of "no obvious arbitrage" (NOA) already leads to a preliminary "local" result.

Theorem 1.4 *Let S satisfy Assumption 1.1. Then (NOA) holds iff there exists a sequence $(\tau_n)_{n=1}^{\infty}$ of $[0,T]$-valued stopping times, increasing to T, such that each stopped process S^{τ_n} admits an ε-consistent price system, for all $\varepsilon > 0$.*

In order to obtain an ε-consistent price system for the original process S rather than for its localizations, we need sharper concepts of no-arbitrage than just the above obvious one. In order to formulate these we need to formalize what we mean by a trading strategy.

Definition 1.5 *A trading strategy is a predictable finite-variation $\mathbb{R}$-valued process $\theta = (\theta_t)_{0\le t\le T}$ such that $\theta_0 = \theta_T = 0$. Its ($\mathbb{R}_+$-valued) ω-wise total variation process is denoted by* $\mathrm{Var}_s(\theta)$*:*

$$\mathrm{Var}_s(\theta) = \sup_{0\le t_0\le \ldots \le t_n = s} \sum_{i=1}^{n} |\theta_{t_i} - \theta_{t_{i-1}}|.$$

We then denote by $V^\varepsilon(\theta)$ the random variable

$$V^\varepsilon(\theta) := \int_0^T \theta_t dS_t - \varepsilon \int_0^T S_t d\mathrm{Var}_t(\theta), \tag{3}$$

and, for $0 \le t \le T$ we define the random variables $V_t^\varepsilon(\theta)$ as

$$V_t^\varepsilon(\theta) = V^\varepsilon(\theta 1_{(0,t)}), \tag{4}$$

*so that $V^\varepsilon(\theta) = V_T^\varepsilon(\theta)$. Given $M > 0$, the strategy θ is M-*admissible *if*

$$V_t^\varepsilon(\theta) \ge -M(1+S_t) \qquad a.s. \text{ for all } 0 \le t \le T$$

and denote by $\mathcal{A}_M^{\mathrm{adm}}(\varepsilon)$ the set of such θ. Define also $\mathcal{A}^{\mathrm{adm}}(\varepsilon) := \cup_{M>0}\mathcal{A}_M^{\mathrm{adm}}(\varepsilon)$.

When no ambiguity arises, we shall often drop ε in the notation and write $V(\theta), V_t(\theta)$.

Remark 1.6 Note that the integrals in (3) are a.s. well-defined pointwise, in the Riemann-Stieltjes sense, since $\mathrm{Var}_T(\theta)$ is a.s. finite. Indeed, for the second integral simply observe that S_t is continuous. The same argument applies to the first integral, after an integration by parts.

Remark 1.7 In the present setting of *continuous* processes S it is possible to consider only left-continuous predictable (or equivalently, right-continuous adapted) processes θ of bounded variation. The above definition is adopted from [1], and extends to asset prices with jumps.

Remark 1.8 The random variable $V(\theta)$ represents the final gain or loss when applying the trading strategy θ of holding θ_t shares at time t: during the interval $[t, t+dt]$ the value of the position in stock (without considering transaction costs) changes by $\theta_t dS_t$, while the transaction cost $\varepsilon S_t d\mathrm{Var}_t(\theta)$ is charged to the cash account.

The condition $\theta_0 = \theta_T = 0$ prescribes that a strategy must begin and end with a cash position only. Similarly, the term $1_{(0,t)}$ in the definition of $V_t(\theta)$ accounts for the liquidation cost.

Remark 1.9 An "obvious arbitrage" in the sense of Definition 1.2 is realized either by the strategy $\theta = 1_{]\!]\sigma,\tau]\!]}$ or by $\theta = -1_{]\!]\sigma,\tau]\!]}$.

Definition 1.10 *S admits arbitrage with ε-transaction costs if there is $\theta \in \mathcal{A}^{\mathrm{adm}}(\varepsilon)$ such that $V^\varepsilon(\theta) \geq 0$ a.s. and $\mathbf{P}[V^\varepsilon(\theta) > 0] > 0$.*

The following result characterizes absence of arbitrage in terms of consistent price systems. A crucial feature of this theorem is that both equivalent statements contain the quantifier "for all ε".

An attractive aspect of this result is that it is easier and cleaner than its frictionless counterpart (see e.g. [6]) as it does not involve limits (free lunches).

Theorem 1.11 (Fundamental Theorem) *Let S satisfy Assumption 1.1. The following assertions are equivalent:*

(i) For each $\varepsilon > 0$ there exists an ε-consistent price system.

(ii) For each $\varepsilon > 0$, there is no arbitrage for ε-transaction costs.

The paper is organized as follows: in section 2 Theorem 1.4 is shown, in section 3 we prove Theorem 1.11. Our proof relies on the paper [1], section 4 translates results of that paper to the setting we use here. Section 5 compares the cases with and without a numéraire. The Appendix contains some related (counter)examples.

2 The Local Theorem

This section contains the proof of Theorem 1.4. The localization will be done, for a given continuous price process $S = (S_t)_{0 \le t \le T}$ and fixed $\varepsilon > 0$, using the following sequence of $[0,T] \cup \{\infty\}$-valued stopping times. Define:

$$\overline{\rho}_0^\varepsilon = 0 \tag{5}$$
$$\overline{\rho}_n^\varepsilon = \inf\{t > \overline{\rho}_{n-1}^\varepsilon \mid S_t/S_{\overline{\rho}_{n-1}^\varepsilon} \ge 1+\varepsilon/3 \text{ or } S_t/S_{\overline{\rho}_{n-1}^\varepsilon} \le \tfrac{1}{1+\varepsilon/3}\}, \quad n \ge 1. \tag{6}$$

Adopting the usual convention that $\inf \emptyset = +\infty$, we obtain a sequence of stopping times increasing a.s. to ∞. Hence, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}[\overline{\rho}_n^\varepsilon < \infty] = 0.$$

Finally, define $\rho_n^\varepsilon := \overline{\rho}_n^\varepsilon \wedge T$.

Proposition 2.1 *Let S satisfy Assumption 1.1 and (NOA). For each $0 < \varepsilon < 1$ and $n \in \mathbb{N}$, the stopped process $S^{\rho_n^\varepsilon}$ admits an ε-consistent price system.*

Proof The proof borrows from an argument in [10]. We construct a process $(\widetilde{S}_t)_{0 \le t \le \rho_n^\varepsilon}$ based on and adapted to the stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$, as well as a measure $\mathbf{Q}_n$ on $\mathcal{F}_{\rho_n^\varepsilon}$ equivalent to $\mathbf{P}|_{\mathcal{F}_{\rho_n^\varepsilon}}$ such that $\tilde{S}_0 = S_0$ and

$$1-\varepsilon \le \frac{\widetilde{S}_t}{S_t} \le 1+\varepsilon, \quad 0 \le t \le \rho_n^\varepsilon \text{ and } \tilde{S}_{\rho_n^\varepsilon} = S_{\rho_n^\varepsilon} \text{ on } \{\rho_n^\varepsilon < T\}.$$

We proceed inductively on $i = 1, \dots, n$. For the first step recall that $\mathcal{F}_0$ is trivial, and divide Ω into three sets:

$$\begin{aligned} A_+ &= \{\rho_1^\varepsilon < T,\ S_{\rho_1^\varepsilon} = S_0(1+\varepsilon/3)\}, \\ A_- &= \{\rho_1^\varepsilon < T,\ S_{\rho_1^\varepsilon} = S_0/(1+\varepsilon/3)\}, \\ A_0 &= \Omega \setminus (A_+ \cup A_-) \subseteq \{\rho_1^\varepsilon = T\}. \end{aligned}$$

The crucial observation is that the cases $\mathbf{P}[A_+] = 1$ or $\mathbf{P}[A_-] = 1$ are excluded, as they would yield an obvious arbitrage, which is ruled out by assumption. We now distinguish between three cases:

Case 1: $\mathbf{P}[A_0] = 0$. As just noticed, in this case we necessarily have $\mathbf{P}[A_+] > 0$ and $\mathbf{P}[A_-] > 0$ and may define

$$Z_{\rho_1^\varepsilon} = \frac{\mu}{P[A_+]} 1_{A_+} + \frac{1-\mu}{P[A_-]} 1_{A_-}. \tag{7}$$

where $\mu = 1/(\varepsilon/3 + 2)$, and therefore:

$$\mu(1+\varepsilon/3) + \frac{1-\mu}{1+\varepsilon/3} = 1,$$

Define the probability $\mathbf{Q}_1$ on $\mathcal{F}_{\rho_1^\varepsilon}$ by $d\mathbf{Q}_1/d\mathbf{P} = Z_{\rho_1^\varepsilon}$, which is equivalent to $\mathbf{P}|_{\mathcal{F}_{\rho_1^\varepsilon}}$. Then, the process

$$\widetilde{S}_t = \mathbf{E}_{\mathbf{Q}_1}[S_{\rho_1^\varepsilon} \mid \mathcal{F}_t], \quad 0 \le t \le \rho_1^\varepsilon.$$

is obviously a $\mathbf{Q}_1$-martingale, and $\widetilde{S}_0 = \mathbf{E}_{\mathbf{Q}_1}[S_{\rho_1^\varepsilon}] = S_0$ by construction. Observe also that $\widetilde{S}_{\rho_1^\varepsilon} = S_{\rho_1^\varepsilon}$ almost surely. For the intermediate times $t \in \rrbracket 0, \rho_1^\varepsilon \llbracket$ we have the obvious estimate

$$1 - \varepsilon \le \frac{1}{(1+\varepsilon/3)^2} \le \frac{\widetilde{S}_t}{S_t} \le (1+\varepsilon/3)^2 \le 1 + \varepsilon, \quad 0 \le t \le \rho_1^\varepsilon. \tag{8}$$

Indeed, $\tilde{S}_{\rho_1^\varepsilon} = S_{\rho_1^\varepsilon} \in \left[\frac{S_0}{1+\varepsilon/3}, S_0(1+\varepsilon/3)\right]$ and hence, by the martingale property of $\tilde{S}$, $\widetilde{S}_t \in \left[\frac{S_0}{1+\varepsilon/3}, S_0(1+\varepsilon/3)\right]$ which yields (8), by the definition of ρ_1^ε.

Case 2: $0 < \mathbf{P}[A_0] < 1$. Note that $\rho_1^\varepsilon = T$ on A_0. Consider the random variable $S_{\rho_1^\varepsilon}$ which takes values in $\left[\frac{S_0}{1+\varepsilon/3}, S_0(1+\varepsilon/3)\right]$. We again want to construct a consistent price system.

Subcase 2a. Assume that $\mathbf{P}[A_+] > 0$ but $\mathbf{P}[A_-] = 0$. Let us replace $S_{\rho_1^\varepsilon}$ by an $\mathcal{F}_{\rho_1^\varepsilon}$-measurable random variable $\check{S}_{\rho_1^\varepsilon}$ by leaving it unchanged on A_+ while changing its values on A_0. More concretely, we define $\check{S}_{\rho_1^\varepsilon}$ to equal $S_0(1+\varepsilon/3)$ on A_+ and $S_0(1+\varepsilon/3)^{-1}$ on A_0 and let

$$\frac{d\mathbf{Q}_1}{d\mathbf{P}|_{\mathcal{F}_{\rho_1^\varepsilon}}} := 1_{A_+}\mu/\mathbf{P}[A_+] + 1_{A_-}(1-\mu)/\mathbf{P}[A_0],$$

where μ is chosen so that $\mathbf{E}_{\mathbf{Q}_1}[\check{S}_{\rho_1^\varepsilon}] = S_0$. We have

$$\frac{1}{1+\varepsilon/3} \le \frac{\check{S}_{\rho_1^\varepsilon}}{S_0} \le 1 + \varepsilon/3.$$

Define

$$\widetilde{S}_t = \mathbf{E}[\check{S}_{\rho_1^\varepsilon} \mid \mathcal{F}_t], \quad 0 \le t \le \rho_1^\varepsilon. \tag{9}$$

We again get $\widetilde{S}_0 = S_0$, but now we have $\widetilde{S}_{\rho_1^\varepsilon} = S_{\rho_1^\varepsilon}$ only on A_+. However, we will see that this is sufficient to carry out the inductive procedure. As regards the intermediate values $t \in]\!]0, \rho_1^\varepsilon[\![$ we still have the estimate (8).

The case where $\mathbf{P}[A_+] = 0$ and $\mathbf{P}[A_-] > 0$ is handled in the same fashion.

Subcase 2b. If $\mathbf{P}[A_+] > 0$, $\mathbf{P}[A_-] > 0$ then we proceed in a similar way. We choose $\check{S}_{\rho_1^\varepsilon}$ such that it equals S_0 on A_0 and $S_{\rho_1^\varepsilon}$ on $A_+ \cup A_-$, and define

$$\frac{d\mathbf{Q}_1}{d\mathbf{P}|_{\mathcal{F}_{\rho_1^\varepsilon}}} := 1_{A_0}\frac{\alpha}{\mathbf{P}[A_0]} + 1_{A_+}\frac{\mu(1-\alpha)}{\mathbf{P}[A_+]} + 1_{A_-}\frac{(1-\mu)(1-\alpha)}{\mathbf{P}[A_-]}.$$

Here $0 < \alpha < 1$ is an arbitrary constant, say $\alpha = 1/2$, while μ is determined by the condition:

$$\mathbf{E}_{\mathbf{Q}_1}[\check{S}_{\rho_1^\varepsilon}] = S_0.$$

Again, we define $\tilde{S}_t$ as in (9) and (8) will hold true.

Case 3: If $\mathbf{P}[A_0] = 1$ then define $\widetilde{S}_{\rho_1^\varepsilon} := S_0$ and $\mathbf{Q}_1 := \mathbf{P}|_{\mathcal{F}_{\rho_1^\varepsilon}}$. This completes the argument for $i = 1$.

For the general induction step, suppose that for the stopping time ρ_{i-1}^ε defined in (6) there is a process $(\widetilde{S}_t^{(i-1)})_{0\le t\le\rho_{i-1}^\varepsilon}$ and a probability measure $\mathbf{Q}_{i-1}$ on $\mathcal{F}_{\rho_{i-1}^\varepsilon}$, equivalent to $\mathbf{P}|_{\mathcal{F}_{\rho_{i-1}^\varepsilon}}$ such that $S_0 = \tilde{S}_0$,

$$1 - \varepsilon \le \frac{\widetilde{S}_t^{(i-1)}}{S_t} \le 1 + \varepsilon, \quad 0 \le t \le \rho_{i-1}^\varepsilon, \tag{10}$$

$$\big(\widetilde{S}_t^{(i-1)}\big)_{0\le t\le\rho_{i-1}^\varepsilon} \text{ is a } \mathbf{Q}_{i-1}\text{-martingale.} \tag{11}$$

$$\tilde{S}_{\rho_{i-1}^\varepsilon} = S_{\rho_{i-1}^\varepsilon} \text{ on } \{\rho_{i-1}^\varepsilon < T\}. \tag{12}$$

Define the partition of $\{\rho_{i-1}^\varepsilon < T\}$ into three sets

$$\begin{aligned}
A_+^i &= \{\rho_{i-1}^\varepsilon < T\} \cap \{S_{\rho_i^\varepsilon} = S_{\rho_{i-1}^\varepsilon}(1+\varepsilon/3)\},\\
A_-^i &= \{\rho_{i-1}^\varepsilon < T\} \cap \{S_{\rho_i^\varepsilon} = S_{\rho_{i-1}^\varepsilon}/(1+\varepsilon/3)\},\\
A_0^i &= \{\rho_{i-1}^\varepsilon < T\} \cap (\Omega \setminus (A_+^i \cup A_-^i)) \subseteq \{\rho_{i-1}^\varepsilon < T\} \cap \{\rho_i^\varepsilon = T\}.
\end{aligned}$$

We proceed as in the first step, conditionally on $\mathcal{F}_{\rho_{i-1}^\varepsilon}$, reasoning on the sets:

$$\begin{aligned}
B_1^i &= \{\rho_{i-1}^\varepsilon < T\} \cap \{\mathbf{P}[A_0^i \mid \mathcal{F}_{\rho_{i-1}^\varepsilon}] = 0\}\\
B_2^i &= \{\rho_{i-1}^\varepsilon < T\} \cap \{\mathbf{P}[A_0^i \mid \mathcal{F}_{\rho_{i-1}^\varepsilon}] \in (0,1)\}\\
B_3^i &= \{\rho_{i-1}^\varepsilon < T\} \cap \{\mathbf{P}[A_0^i \mid \mathcal{F}_{\rho_{i-1}^\varepsilon}] = 1\}
\end{aligned}$$

which correspond to the three cases considered in the first step.

Again, the crucial observation is that there cannot be a set $B \in \mathcal{F}_{\rho^\varepsilon_{i-1}}$ with $B \subseteq \{\rho^\varepsilon_{i-1} < T\}$ and $\mathbf{P}[B] > 0$, such that $B \subseteq A^i_+$ or $B \subseteq A^i_-$ almost surely. Indeed, otherwise an obvious arbitrage arises for the stopping times $\sigma = \rho^\varepsilon_{i-1} 1_B + \infty\, 1_{\Omega\setminus B}$ and $\tau = \rho^\varepsilon_i 1_B + \infty\, 1_{\Omega\setminus B}$.

Case 1: As just remarked, on B^i_1 the absence of obvious arbitrage implies that $P[A^i_+|\mathcal{F}_{\rho^\varepsilon_{i-1}}] = 1$ or $P[A^i_-|\mathcal{F}_{\rho^\varepsilon_{i-1}}] = 1$ can hold only on a set of 0 probability. So we define $Z_{\rho^\varepsilon_i}$ on B^i_1 similarly as in (7) by

$$Z_{\rho^\varepsilon_i} 1_{B^i_1} = \left(\frac{\mu}{\mathbf{P}[A^i_+|\mathcal{F}_{\rho^\varepsilon_{i-1}}]} 1_{A^i_+} + \frac{1-\mu}{\mathbf{P}[A^i_-|\mathcal{F}_{\rho^\varepsilon_{i-1}}]} 1_{A^i_-} \right) 1_{B^i_1}.$$

Case 2: Turning to the set B^i_2, we again have to modify the random variable $S_{\rho^\varepsilon_i}$ on A^i_0 to obtain an $\mathcal{F}_{\rho^\varepsilon_i}$-measurable $\check{S}_{\rho^\varepsilon_i}$ similarly as above. To avoid measurable selection issues, we provide an explicit construction. Set:

$$\begin{aligned} H^i_1 &:= B^i_2 \cap \{\mathbf{P}[A^i_-|\mathcal{F}_{\rho^\varepsilon_{i-1}}] = 0\}, \\ H^i_2 &:= B^i_2 \cap \{\mathbf{P}[A^i_+|\mathcal{F}_{\rho^\varepsilon_{i-1}}] = 0\}, \\ H^i_3 &:= B^i_2 \setminus (H^i_1 \cup H^i_2). \end{aligned}$$

Subcase 2a: On H^i_1 we set

$$\check{S}_{\rho^\varepsilon_i} 1_{H^i_1} := S_{\rho^\varepsilon_{i-1}} (1+\varepsilon/3) 1_{A^i_+ \cap H^i_1} + S_{\rho^\varepsilon_{i-1}} (1+\varepsilon/3)^{-1} 1_{A^i_0 \cap H^i_1}$$

and define

$$Z_{\rho^\varepsilon_i} 1_{H^i_1} = \frac{\mu}{\mathbf{P}[A^i_+|\mathcal{F}_{\rho^\varepsilon_{i-1}}]} 1_{A^i_+ \cap H^i_1} + \frac{1-\mu}{\mathbf{P}[A^i_0|\mathcal{F}_{\rho^\varepsilon_{i-1}}]} 1_{A^i_0 \cap H^i_1},$$

where μ is such that

$$\mathbf{E}[Z_{\rho^\varepsilon_i} \check{S}_{\rho^\varepsilon_i} | \mathcal{F}_{\rho^\varepsilon_{i-1}}] 1_{H^i_1} = S_{\rho^\varepsilon_{i-1}} 1_{H^i_1}$$

holds true. On H^i_2 we proceed similarly.

Subcase 2b: Define $Z_{\rho^\varepsilon_i}$ and $\check{S}_{\rho^\varepsilon_i}$ on H^i_3 by

$$\begin{aligned} Z_{\rho^\varepsilon_i} 1_{H^i_3} &:= 1_{A^i_0 \cap H^i_3} \frac{\alpha}{\mathbf{P}[A^i_0|\mathcal{F}_{\rho^\varepsilon_{i-1}}]} + 1_{A^i_+ \cap H^i_3} \frac{\mu(1-\alpha)}{\mathbf{P}[A^i_+|\mathcal{F}_{\rho^\varepsilon_{i-1}}]} + 1_{A^i_- \cap H^i_3} \frac{(1-\mu)(1-\alpha)}{\mathbf{P}[A^i_-|\mathcal{F}_{\rho^\varepsilon_{i-1}}]}, \\ \check{S}_{\rho^\varepsilon_i} 1_{H^i_3} &:= S_{\rho^\varepsilon_i} 1_{(A^i_+ \cup A^i_-) \cap H^i_3} + S_{\rho^\varepsilon_{i-1}} 1_{A^i_0 \cap H^i_3}, \end{aligned}$$

with an arbitrary $0 < \alpha < 1$, say $\alpha := 1/2$ and μ such that

$$\mathbf{E}[Z_{\rho^\varepsilon_i} \check{S}_{\rho^\varepsilon_i} | \mathcal{F}_{\rho^\varepsilon_{i-1}}] 1_{H^i_3} = S_{\rho^\varepsilon_{i-1}} 1_{H^i_3}.$$

Case 3: We set $Z_{\rho_i^\varepsilon} := 1$ on B_3^i.

We now have defined the conditional density $Z_{\rho_i^\varepsilon}$ on B_3^i as well as on B_1^i and B_2^i. To put the pieces together, define the probability measure $\mathbf{Q}_i$ on $\mathcal{F}_{\rho_i^\varepsilon}$ by

$$\frac{d\mathbf{Q}_i}{d\mathbf{P}|_{\mathcal{F}_{\rho_i^\varepsilon}}} = \frac{d\mathbf{Q}_{i-1}}{d\mathbf{P}|_{\mathcal{F}_{\rho_{i-1}^\varepsilon}}} \left[Z_{\rho_i^\varepsilon} 1_{\{\rho_{i-1}^\varepsilon < T\}} + 1_{\{\rho_{i-1}^\varepsilon = T\}} \right]$$

and

$$\widetilde{S}^{(i)}_{\rho_i^\varepsilon} = S_{\rho_i^\varepsilon} 1_{B_1^i} + \check{S}_{\rho_i^\varepsilon} 1_{B_2^i} + \widetilde{S}^{(i-1)}_T 1_{\{\rho_{i-1}^\varepsilon = T\}} + S_{\rho_{i-1}^\varepsilon} 1_{B_3^i}.$$

Letting

$$\widetilde{S}^{(i)}_t = \mathbf{E}_{\mathbf{Q}_i} \left[\widetilde{S}^{(i)}_{\rho_i^\varepsilon} \,\middle|\, \mathcal{F}_t \right], \quad 0 \le t \le \rho_i^\varepsilon,$$

we have completed the induction step, as conditions (10), (11) and (12) are satisfied with $i-1$ replaced by i.

Summing up, we have constructed, for each $n \in \mathbb{N}$, a probability measure $\mathbf{Q}_n$ on $\mathcal{F}_{\rho_n^\varepsilon}$, equivalent to $\mathbf{P}|_{\mathcal{F}_{\rho_n^\varepsilon}}$ and a $\mathbf{Q}_n$-martingale $(\widetilde{S}^{(n)}_t)_{0 \le t \le \rho_n^\varepsilon}$ such that

$$1 - \varepsilon \le \frac{\widetilde{S}^{(n)}_t}{S_t} \le 1 + \varepsilon.$$

We may extend each $\mathbf{Q}_n$ to $\mathcal{F}$ by setting

$$\mathbf{Q}_n[A] := \int_A \frac{d\mathbf{Q}_n}{d\mathbf{P}|_{\mathcal{F}_{\rho_n^\varepsilon}}} d\mathbf{P}, \quad A \in \mathcal{F},$$

which completes the proof. ■

Remark 2.2 The above construction verifies a concatenation property: on $[\![0, \rho_{n-1}^\varepsilon]\!]$ the processes $\widetilde{S}^{(n)}$ and $\widetilde{S}^{(n-1)}$ coincide, and $\mathbf{Q}_{n-1}|_{\mathcal{F}_{\rho_{n-1}^\varepsilon}}$ equals $\mathbf{Q}_n|_{\mathcal{F}_{\rho_{n-1}^\varepsilon}}$.

Having proved Proposition 2.1, the remainder of the proof of Theorem 1.4 is standard.

Proof of Theorem 1.4 First, suppose that there is no obvious arbitrage. Fix a sequence $(\varepsilon_k)_{k \ge 1}$ in $\mathbb{R}_+$ tending to zero and find, for $n \ge 1$, an increasing sequence of integers $(m_{n,k})_{k=1}^\infty$ such that

$$\mathbf{P}\left[\rho^{\varepsilon_k}_{m_{n,k}} < T\right] < 2^{-(n+k)}.$$

Letting $\tau_n = \bigwedge_{k=1}^\infty \rho^{\varepsilon_k}_{m_{n,k}}$ we find that

$$\mathbf{P}[\tau_n < T] < 2^{-n}.$$

Clearly $(\tau_n)_{n=1}^{\infty}$ is a sequence of stopping times increasing a.s. to T and, for each $n \geq 1$ and $\varepsilon > 0$, the stopped process S^{τ_n} admits an ε-consistent price system.

To see the other direction of Theorem 1.4, notice that the convergence of τ_n to T implies that any obvious arbitrage can also be realized by some σ, τ such that $\tau \leq \tau_k$ on $\{\tau < \infty\}$ for some k. But this is clearly excluded by the existence of consistent price systems for arbitrary ε, see the second part of the proof of Theorem 1.11 below. ■

3 Duality Theory

The previous section established that, under the no obvious arbitrage (NOA) condition, there are localizations $(S^{\tau_n})_{n\geq 1}$ admitting ε-consistent price systems. Thus the duality theory developed in [1] or [17] applies to the stopped processes $(S^{\tau_n})_{n\geq 1}$, and this section employs dual arguments to construct consistent price systems for S.

In this section, fix a process $S = (S_t)_{0\leq t\leq T}$ as in Theorem 1.4, and the associated stopping times $(\tau_n)_{n=1}^{\infty}$ such that S^{τ_n} admits an ε-consistent price systems for all $\varepsilon > 0$ and $n \geq 1$.

If $\tau_n \geq T$ a.s. for some n, then S already has ε-consistent price systems, and Theorem 1.11 immediately follows from Theorem 1.4. To exclude this trivial case, we make throughout this section the following:

Assumption 3.1 $\mathbf{P}[\tau_n < T] > 0$ *for all* $n \geq 1$.

Denote by $\mathcal{T}$ the set of $[0, T]$-valued stopping times σ such that $\mathbf{P}[\sigma < T] > 0$ and $\sigma \leq \tau_n$ a.s. on $\{\sigma < T\}$ for some $n \geq 1$. For example, $\tau_k \in \mathcal{T}$ for each k, due to Assumption 3.1.

Lemma 3.2 *Let S satisfy Assumption 1.1 and (NOA), let $\sigma \in \mathcal{T}$ and $\sigma \leq \tau_n$ on $\{\sigma < T\}$. The process ${}^{\sigma}S^{\tau_n} = (S_t)_{\sigma\leq t\leq\tau_n}$ also admits an ε-consistent price system, for all $\varepsilon > 0$. More precisely, for any $\varepsilon > 0$ and any probability $\mathbf{R}$ on $\mathcal{F}_\sigma$ with $\mathbf{R} \sim 1_{\{\sigma<T\}}\mathbf{P}|_{\mathcal{F}_\sigma}$, there exists a probability $\mathbf{Q}$ on $\mathcal{F}_{\tau_n}$ and a process $\widetilde{S} = (\widetilde{S}_t)_{\sigma\leq t\leq\tau_n}$ such that:*

(i) $\mathbf{Q}\big|_{\mathcal{F}_\sigma} = \mathbf{R}$,

(ii) *$\widetilde{S}$ is a martingale under $\mathbf{Q}$ on $[\![\sigma, \tau_n]\!]$, i.e., for each stopping time ρ s.t. $\sigma \leq \rho \leq \tau_n$ we have*
$$\mathbf{E}_{\mathbf{Q}}[\widetilde{S}_{\tau_n}|\mathcal{F}_\rho] = \widetilde{S}_\rho,$$

(iii) $\widetilde{S}_\sigma = S_\sigma$ *and*

$$\frac{1}{1+\varepsilon} \le \frac{\widetilde{S}_t}{S_t} \le 1+\varepsilon, \quad \textit{for } \sigma \le t \le \tau_n. \tag{13}$$

Proof Let $0 < \varepsilon < 1$ and take, for $\delta = \varepsilon/3$, a δ-consistent price system $((\widetilde{S}_t^{(\delta)})_{0\le t\le\tau_n}, \mathbf{Q}^{(\delta)})$ for the process $(S_t)_{0\le t\le\tau_n}$ provided by Theorem 1.4. Define

$$\widetilde{S}_t = \widetilde{S}_t^{(\delta)} \frac{S_\sigma}{\widetilde{S}_\sigma^{(\delta)}}, \quad \sigma \le t \le \tau_n$$

and

$$\frac{d\mathbf{Q}}{d\mathbf{P}} := \frac{d\mathbf{Q}^{(\delta)}}{d\mathbf{P}} \frac{d\mathbf{R}/d(\mathbf{P}|\mathcal{F}_\sigma)}{E[d\mathbf{Q}^{(\delta)}/d\mathbf{P}|\mathcal{F}_\sigma]}.$$

Then $(\widetilde{S}, \mathbf{Q})$ satisfies (i), (ii) and (iii). ■

Definition 3.3 *For* $\varepsilon > 0$ *and a stopping time* $\sigma \in \mathcal{T}$ *satisfying* $\sigma \le \tau_n$ *on* $\{\sigma < T\}$*, define* $\mathrm{CPS}(\sigma,\varepsilon,n)$ *as the family of all pairs* $(\widetilde{S}, \mathbf{Q})$ *such that* $\mathbf{Q}$ *is a probability on* $\mathcal{F}_{\tau_n}$*,* $\mathbf{Q}|_{\mathcal{F}_\sigma} \sim 1_{\{\sigma<T\}}\mathbf{P}|_{\mathcal{F}_\sigma}$*, (ii) of Lemma 3.2 and* (13) *above are satisfied. The subfamily of pairs satisfying* (13) *with strict inequalities is denoted by* $\mathrm{SCPS}(\sigma,\varepsilon,n)$*. (Cf. Definition 1.3.)*

Lemma 3.2 states that $CPS(\sigma,\varepsilon,n)$ is nonempty for each $\sigma \in \mathcal{T}$ and for each n such that $\sigma \le \tau_n$ on $\{\sigma < T\}$.

Definition 3.4 *Let S satisfy the Assumptions 1.1, 3.1 and (NOA). For* $\varepsilon > 0$ *and* $\sigma \in \mathcal{T}$ *define*[1]

$$F(\sigma,\varepsilon) = \lim_{n\to\infty} \inf_{(\widetilde{S},\mathbf{Q})\in\mathrm{CPS}(\sigma,\varepsilon,n)} \mathbf{Q}[\tau_n < T], \tag{14}$$

$$G(\sigma,\varepsilon) = \lim_{n\to\infty} \inf_{(\widetilde{S},\mathbf{Q})\in\mathrm{CPS}(\sigma,\varepsilon,n)} \mathbf{E}_\mathbf{Q}\left[\frac{\widetilde{S}_{\tau_n}}{\widetilde{S}_\sigma} 1_{\{\tau_n<T\}}\right], \tag{15}$$

and

$$F(\varepsilon) = \sup_{\sigma\in\mathcal{T}} F(\sigma,\varepsilon), \quad G(\varepsilon) = \sup_{\sigma\in\mathcal{T}} G(\sigma,\varepsilon), \tag{16}$$

These definitions beg for an intuitive explanation. For instance, consider $\sigma \equiv 0$ in (14). Then, for a fixed n, the right hand side of (14) estimates the minimum $\mathbf{Q}$-probability assigned by any ε-consistent price system $(\widetilde{S}, \mathbf{Q})$ to the event $\{\tau_n < T\}$. Since $\lim_{n\to\infty} \mathbf{P}[\tau_n < T] = 0$, a strictly positive value of

[1]The existence of the limits follows by the argument of Claim 1 in Lemma 3.5.

$F(0,\varepsilon)$ indicates that, for $(\widetilde{S},\mathbf{Q}) \in \mathrm{CPS}(0,\varepsilon,n)$ the probabilities $\mathbf{Q}$ become increasingly singular with respect to $\mathbf{P}$ as $n \to \infty$. This intuitive idea can be made precise by the notion of contiguity (see e.g. [13]).

Similarly, $G(0,\varepsilon)$ measures the limiting probability (as $n \to \infty$) of the event $\{\tau_n < T\}$ assigned by the measure $\mathbf{R}$ defined by $d\mathbf{R}/d\mathbf{Q} = \widetilde{S}_{\tau_n}/\widetilde{S}_0$.

The idea is that $F(\varepsilon) = G(\varepsilon) = 0$, for all $\varepsilon > 0$, corresponds to the "regular" case, where there are ε-consistent price systems and — dually — there are no free lunches. On the contrary, the case $F(\varepsilon) > 0$ or $G(\varepsilon) > 0$, for some $\varepsilon > 0$ corresponds to the "singular" case, where there are no ε-consistent price system and — again dually — there are free lunches. We shall now show that this heuristic idea can indeed be made precise: in general, however, instead of considering $\sigma = 0$ only, one needs to consider arbitrary stopping times $\sigma \in \mathcal{T}$, which leads to definition (16) above. The proof of Proposition A.2 in the Appendix shows a concrete example where no obvious arbitrage exists, and yet $F(\varepsilon) = 1$.

Lemma 3.5 *Let S be as in Definition 3.4. Let $\sigma \in \mathcal{T}$ such that $\sigma \le \tau_{n_0}$ a.s. on $\{\sigma < T\}$. Let $\varepsilon > 0$, $\delta > 0$, and consider a probability measure $\mathbf{Q}_\sigma$ on $\mathcal{F}_\sigma$ equivalent to $1_{\{\sigma<T\}}\mathbf{P}|_{\mathcal{F}_\sigma}$.*

Then there is $n \ge n_0$ and $(\widetilde{S},\mathbf{Q}) \in \mathrm{CPS}(\sigma,\varepsilon,n)$ such that $\mathbf{Q}|_{\mathcal{F}_\sigma} = \mathbf{Q}_\sigma$ and $\mathbf{Q}[\tau_n < T] < F(\varepsilon) + \delta$.

There exists also $l \ge n_0$ and $(\widetilde{S},\mathbf{Q}) \in \mathrm{CPS}(\sigma,\varepsilon,l)$ such that $\mathbf{Q}|_{\mathcal{F}_\sigma} = \mathbf{Q}_\sigma$ and $\mathbf{E}_{\mathbf{Q}}\big[\frac{\widetilde{S}_{\tau_l}}{\widetilde{S}_\sigma} 1_{\{\tau_l<T\}}\big] < G(\varepsilon) + \delta$.

Proof We shall concentrate below on the statement related to F and start defining conditional versions of $F(\sigma,\varepsilon)$, which are $\mathcal{F}_\sigma$-measurable functions defined on the set $\{\sigma < T\}$:

$$f(\sigma,\varepsilon,n) \quad = \quad \operatorname*{ess\,inf}_{(\widetilde{S},\mathbf{Q})\in\mathrm{CPS}(\sigma,\varepsilon,n)} \mathbf{Q}[\tau_n < T \mid \mathcal{F}_\sigma], \quad n \ge n_0. \tag{17}$$

Claim 0: We may replace CPS by SCPS in the above definition, i.e.

$$f(\sigma,\varepsilon,n) = \operatorname*{ess\,inf}_{(\widetilde{S},\mathbf{Q})\in\mathrm{SCPS}(\sigma,\varepsilon,n)} \mathbf{Q}[\tau_n < T \mid \mathcal{F}_\sigma], \quad n \ge n_0.$$

Note that by Theorem 1.4 the set $SCPS(\sigma,\varepsilon,n)$ of ε-strictly consistent price systems (see Definition 3.3 above) is nonempty as any $\varepsilon/2$-consistent price system is ε-strictly consistent. Fix $(\hat{S},\hat{\mathbf{Q}}) \in SCPS(\sigma,\varepsilon,n)$. For each $0 < \eta < 1$ and each $(\tilde{S},\mathbf{Q}) \in CPS(\sigma,\varepsilon,n)$, set:

$$\begin{aligned} S^\eta_T &:= \frac{\eta\hat{q}\hat{S}_T + (1-\eta)q\tilde{S}_T}{\eta\hat{q} + (1-\eta)q}, \\ \mathbf{Q}_\eta &:= \eta\hat{\mathbf{Q}} + (1-\eta)\mathbf{Q}, \end{aligned}$$

where $\hat{q} = d\hat{\mathbf{Q}}/d\mathbf{P}$ and $q = d\mathbf{Q}/d\mathbf{P}$. Defining $S^\eta_t := E[S^\eta_T|\mathcal{F}_t]$ we clearly have that $(S^\eta, \mathbf{Q}_\eta) \in SCPS(\sigma, \varepsilon, n)$. Setting $\eta_k := 1/k$, the corresponding $(S^{\eta_k}, \mathbf{Q}_{\eta_k})$ are in $SCPS(\sigma, \varepsilon, n)$ and a.s.

$$\mathbf{Q}_{\eta_k}[\tau_n < T|\mathcal{F}_\sigma] \to \mathbf{Q}[\tau_n < T|\mathcal{F}_\sigma], \quad k \to \infty.$$

Claim 1: For fixed σ and $\varepsilon > 0$, the sequence $(f(\sigma, \varepsilon, n))_{n=1}^\infty$ of $\mathcal{F}_\sigma$-measurable functions is decreasing.

To see this, take any $(\tilde{S}, \mathbf{Q}) \in SCPS(\sigma, \varepsilon, n)$. Corollary 4.7 below shows that there exists $(\tilde{S}', \mathbf{Q}') \in CPS(\sigma, \varepsilon, n+1)$ such that the restriction of $\tilde{S}'$ to $[\![0, \tau_n]\!]$ equals $\tilde{S}$ and $\mathbf{Q}'|_{\mathcal{F}_{\tau_n}} = \mathbf{Q}$. Thus

$$\mathbf{Q}[\tau_n < T|\mathcal{F}_\sigma] = \mathbf{Q}'[\tau_n < T|\mathcal{F}_\sigma] \geq \mathbf{Q}'[\tau_{n+1} < T|\mathcal{F}_\sigma],$$

and the claim follows.

Since $\big(f(\sigma, \varepsilon, n)\big)_{n \geq n_0}$ is a decreasing sequence of (equivalence classes of) $\mathcal{F}_\sigma$-measurable $[0, 1]$-valued functions we conclude that

$$f(\sigma, \varepsilon) \;=\; \lim_{n\to\infty} f(\sigma, \varepsilon, n) \tag{18}$$

is a well-defined $\mathcal{F}_\sigma$-measurable function.
Claim 2:

$$f(\sigma, \varepsilon) \leq F(\varepsilon), \quad \text{a.s. on } \{\sigma < T\}$$

Indeed, otherwise there is an $\mathcal{F}_\sigma$-measurable set $A \subseteq \{\sigma < T\}$, $\mathbf{P}[A] > 0$, and $\alpha > 0$ such that

$$f(\sigma, \varepsilon)1_A \geq (F(\varepsilon) + \alpha)1_A.$$

Define the stopping time ρ by

$$\rho = \sigma 1_A + T 1_{\Omega\setminus A}$$

and observe that, for each $n \geq n_0$ and $(\widetilde{S}, \mathbf{Q}) \in \mathrm{CPS}(\rho, \varepsilon, n)$

$$\mathbf{Q}[\tau_n < T] = \mathbf{E}_{\mathbf{Q}}\big[\mathbf{Q}[\tau_n < T \mid \mathcal{F}_\rho]\big] \geq F(\varepsilon) + \alpha,$$

since $\mathbf{Q}$ is by definition concentrated on $\{\rho < T\}$, and this gives a contradiction to (14) and (16).

Claim 3: For each $n \geq n_0$ and $\eta > 0$ there is $(\widetilde{S}^{0,n}, \mathbf{Q}^{0,n}) \in \mathrm{CPS}(\sigma, \varepsilon, n)$ such that

$$\mathbf{Q}^{0,n}[\tau_n < T \mid \mathcal{F}_\sigma] \leq f(\sigma, \varepsilon, n) + \eta, \quad a.s. \tag{19}$$

and

$$\mathbf{Q}^{0,n}|_{\mathcal{F}_\sigma} = \frac{1_{\{\sigma<T\}}}{\mathbf{P}[\sigma < T]}\mathbf{P}\bigg|_{\mathcal{F}_\sigma}. \tag{20}$$

To see this, notice that, by the definition of the essential infimum in (17), for any $\eta > 0$ there is a sequence $(\widetilde{S}^{k,n}, \mathbf{Q}^{k,n})_{k=1}^{\infty} \in CPS(\sigma, \varepsilon, n)$ as well as a partition $(A^{k,n})_{k=1}^{\infty}$ of $\{\sigma < T\}$ into $\mathcal{F}_\sigma$-measurable sets such that

$$\sum_{k=1}^{\infty} \mathbf{Q}^{k,n}[\tau_n < T \mid \mathcal{F}_\sigma] 1_{A^{k,n}} \le f(\sigma, \varepsilon, n) + \eta.$$

We construct $(\widetilde{S}^{0,n}, \mathbf{Q}^{0,n})$ by pasting together all $(\widetilde{S}^{k,n}, \mathbf{Q}^{k,n})$. Define $\widetilde{S}^{0,n} = (\widetilde{S}_t^{0,n})_{\sigma \le \tau \le \tau_n}$ by

$$\widetilde{S}_t^{0,n} = \sum_{k=1}^{\infty} \widetilde{S}_t^{k,n} 1_{A^{k,n}}, \quad \text{for } \sigma \le t \le \tau_n.$$

To define $\mathbf{Q}^{0,n}$, first introduce the Radon-Nikodym derivatives

$$\varphi^{k,n} = \frac{d\mathbf{Q}^{k,n}|_{\mathcal{F}_\sigma}}{d(\mathbf{P}|\mathcal{F}_\sigma)},$$

which are $\mathcal{F}_\sigma$-measurable functions, strictly positive on $\{\sigma < T\}$, and set

$$\frac{d\mathbf{Q}^{0,n}}{d\mathbf{P}} = \frac{1}{\mathbf{P}[\sigma < T]} \left[\sum_{k=1}^{\infty} \frac{d\mathbf{Q}^{k,n}}{d\mathbf{P}} \frac{1}{\varphi^{k,n}} 1_{A^{k,n}} \right].$$

Then $(\widetilde{S}^{0,n}, \mathbf{Q}^{0,n})$ is indeed an element of CPS(σ, ε, n) verifying Claim 3.

To finish the proof, fix the stopping time σ and the probability measure $\mathbf{Q}_\sigma$ as in the statement of the lemma. For $n \ge n_0$ and $\eta = n^{-1}$ apply Claim 3 to find $(\widetilde{S}^{0,n}, \mathbf{Q}^{0,n}) \in$ CPS(σ, ε, n) verifying (19) and (20). Define $\widetilde{\mathbf{Q}}^{0,n}$ by

$$\frac{d\widetilde{\mathbf{Q}}^{0,n}}{d\mathbf{P}} := \frac{d\mathbf{Q}^{0,n}}{d\mathbf{P}} \frac{d\mathbf{Q}_\sigma / d(\mathbf{P}|\mathcal{F}_\sigma)}{\mathbf{E}[d\mathbf{Q}^{0,n}/d\mathbf{P}|\mathcal{F}_\sigma]}$$

so that $(\widetilde{S}^{0,n}, \widetilde{\mathbf{Q}}^{0,n})$ again is in CPS(σ, ε, n) and the restriction of $\widetilde{\mathbf{Q}}^{0,n}$ to $\mathcal{F}_\sigma$ equals $\mathbf{Q}_\sigma$. By monotone convergence (Claim 1) and Claim 2 we conclude that

$$\begin{aligned}
\limsup_{n\to\infty} \widetilde{\mathbf{Q}}^{0,n}[\tau_n < T] &= \limsup_{n\to\infty} \mathbf{E}_{\widetilde{\mathbf{Q}}^{0,n}} \left[\widetilde{\mathbf{Q}}^{0,n}[\tau_n < T \mid \mathcal{F}_\sigma] \right] \\
&\le \mathbf{E}_{\mathbf{Q}_\sigma}[\lim_{n\to\infty} f(\sigma, \varepsilon, n) + n^{-1}] \\
&= \mathbf{E}_{\mathbf{Q}_\sigma}[f(\sigma, \varepsilon)] \le F(\varepsilon).
\end{aligned}$$

Now the same arguments can be carried out for G; this time the analogue of $f(\sigma, \varepsilon, n)$ should be

$$g(\sigma, \varepsilon, n) := \text{ess.} \inf_{(\tilde{S}, \mathbf{Q}) \in CPS(\sigma, \varepsilon, n)} \mathbf{E}_{\mathbf{Q}} \left[\frac{\tilde{S}_{\tau_n}}{\tilde{S}_\sigma} 1_{\{\tau_n < T\}} | \mathcal{F}_\sigma \right].$$

■

Lemma 3.6 *In the setting of Definition 3.4 we have, for $\varepsilon_1, \varepsilon_2 > 0$, the functional inequalities*

(i) $F(\varepsilon_1)F(\varepsilon_2) \geq F((1+\varepsilon_1)(1+\varepsilon_2) - 1)$,

(ii) $G(\varepsilon_1)G(\varepsilon_2) \geq G((1+\varepsilon_1)(1+\varepsilon_2) - 1)$.

Hence either $F(\varepsilon) = 0$ (resp. $G(\varepsilon) = 0$) for all $\varepsilon > 0$, or there is a constant $c_F > 0$ (resp. $c_G > 0$) such that $F(\varepsilon) \geq 1 - c_F\varepsilon$ (resp. $G(\varepsilon) \geq 1 - c_G\varepsilon$).

Proof Fix $\sigma \in \mathcal{T}$ such that $\sigma \leq \tau_n$ on $\{\sigma < T\}$. For $\delta > 0$, we want to find $m \geq n$ and $(\widetilde{S}, \mathbf{Q}) \in \text{CPS}(\sigma, (1+\varepsilon_1)(1+\varepsilon_2) - 1, m)$ such that

$$\mathbf{Q}[\tau_m < T] < (F(\varepsilon_1) + \delta)(F(\varepsilon_2) + \delta) \tag{21}$$

Indeed, first apply the previous Lemma 3.5 to find $k \geq n$ and $(\widetilde{S}^1, \mathbf{Q}^1) \in \text{CPS}(\sigma, \varepsilon_1, k)$ such that

$$\mathbf{Q}^1[\tau_k < T] < F(\varepsilon_1) + \delta.$$

Using the Lemma again, find $m > k$ and $(\widetilde{S}^2, \mathbf{Q}^2) \in \text{CPS}(\tau_k, \varepsilon_2, m)$ such that $\mathbf{Q}^2|_{\mathcal{F}_{\tau_k}}$ equals the normalized restriction of $\mathbf{Q}^1$ to the set $\{\tau_k < T\}$ and

$$\mathbf{Q}^2[\tau_m < T] < F(\varepsilon_2) + \delta.$$

To concatenate these two objects, define the probability measure $\mathbf{Q}$ by:

$$\frac{d(\mathbf{Q}|\mathcal{F}_t)}{d\mathbf{P}} = \begin{cases} \frac{d\mathbf{Q}^1|\mathcal{F}_t}{d\mathbf{P}}, & \text{for } \sigma \leq t \leq \tau_k \\ \frac{d(\mathbf{Q}^2|\mathcal{F}_t)/d\mathbf{P}}{d(\mathbf{Q}^2|\mathcal{F}_{\tau_k})/d\mathbf{P}}(d\mathbf{Q}^1/d\mathbf{P}), & \text{for } \tau_k \leq t \leq \tau_m. \end{cases} \tag{22}$$

Note that in the present case

$$\mathbf{Q}[A] = \mathbf{Q}^1[A \cap \{\tau_k = T\}] + \mathbf{Q}^1[\tau_k < T]\, \mathbf{Q}^2[A], \ A \in \mathcal{F}, \tag{23}$$

and define the process $\widetilde{S} = (\widetilde{S}_t)_{\sigma \le t \le \tau_m}$ by

$$\widetilde{S}_t = \begin{cases} \widetilde{S}^1_t, & \text{for } \sigma \le t \le \tau_k \\ \widetilde{S}^2_t \frac{\widetilde{S}^1_{\tau_k}}{\widetilde{S}^2_{\tau_k}}, & \text{for } \tau_k \le t \le \tau_m. \end{cases} \tag{24}$$

We find that $(\mathbf{Q}, \widetilde{S}) \in \mathrm{CPS}(\sigma, (1+\varepsilon_1)(1+\varepsilon_2)-1, m)$ and satisfies (21), which proves (i). We may rewrite inequality (i) by stating that, for $\eta_1, \eta_2 > 0$,

$$\log(F(e^{\eta_1} - 1)) + \log(F(e^{\eta_2} - 1)) \ge \log(F(e^{\eta_1+\eta_2} - 1)).$$

Letting $h(\eta) = \log(F(e^{\eta}-1))$, which takes its values in $[-\infty, 0]$ and decreases monotonically, we therefore obtain the functional inequality

$$h(\eta_1) + h(\eta_2) \ge h(\eta_1 + \eta_2), \quad \eta_1, \eta_2 > 0. \tag{25}$$

If $h(\eta) \equiv -\infty$, i.e. $F(\varepsilon) = 0$, for all $\varepsilon > 0$, then this inequality is clearly satisfied. This corresponds to the "regular case" (see the discussion after Definition 3.4). Otherwise we have the following:

Claim. *If $h(\eta_0) > -\infty$ for some $\eta_0 > 0$ then there is a constant $C_F \ge 0$ such that*

$$h(\eta) \ge -C_F \eta, \quad \text{for } 0 < \eta \le \eta_0.$$

Indeed, it is immediately seen by induction that $h(l\eta) \le l h(\eta)$ for all $\eta > 0$ and all natural numbers $l \ge 1$. This implies that

$$\frac{h(\eta)}{\eta} \ge \frac{h(l\eta)}{l\eta}, \ \eta > 0, \ l \ge 1.$$

Now fix $\eta_0 > 0$ and set $-C_F := \inf_{\eta \in (\eta_0/2, \eta_0]}(h(\eta)/\eta)$, which is finite since h is decreasing and $h(\eta_0) > -\infty$. For $k \ge 0$ and $\eta \in I_k := (\eta_0/2^{k+1}, \eta_0/2^k]$ it follows that

$$\frac{h(\eta)}{\eta} \ge \frac{h(2^k \eta)}{2^k \eta} \ge -C_F,$$

which proves the claim since $\cup_{k \ge 0} I_k = (0, \eta_0]$.

The above claim implies that there is $\varepsilon_0 > 0$ and $c_F \ge 0$ such that

$$F(\varepsilon) \ge 1 - c_F \varepsilon, \quad \text{for } 0 < \varepsilon \le \varepsilon_0. \tag{26}$$

Since $F(\varepsilon) \in [0, 1]$, it follows that (26) holds for all $\varepsilon > 0$ (by possibly passing to a different constant $c_F > 0$). This completes the proof of the assertion pertaining to the function F.

The arguments for G again are similar to those for F: apply Lemma 3.5 to get $(\tilde{S}^1, \mathbf{Q}^1)$, $(\tilde{S}^2, \mathbf{Q}^2)$ such that

$$\mathbf{E}_{\mathbf{Q}^1}\left[\frac{\tilde{S}^1_{\tau_k}}{\tilde{S}^1_\sigma} 1_{\{\tau_k<T\}}\right] < G(\varepsilon_1) + \delta, \quad \mathbf{E}_{\mathbf{Q}^2}\left[\frac{\tilde{S}^2_{\tau_m}}{\tilde{S}^2_{\tau_k}} 1_{\{\tau_m<T\}}\right] < G(\varepsilon_2) + \delta$$

and

$$\frac{d(\mathbf{Q}^2|\mathcal{F}_{\tau_k})}{d\mathbf{P}} = \frac{(d\mathbf{Q}^1/d\mathbf{P})(\tilde{S}^1_{\tau_k}/\tilde{S}^1_\sigma)1_{\{\tau_k<T\}}}{\mathbf{E}[(d\mathbf{Q}^1/d\mathbf{P})(\tilde{S}^1_{\tau_k}/\tilde{S}^1_\sigma)1_{\{\tau_k<T\}}]}. \tag{27}$$

Denote by $(\tilde{S}, \mathbf{Q})$ the result of the "concatentation" (as in (22) and (24)). This time we have, by plugging in the definitions and using (27),

$$\mathbf{E}_{\mathbf{Q}}\left[\frac{\tilde{S}_{\tau_m}}{\tilde{S}_\sigma} 1_{\{\tau_m<T\}}\right] = \mathbf{E}\left[\frac{d\mathbf{Q}^2}{d\mathbf{P}} \frac{\tilde{S}^2_{\tau_m}}{\tilde{S}^2_{\tau_k}} 1_{\{\tau_m<T\}}\right] \mathbf{E}\left[\frac{d\mathbf{Q}^1}{d\mathbf{P}} \frac{\tilde{S}^1_{\tau_k}}{\tilde{S}^1_\sigma} 1_{\{\tau_k<T\}}\right] \leq (G(\varepsilon_1)+\delta)(G(\varepsilon_2)+\delta),$$

and the rest of the arguments are identical to those for F. ■

Proposition 3.7 *Suppose that S is as in Definition 3.4. Then at least one of the following three statements holds true.*

(i) For each $0 < \varepsilon < 1$, there is an ε-consistent price system $(\widetilde{S}, \mathbf{Q})$ for S.

(ii) There is $c_F > 0$ such that $F(\varepsilon) \geq 1 - c_F\varepsilon$, for each $\varepsilon > 0$.

(iii) There is $c_G > 0$ such that $G(\varepsilon) \geq 1 - c_G\varepsilon$, for each $\varepsilon > 0$.

Proof In view of the preceding lemma we have to show that $F \equiv G \equiv 0$ implies the existence of an ε-consistent price system, for each $0 < \varepsilon < 1$.

Fix $0 < \varepsilon$ and let $\varepsilon_k = (1+\varepsilon)^{2^{-(k+1)}} - 1$ so that $\prod_{k=0}^\infty (1+\varepsilon_k) = 1+\varepsilon$. Using Lemma 3.5 just like in the previous Lemma, we inductively construct an increasing sequence $(n_k)_{k=0}^\infty$, starting at $n_0 = 0$, and $(\widetilde{S}^{(n_k)}, \mathbf{Q}^{(n_k)})_{k=0}^\infty \in \mathrm{CPS}(\tau_{n_k}, \varepsilon_k, n_{k+1})$, where $\tau_{n_0} = 0$, and concatenate these objects similarly as in the proof of the previous Lemma 3.6:

$$\frac{d(\overline{Q}^{(n_K)}|\mathcal{F}_t)}{d\mathbf{P}} = \begin{cases} \frac{d(\overline{Q}^{(n_{K-1})}|\mathcal{F}_t)}{d\mathbf{P}}, & \text{for } \sigma \leq t \leq \tau_{n_K} \\ \frac{d(\mathbf{Q}^{(n_K)}|\mathcal{F}_t)/d\mathbf{P}}{d(\mathbf{Q}^{(n_K)}|\mathcal{F}_{\tau_{n_K}})/d\mathbf{P}}(d\overline{Q}^{(n_{K-1})}/d\mathbf{P}), & \text{for } \tau_{n_K} \leq t \leq \tau_{n_{K+1}}, \end{cases} \tag{28}$$

$$\overline{S}_t^{(n_K)} \quad := \quad \begin{cases} \overline{S}_t^{(n_{K-1})}, & \text{for } 0 \le t \le \tau_{n_K} \\ \tilde{S}_t^{(n_K)} \frac{\overline{S}_{\tau_{n_K}}^{(n_{K-1})}}{\tilde{S}_{\tau_{n_K}}^{(n_K)}}, & \text{for } \tau_{n_K} \le t \le \tau_{n_{K+1}}. \end{cases}$$

We choose $(\widetilde{S}^{(n_k)}, \mathbf{Q}^{(n_k)})_{k=0}^{\infty}$ in such a way that for k odd we have (by the argument of Lemma 3.5 pertaining to F)

$$\mathbf{Q}^{(n_k)}|\mathcal{F}_{\tau_{n_k}} = \frac{1_{\{\tau_{n_k}<T\}}\overline{Q}^{(n_{k-1})}}{\overline{Q}^{(n_{k-1})}[\tau_{n_k} < T]}$$

and for k even (by the arguments pertaining to G)

$$(d\mathbf{Q}^{(n_k)}|\mathcal{F}_{\tau_{n_k}})/d\mathbf{P} = \frac{1_{\{\tau_{n_k}<T\}}(d\overline{Q}^{(n_{k-1})}/d\mathbf{P})(\tilde{S}_{\tau_{n_k}}^{(n_{k-1})}/\tilde{S}_{\tau_{n_{k-1}}}^{(n_{k-1})})}{\mathbf{E}[1_{\{\tau_{n_k}<T\}}(d\overline{Q}^{(n_{k-1})}/d\mathbf{P})(\tilde{S}_{\tau_{n_k}}^{(n_{k-1})}/\tilde{S}_{\tau_{n_{k-1}}}^{(n_{k-1})})]}.$$

It follows, as in Lemma 3.6, that

$$\overline{Q}^{(n_K)}[\tau_{n_{K+1}} < T] \quad \le \quad 2^{1-K/2} \tag{29}$$

$$\mathbf{E}_{\overline{Q}^{(n_K)}}\left[\frac{\overline{S}_{\tau_{n_{K+1}}}^{(n_K)}}{\overline{S}_0^{(n_K)}} 1_{\{\tau_{n_{K+1}}<T\}}\right] \quad \le \quad 2^{1-K/2}. \tag{30}$$

For each $K \ge 0$ we thus obtain

$$\left(\overline{S}^{(n_K)}, \overline{Q}^{(n_K)}\right) \in \mathrm{CPS}\left(0, \prod_{k=0}^{K}(1+\varepsilon_k) - 1, n_{K+1}\right),$$

and the restriction of $\left(\overline{S}^{(n_K)}, \overline{Q}^{(n_K)}\right)$ to $[\![0, \tau_{n_K}]\!]$ equals $\left(\overline{S}^{(n_{K-1})}, \overline{Q}^{(n_{K-1})}\right)$ for $K \ge 1$.

The probability measure $\overline{Q}^{(n_K)}|_{\mathcal{F}_{\tau_{n_{K+1}}}}$ is equivalent to $\mathbf{P}|_{\mathcal{F}_{\tau_{n_{K+1}}}}$.

As $\mathbf{Q}^{(n_L)}[\tau_{n_K} = T] = 0$ for $L \ge K$, we have

$$\frac{d\overline{Q}^{(n_L)}}{d\mathbf{P}} = \frac{d\overline{Q}^{(n_{K-1})}}{d\mathbf{P}} > 0$$

on $\{\tau_{n_K} = T\}$ for each $L \ge K-1$.

As $\{\tau_{n_K} = T\} \uparrow \Omega$ a.s. as $K \to \infty$, we may define $\mathbf{Q}$ by

$$\frac{d\mathbf{Q}}{d\mathbf{P}} := \lim_{K\to\infty} \frac{d\overline{Q}^{(n_K)}}{d\mathbf{P}},$$

and we get an equivalent measure $\mathbf{Q} \sim \mathbf{P}$. Define also $\tilde{S}_t := \overline{S}_t^{(n_K)}$ for $0 \le t \le \tau_{n_{K+1}}$.

The process $\tilde{S}$ satisfies

$$1/(1+\varepsilon) \le \frac{\widetilde{S}_t}{S_t} \le 1+\varepsilon, \text{ a.s.} \quad 0 \le t \le T. \tag{31}$$

It remains to verify that $\mathbf{Q}$ is a probability measure and $\tilde{S}$ is a $\mathbf{Q}$-martingale. The first statement follows from:

$$\begin{aligned} \mathbf{Q}(\Omega) = \mathbf{E}\left[\lim_{K\to\infty} \frac{d\mathbf{Q}}{d\mathbf{P}} 1_{\{\tau_{n_{K+1}}=T\}}\right] &= \\ \lim_{K\to\infty} \mathbf{E}\left[\frac{d\overline{Q}^{(n_K)}}{d\mathbf{P}} 1_{\{\tau_{n_{K+1}}=T\}}\right] &= \lim_{K\to\infty} \overline{Q}^{(n_K)}[\tau_{n_{K+1}} = T] = 1, \end{aligned}$$

where we used monotone convergence in the second equality and (29). The second statement follows similarly.

To sum up, we have constructed for all $\varepsilon > 0$ a probability $\mathbf{Q} \sim \mathbf{P}$ and a $\mathbf{Q}$-martingale $\tilde{S}$ satisfying (31), which implies (i) in the statement of the Proposition. ■

Remark 3.8 We stress again that the above proof crucially depends on the quantifier "for each $\varepsilon > 0$".

We can now formulate and prove a version of Theorem 1.11. The argument relies on the superhedging theorem from [1]. This result characterizes the initial endowments from which a given contingent claim can be hedged in a continuous-time financial market model with transaction costs. The first theorem of this type was obtained in [4]. The geometric framework developed in [15] was applied in a series of papers [16], [17] and [1] at increasing levels of generality. We need the superhedging result to produce trading strategies with given payoffs which form a "free lunch", i.e. an asymptotic form of arbitrage in a suitable sense.

We have to use admissible ε-self-financing portfolio processes in the sense of [1, Def. 7] instead of those in Definition 1.5. We shall elaborate in section 4 below upon the relationship between these two classes of strategies.

Recall the following definition ([15], [22]).

Definition 3.9 *For a given $\mathbb{R}_+$-valued process $S = (S_t)_{0\le t\le T}$, and $\varepsilon > 0$, we denote by $\widehat{K}_t^\varepsilon$ the* solvency cone at time t, *defined as:*

$$\widehat{K}_t^\varepsilon = \text{cone}\left\{S_t(1+\varepsilon)e_1 - e_2, -e_1 + \frac{1}{S_t(1-\varepsilon)}e_2\right\}.$$

We denote by $(\widehat{K}_t^\varepsilon)^*$ *its polar cone, given by:*

$$\begin{aligned}(\widehat{K}_t^\varepsilon)^* &= \left\{(w_1,w_2)\in\mathbb{R}_+^2 \,\middle|\, S_t(1-\varepsilon)\le \frac{w_2}{w_1}\le S_t(1+\varepsilon)\right\}\\ &= \{w\in\mathbb{R}^2 \mid \langle x,w\rangle \ge 0,\ \textit{for } x\in \widehat{K}_t^\varepsilon\}.\end{aligned}$$

Here $e_1=(1,0)$, $e_2=(0,1)$ denote the unit vectors in $\mathbb{R}^2$, and the vectors in $\mathbb{R}^2$ describe the holdings (in physical units) in the bond and the stock. The random cone $\widehat{K}_t^\varepsilon$ consists of the investment positions in physical units at time t which can be liquidated to zero. By $\langle\,.\,,\,.\,\rangle$ we denote the inner product in $\mathbb{R}^2$. We shall also use the notation $\mathbf{1}:=(1,1)$.

Definition 3.10 *We define $\mathcal{Z}$ (resp. $\mathcal{Z}^s$) as the set of* $\mathbf{P}$*-martingales such that for all t, $Z_t\in(\widehat{K}_t^\varepsilon)^*\setminus\{0\}$ a.s. (resp. $Z_t\in\operatorname{int}(\widehat{K}_t^\varepsilon)^*$ a.s.).*

The following statement is straightforward to check:

Proposition 3.11 *Define the measure $\mathbf{Q}(Z)$ by $d\mathbf{Q}(Z)/d\mathbf{P}:=Z_T^1/EZ_T^1$. Then $Z\in\mathcal{Z}$ (resp. $Z\in\mathcal{Z}^s$) iff $((Z_t^2/Z_t^1)_{0\le t\le T},\mathbf{Q}(Z))$ is a consistent (resp. strictly consistent) price system.* ∎

Thus we are entitled to call elements of $\mathcal{Z}$ (resp. $\mathcal{Z}^s$) consistent (resp. strictly consistent) price processes without introducing any ambiguity in the terminology.

Definition 3.12 *Let $(Y_t)_{0\le t\le T}$ be a progressively measurable $\mathbb{R}^d$-valued process with right- and left-hand limits. Denote by $\Delta Y_t:=Y_t-Y_{t-}$, $\Delta_+Y_t:=Y_{t+}-Y_t$ and $Y_t^c:=Y_t-\sum_{0<s\le t}\Delta Y_s-\sum_{0\le s<t}\Delta_+Y_s$.*

We recall the total variation process $\mathrm{Var}(Y)$ of a *càdlàg* finite variation process Y as defined in e.g. [20]. Here we need an extension to the non-càdlàg setting. The proof of this technical lemma is omitted.

Lemma 3.13 *Under the usual conditions on the filtration, if $(Y_t)_{0\le t\le T}$ is a predictable process with finite variation paths (hence also with left and right limits) then*

$$\mathrm{Var}_s(Y):=\mathrm{Var}_s(Y^c)+\sum_{0<u\le s}|\Delta Y_u|+\sum_{0\le u<s}|\Delta_+Y_u|.$$

for $s\in[0,T]$ defines a predictable process such that

$$\mathrm{Var}_s(Y)=\sup_{0\le t_1\le\ldots\le t_n\le s}\sum_{i=1}^n|Y_{t_i}-Y_{t_{i-1}}|,\ \textit{for all } s\in[0,T],$$

almost surely.

We recall Definition 7 from [1], which we adapt to the present setting.

Definition 3.14 *Suppose that the process $(S_t)_{0\le t\le T}$ satisfies Assumption 1.1 and admits ε-strictly consistent price systems, for each $\varepsilon > 0$. For fixed $\varepsilon > 0$, an $\mathbb{R}^2$-valued process $\widehat{V} = (\widehat{V}_t)_{t\in[0,T]}$, is called an ε-*self-financing portfolio process *if $\widehat{V}_0 = 0$ and it satisfies the following properties:*

(i) it is predictable *and a.e. path has finite variation,*

(ii) for every pair of stopping times $0 \le \sigma \le \tau \le T$, we have

$$\widehat{V}_\tau - \widehat{V}_\sigma \in -\widehat{\mathcal{K}}^{\varepsilon}_{\sigma,\tau} \quad a.s.$$

where

$$\widehat{\mathcal{K}}^{\varepsilon}_{\sigma,\tau}(\omega) := \overline{\mathrm{conv}}\left(\bigcup_{\sigma(\omega)\le u<\tau(\omega)} \widehat{K}^{\varepsilon}_u(\omega)\right),$$

the bar denoting closure in $\mathbb{R}^2$.

The set of ε-self-financing portfolio processes is denoted by $\widehat{\mathcal{V}} = \widehat{\mathcal{V}}(\varepsilon)$. Define the partial order $\succeq_t$ between random variables ξ, ζ as

$$\xi \succeq_t \zeta \iff \xi - \zeta \in \widehat{K}^{\varepsilon}_t \ a.s.$$

An ε-self-financing portfolio process $\widehat{V}$ is admissible *if it satisfies the following additional property:*

(iii) there is a constant $M > 0$ such that $\widehat{V}_T \succeq_T -M\mathbf{1}$ and $\langle Z_\tau, \widehat{V}_\tau\rangle \ge -M\langle Z_\tau, \mathbf{1}\rangle$ a.s. for all $[0,T]$-valued stopping times τ and for all $Z \in \mathcal{Z}^s$.

We denote by $\widehat{\mathcal{V}}^{\mathrm{adm}}_M = \widehat{\mathcal{V}}^{\mathrm{adm}}_M(\varepsilon)$ the set of all such portfolio processes $\widehat{V}$ and by $\widehat{\mathcal{A}}^{\mathrm{adm}}_M = \widehat{\mathcal{A}}^{\mathrm{adm}}_M(\varepsilon)$ the set of all terminal random variables $\widehat{V}_T$, $\widehat{V} \in \widehat{\mathcal{V}}^{\mathrm{adm}}_M$. We also use the notation $\widehat{\mathcal{V}}^{\mathrm{adm}} := \cup_M \widehat{\mathcal{V}}^{\mathrm{adm}}_M$ and $\widehat{\mathcal{A}}^{\mathrm{adm}} := \cup_M \widehat{\mathcal{A}}^{\mathrm{adm}}_M$.

Remark 3.15 The above geometric setting is useful especially in the multi-asset case. The random cones $\widehat{K}^{\varepsilon}_t$ represent solvent positions at time t. Thus condition (ii) says (apart from technicalities) that we are allowed to rebalance our portfolios only in a self-financing way, see [1] for more explanations. The above portfolio processes and admissibility concept are essentially equivalent to the ones defined in Definition 1.5 as we will see in section 4.

Definition 3.16 *A process S satisfying Assumption 1.1 admits a* free lunch with bounded risk for sufficiently small transaction costs *if there is $\varepsilon > 0$ and a sequence $(h_n)_{n=1}^{\infty}$ in $\widehat{\mathcal{A}}^{\mathrm{adm}}(\varepsilon)$ with*

$$\begin{array}{rrcll} & h_n & \succeq_T & -\mathbf{1}, & a.s. \\ \text{and} & \lim_{n\to\infty} h_n & = & h_0, & a.s. \end{array}$$

where h_0 is an $\mathbb{R}_+^2$-valued random variable such that $\mathbf{P}[h_0 \neq 0] > 0$.

Theorem 3.17 *Let S be as in Definition 3.4. The following assertions are equivalent:*

(i) For each $\varepsilon > 0$ there exists an ε-consistent price system.

(ii) There is no free lunch with bounded risk for arbitrarily small transaction costs.

If these two equivalent conditions fail, then one of the following more precise statements holds true.

(a) There exists a constant $c_F > 0$ such that for all $0 < \varepsilon < 1/c_F$ there exists a sequence $(f_n(\varepsilon), 0)_{n=1}^{\infty}$ in $\widehat{\mathcal{A}}^{\mathrm{adm}}(\varepsilon)$ with

$$\begin{array}{rrcll} & f_n(\varepsilon) & \geq & -1 & a.s. \\ \text{and} & \lim_{n\to\infty} f_n(\varepsilon) & = & f(\varepsilon) := [(c_F\varepsilon)^{-1} - 1]1_B & a.s., \end{array}$$

for some $B \in \mathcal{F}_T$ with $\mathbf{P}(B) > 0$.

(b) There exists a constant $c_G > 0$ such that for all $0 < \varepsilon < 1/c_G$ there exists a sequence $(0, g_n(\varepsilon))_{n=1}^{\infty}$ in $\widehat{\mathcal{A}}^{\mathrm{adm}}(\varepsilon)$ with

$$\begin{array}{rrcll} & g_n(\varepsilon) & \geq & -1 & a.s. \\ \text{and} & \lim_{n\to\infty} g_n(\varepsilon) & = & g(\varepsilon) := [(c_G\varepsilon)^{-1} - 1]1_B & a.s., \end{array}$$

for some $B \in \mathcal{F}_T$ with $\mathbf{P}(B) > 0$.

Proof (i)$\Rightarrow$(ii): Fix $\varepsilon > 0$ and suppose there is a free lunch $(h_n)_{n=1}^{\infty} = (f_n, g_n)_{n=1}^{\infty}$ as in Definition 3.16. Supposing there is an ε-consistent price system $(\widetilde{S}, \mathbf{Q})$ we have to arrive at a contradiction. Let $\widehat{V}^n$ be a portfolio process such that $\widehat{V}_T^n = h_n$.

Lemma 8 of [1] implies that the process $\langle Z_s, \widehat{V}_s^n \rangle$ is a supermartingale for each $Z \in \mathcal{Z}$. Hence

$$\mathbf{E}_{\mathbf{Q}}[\langle h_n, (1, \widetilde{S}_T)\rangle] = \mathbf{E}[\langle h_n, Z_T\rangle] \leq 0,$$

where Z corresponds to $(\tilde{S}, \mathbf{Q})$ as in Proposition 3.11 above.

On the other hand, by Fatou's lemma

$$\lim_{n\to\infty} \mathbf{E}_{\mathbf{Q}}[\langle h_n, (1, \widetilde{S}_T)\rangle] \geq \mathbf{E}_{\mathbf{Q}}[\langle h_0, (1, \tilde{S}_T)\rangle] > 0,$$

which yields the desired contradiction.

(ii)⇒(i): If (i) fails, Proposition 3.7 implies that either (ii) or (iii) in that proposition hold true. We shall show that (ii) implies (a), the argument for (iii) implying (b) being analogous.

Consider $c_F > 0$ in (ii), and fix $0 < \varepsilon < c_F^{-1}$. Find a stopping time $\sigma \in \mathcal{T}$ with $\sigma \leq \tau_{n_0}$ on $\{\sigma < T\}$ for some $n_0 \geq 1$ such that $F(\sigma, \varepsilon) > 1 - c_F\varepsilon$. For $n \geq n_0$ define f_n by

$$f_n = -1_{\{\sigma<T\}} + (c_F\varepsilon)^{-1} 1_{\{\sigma<T,\tau_n=T\}}. \tag{32}$$

For any ε-consistent price system $(\widetilde{S}, \mathbf{Q}) \in CPS(\sigma, \varepsilon, \tau_n)$ we know that $\mathbf{Q}[\tau_n = T] \leq 1 - F(\sigma, \varepsilon) < c_F\varepsilon$ so that

$$\mathbf{E}_{\mathbf{Q}}[\langle (1, \widetilde{S}_T), (f_n, 0)\rangle] \leq -1 + c_F\varepsilon(c_F\varepsilon)^{-1} = 0.$$

At this stage we apply Theorem 15 in [1][2]. For the convenience of the reader we reformulate it in the context of the present setting.

Corollary 3.18 *Let $\mathcal{A}^{\mathrm{adm}}_{\sigma,\tau_n}$ denote the set of elements in $\widehat{\mathcal{A}}^{\mathrm{adm}}(\varepsilon)$ which are terminal values of ε-self-financing portfolios $\widehat{V}$ satisfying $\widehat{V} = 0$ on $[\![0, \sigma]\!]$ and $\widehat{V} = \widehat{V}_{\tau_n}$ on $[\![\tau_n, T]\!]$. Then $G \in \mathcal{A}^{\mathrm{adm}}_{\sigma,\tau_n}$ if and only if*

$$\mathbf{E}_{\mathbf{Q}}[\langle (1, \tilde{S}_{\tau_n}), G\rangle] \leq 0$$

for all consistent (equivalently, for all strictly consistent) price systems $(\tilde{S}, \mathbf{Q}) \in CPS(\sigma, \varepsilon, \tau_n)$.

The superhedging theorem implies that $(f_n, 0)$ is indeed in $\widehat{\mathcal{A}}^{\mathrm{adm}}(\varepsilon)$. Observing that $\mathbf{P}[\tau_n = T]$ tends to 1, assertion (a) follows choosing $B := \{\sigma < T\}$. ■

Proof of Theorem 1.11 In view of Theorem 3.17 it suffices to show that the existence of an ε-free lunch implies the existence of an arbitrage, and then $(ii) \to (i)$ of Theorem 1.11 follows. We first assume that (a) of Theorem 3.17 holds true.

[2]Assumption 2 of that paper requires a certain continuity property of the filtration but as explained there, this was assumed only to avoid cumbersome notation.

Let $\widehat{V}(n)$ be an ε-self financing process realizing

$$\widehat{V}_{\tau_n}(n) = (f_n, 0), \tag{33}$$

provided by Corollary 3.18, where f_n is as in (32). We recall that $\widehat{V}(n)$ is of the form

$$\widehat{V}_t(n) = 0,\ 0 \le t \le \sigma, \quad \widehat{V}_t(n) := \widehat{V}_{\tau_n}(n),\ t \ge \tau_n.$$

Recall also that for a.e. ω, $\tau_k(\omega) = T$ for $k \ge \bar{k}(\omega)$ large enough. It follows from Lemma 11 and Proposition 13 of [1] that there are convex combinations of the sequence of stopped processes

$$\left(\widehat{V}_t^{\tau_1}(n)\right)_{n\ge 1}, \quad 0 \le t \le \tau_1,$$

which converge to some predictable finite variation process (on $[\![0,\tau_1]\!]$), denoted by $(\widehat{W}_t^{(1)})_{0\le t\le\tau_1}$. Using a diagonal procedure we may define $(\widehat{W}^{(m)})_{0\le t\le\tau_m}$ for all $m \ge 1$ such that appropriate convex combinations of

$$\left(\widehat{V}_t^{\tau_m}(n)\right)_{n\ge m}, \quad 0 \le t \le \tau_m,$$

converge to $\widehat{W}^{(m)}$ on $[\![0,\tau_m]\!]$ and for $m_1 \le m_2$, the processes $\widehat{W}^{(m_1)}$ and $\widehat{W}^{(m_2)}$ a.s. coincide on $[\![0,\tau_{m_1}]\!]$. Hence we may well define convex combinations of the $\widehat{V}(n)$ converging to a limiting process $\widehat{W}_t, t \in [0,T]$.

$\widehat{W}$ has a.s. finite variation on $[0,T]$ (again because τ_m increases to T in a stationary way, $m \to \infty$). Clearly, $\widehat{W}$ also satisfies (ii) and (iii) of Definition 3.14 as all the $\widehat{V}(n)$ do.

Proposition 4.9 implies that the positive random variable $\widehat{W}_T^1 = f$ is the terminal value of an admissible trading strategy (in the sense of Definition 1.5), hence there is arbitrage.

In the case where (b) of Theorem 3.17 holds we notice that adding a transfer at T we may get the positions

$$\left(\frac{g_n}{S_T(1+\varepsilon)}, 0\right),\ n \ge 0,$$

and the argument for case (a) can be repeated.

Finally we show that the existence of an ε-consistent price system excludes the occurence of arbitrage possibilities for markets with ε-transaction costs.

Let $V_T(\theta) = V_T^\varepsilon(\theta)$ be a.s. nonnegative. Take any consistent price system $(\tilde{S}, \mathbf{Q})$ and estimate

$$\begin{aligned}\int_0^T \theta_t d\tilde{S}_t &= -\int_0^T \tilde{S}_t d\theta_t = \\ V_T(\theta) &+ \varepsilon\int_0^T S_t d\mathrm{Var}_T(\theta) + \int_0^T (S_t - \tilde{S}_t) d\theta_t \ge V_T(\theta) \ge 0,\end{aligned}$$

using Definitions 1.3 and 1.5. The stochastic integral $\int \theta d\tilde{S}$ is bounded from below, hence it is a $\mathbf{Q}$-supermartingale and we get that $\mathbf{E}_{\mathbf{Q}}[V_T(\theta)] \leq 0$, proving that $V_T(\theta)$ equals 0, $\mathbf{Q}$- and hence $\mathbf{P}$-almost everywhere. ■

Remark 3.19 The reader might wonder whether the assumption of equal transaction costs for buying and selling restricts the generality of the model considered. This is clearly not the case. For example, consider the situation of ε-transaction costs for buying, and zero cost for selling, which corresponds to the bid-ask interval $[S_t, S_t(1+\varepsilon)]$. Under Assumption 1.1 and *(ii)* of Theorem 1.11, we claim that there exists a consistent price system, i.e. a probability $\mathbf{Q} \sim \mathbf{P}$ and a $\mathbf{Q}$-martingale $\tilde{S}$ such that

$$S_t \leq \tilde{S}_t \leq S_t(1+\varepsilon) \qquad \text{a.s. for all } t.$$

Indeed, take $\eta > 0$ such that $(1+\eta)/(1-\eta) \leq 1+\varepsilon$. Since absence of arbitrage with bid-ask spread $[S_t, S_t(1+\eta)]$ implies absence of arbitrage with $[S_t(1-\eta), S_t(1+\eta)]$ (with obvious modifications made in the definition of the portfolio value $V^{\eta}(\theta)$), Theorem 1.11 provides a measure $\mathbf{Q}$ and a process $\bar{S}$ such that

$$S_t(1-\eta) \leq \bar{S}_t \leq S_t(1+\eta).$$

Defining $\tilde{S} := \bar{S}/(1-\eta)$ we get the desired consistent price system. This trick is used again in Lemma 4.6 below.

4 Comparing concepts of admissibility

We now explore the relationship between our present Definition 1.5 of admissible strategies and that of [1] (Definition 3.14 above).

We may represent $\widehat{V}$ (see e.g. [17]) as

$$\widehat{V}_t = \widehat{V}^c_t + \sum_{0<s\leq t} \Delta V_s + \sum_{0\leq s<t} \Delta_+ V_s = \int_0^t \dot{\widehat{V}}^c_s d\mathrm{Var}_s(\widehat{V}^c) + \sum_{0<s\leq t} \Delta V_s + \sum_{0\leq s<t} \Delta_+ V_s,$$

where $\widehat{V}^c$ is the continuous part of $\widehat{V}$; $\dot{\widehat{V}}^c_s \in \mathcal{S} \cap (-\widehat{K}^{\varepsilon}_s)$ holds for $\mathrm{Var}_T(\widehat{V}^c)$-a.e. s, $\mathbf{P}$-almost surely, and $\mathcal{S}$ denotes the unit sphere of $\mathbb{R}^2$. This remark is also true for strategies in the sense of Definition 1.5, we may write

$$\theta_t = \int_0^t \dot{\theta}_u d\mathrm{Var}_u(\theta^c) + \sum_{0<u\leq t} \Delta\theta_u + \sum_{0\leq u<t} \Delta_+\theta_u,$$

with $\dot{\theta}_u \in \{\pm 1\}$. Fix $0 < \varepsilon < 1$.

Definition 4.1 *A portfolio process* $\widehat{V} \in \widehat{\mathcal{V}}(\varepsilon)$ *is* cash-settled (c.s.) *if* $\widehat{V}^2_T = 0$. *A c.s. portfolio process* $\widehat{V}$ *is* on the boundary *if, for almost every* ω, $\dot{\widehat{V}}^c_t$ *is in* $-\partial\widehat{K}^\varepsilon_t$ *for* $d\mathrm{Var}_T(\widehat{V}^c)$*-almost every* t*, and for all* t, $\Delta_+\widehat{V}_t, \Delta\widehat{V}_t \in -\partial\widehat{K}^\varepsilon_t$. *Here* $\partial\widehat{K}^\varepsilon_t$ *denotes the boundary of* $\widehat{K}^\varepsilon_t$.

Proposition 4.2 *Let* $V_t(\theta)$ *be the value process of a trading strategy as in Definition 1.5. Then there is a c.s. portfolio process* $\widehat{V}$ *on the boundary such that*

$$\widehat{V}^1_T = V_T(\theta). \tag{34}$$

Conversely, for each c.s. portfolio process on the boundary there is θ *such that* (34) *holds.*

Proof Consider a predictable process h such that $\theta^c_t = \int_0^t h(s)d\mathrm{Var}_s(\theta)$ and $h(s) \in \{\pm 1\}$. Define:

$$l_t \ := \begin{cases} v_{t,-} := (S_t(1-\varepsilon)e_1 - e_2) & \text{if } h(t) = -1, \\ v_{t,+} := (e_2 - S_t(1+\varepsilon)e_1) & \text{if } h(t) = +1, \end{cases}$$

and set

$$d\widehat{V}^c_t = l_t d\mathrm{Var}_t(\theta^c), \qquad \Delta_+\widehat{V}_t = |\Delta_+\theta_t| v_{t,\mathrm{sign}(\Delta_+\theta_t)}, \qquad \Delta\widehat{V}_t = |\Delta\theta_t| v_{t,\mathrm{sign}(\Delta\theta_t)}.$$

Thus (34) will be satisfied. The inverse transformation is equally easy, recalling that $\dot{\widehat{V}}_t$ takes values in $\{v_{t,+}/|v_{t,+}|, v_{t,-}/|v_{t,-}|\}$, $d\mathrm{Var}_T(\widehat{V}^c)$-a.e. and $\Delta\widehat{V}_t, \Delta_+\widehat{V}_t$ are multiples of either $v_{t,+}$ or $v_{t,-}$. ■

The next Lemma is a simple geometric observation.

Lemma 4.3 *Let* $v \in (-\widehat{K}^\varepsilon_t) \cap \mathcal{S}$*. Then there is* $y \in (\mathcal{S} \cap -\partial\widehat{K}^\varepsilon_t) \cup \{0\}$ *such that* $y - v \in \mathbb{R}^2_+$.

Corollary 4.4 *The set of cash positions attainable by c.s. portfolio processes* $\widehat{V}^1_T$ *is equal to the set of positions dominated by some* $V_T(\theta)$.

Proof Using the previous Lemma and the measurable selection theorem, take a predictable process $c_t \in (\mathcal{S} \cap -\partial\widehat{K}^\varepsilon_t) \cup \{0\}$ and predictable processes $d_t, g_t \in -\partial\widehat{K}^\varepsilon_t$ such that

$$\begin{aligned} c_t - \dot{\widehat{V}}^c_t &\in \mathbb{R}^2_+, \\ g_t - \Delta\widehat{V}_t &\in \mathbb{R}^2_+, \\ d_t - \Delta_+\widehat{V}_t &\in \mathbb{R}^2_+, \end{aligned}$$

and $|g_t| = |\Delta \widehat{V}_t|$, $|d_t| = |\Delta_+ \widehat{V}_t|$. Define

$$\begin{aligned} d\widehat{W}_t &:= c_t d\mathrm{Var}_t(\widehat{V}), \\ \Delta \widehat{W}_t &:= g_t, \\ \Delta_+ \widehat{W}_t &:= d_t. \end{aligned}$$

This process is still of finite variation and is a portfolio process on the boundary. Let θ be the strategy corresponding to $\widehat{W}$ (as in Proposition 4.2 above). Then $\widehat{V}^1_T$ will obviously be dominated by $\widehat{W}^1_T$ hence also by $V_T(\theta)$.

Conversely, let $X \leq V_T(\theta)$ for some $V_T(\theta)$. We can find a c.s. $\widehat{V} \in \widehat{\mathcal{V}}(\varepsilon)$ on the boundary yielding terminal wealth $\widehat{V}^1_T = V_T(\theta)$ and adding a final $(-\mathbb{R}_+) \times \{0\}$-valued jump at time T we obtain a c.s. portfolio process with terminal value exactly X. ■

We start with a result of independent interest: in the present setting, the admissibility condition in Definition 3.14 (iii) can be replaced by a more intuitive concept, see Proposition 9 of [1] and pages 130–131 of [17].

Proposition 4.5 *Suppose that the process $S = (S_t)_{0 \leq t \leq T}$ satisfies Assumption 1.1 and admits ε-consistent price systems, for each $\varepsilon > 0$.*

Then, in the setting of Definition 3.14, the admissibility condition (iii) is equivalent to the subsequent condition (iii'):

(iii') there is a threshold $M > 0$ such that almost surely

$$\widehat{V}_t \succeq_t -M\mathbf{1}, \qquad \text{for } \; 0 \leq t \leq T. \tag{35}$$

Note that inequality (35), which can be restated as $\widehat{V}_t + M\mathbf{1} \succeq_t 0$, requires that at each time t the portfolio $\widehat{V}_t$ plus M units of both the bond and the stock is solvent.

We need a preparatory result:

Lemma 4.6 *Fix $\varepsilon > 0$. Under the conditions of Proposition 4.5, for each $0 \leq t \leq T$ and each random variable $f_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{R}, \mathbf{P})$ such that:*

$$(1-\varepsilon) S_t < f_t < (1+\varepsilon) S_t, \qquad a.s$$

and for each $\varepsilon' > \varepsilon$ there is an ε'-strictly consistent price system $\big((\widetilde{S}_u)_{0 \leq u \leq T}, \mathbf{Q}\big)$ such that $\tilde{S}_t = f_t$.

Proof Choose δ such that $\delta+(1+\delta)(\varepsilon+\delta)/(1-\delta)<\varepsilon'$. Take an arbitrary f_t and let $(\tilde{W}_t(\eta),\tilde{\mathbf{Q}}(\eta))$ be fixed $\min\{\eta,\delta\}$-consistent price systems, where η will vary later. We have:

$$1-\eta \le g_t(\eta) := \frac{\tilde{W}_t(\eta)}{S_t} \le 1+\eta, \tag{36}$$

and for $h_t := f_t/S_t$ we have

$$1-\varepsilon < h_t < 1+\varepsilon.$$

We define, for $n\ge 1$, the $\mathcal{F}_t$-measurable events

$$\begin{aligned} A_n^+ &:= (\{1+n\varepsilon/(n+1) > h_t \ge 1+(n-1)\varepsilon/n\}, && (37)\\ A_n^- &:= \{1-(n-1)\varepsilon/n > h_t \ge 1-n\varepsilon/(n+1)\}). && (38)\end{aligned}$$

Now set $\tilde{\mathbf{Q}} := \sum_{n=1}^{\infty} 1_{A_n^+\cup A_n^-}\tilde{\mathbf{Q}}(\varepsilon/(9n+3))$ and

$$\tilde{W}_u := \sum_{n=1}^{\infty} 1_{A_n^+\cup A_n^-}\frac{h_t}{g_t(\frac{\varepsilon}{9n+3})}\tilde{W}_u\left(\frac{\varepsilon}{9n+3}\right),\ t\le u\le T. \tag{39}$$

Since $f_t\in L^1$, it is clear that $\tilde{W}$ is well-defined and is a $\tilde{\mathbf{Q}}$-martingale on $[t,T]$ with

$$\tilde{W}_t = f_t.$$

Moreover, on A_n^+ we have for $u\in[t,T]$,

$$\begin{aligned} 1-\varepsilon &< \frac{1-\varepsilon/(9n+3)}{1+\varepsilon/(9n+3)} \le \frac{h_t}{g_t}\left(1-\frac{\varepsilon}{9n+3}\right) \le \frac{\tilde{W}_u}{S_u} \le \frac{h_t}{g_t}\left(1+\frac{\varepsilon}{9n+3}\right)\\ &\le \left(1+\frac{n\varepsilon}{n+1}\right)\frac{1}{1-\varepsilon/(9n+3)}\left(1+\frac{\varepsilon}{9n+3}\right) < 1+\varepsilon. \end{aligned}$$

Similarly, on A_n^-,

$$\begin{aligned} 1+\varepsilon &> \frac{1+\varepsilon/(9n+3)}{1-\varepsilon/(9n+3)} \ge \frac{h_t}{g_t}\left(1+\frac{\varepsilon}{9n+3}\right) \ge \frac{\tilde{W}_u}{S_u} \ge \frac{h_t}{g_t}\left(1-\frac{\varepsilon}{9n+3}\right)\\ &\ge \left(1-\frac{n\varepsilon}{n+1}\right)\frac{1}{1+\varepsilon/(9n+3)}\left(1-\frac{\varepsilon}{9n+3}\right) > 1-\varepsilon, \end{aligned}$$

hence $(\tilde{\mathbf{Q}},\tilde{W}_u)_{t\le u\le T}$ is an ε-strictly consistent price system on $[t,T]$.

Now set $\tilde{S}_u = \tilde{W}_u,\ t\le u\le T$ and $\tilde{S}_u := E_{\tilde{\mathbf{Q}}(\delta)}(f_t|\mathcal{F}_u)$, $0\le u\le t$.

We have

$$|\tilde{W}_t(\delta)-f_t| < (\varepsilon+\delta)S_t \le \frac{\varepsilon+\delta}{1-\delta}\tilde{W}_t(\delta),$$

consequently, for $u \le t$,

$$|\tilde{S}_u - \tilde{W}_u(\delta)| < \tilde{W}_u(\delta)\frac{\varepsilon+\delta}{1-\delta} \le S_u\frac{(\varepsilon+\delta)(1+\delta)}{1-\delta},$$

thus using (36) for $\eta = \delta$ we get

$$S_u(1-\varepsilon') < \tilde{S}_u < S_u(1+\varepsilon').$$

Define the measure $\mathbf{Q}$ by setting

$$\frac{d\mathbf{Q}}{d\mathbf{P}} := \frac{d\tilde{\mathbf{Q}}(\delta)}{d\mathbf{P}}\frac{Y_T}{Y_t},$$

where $Y_t = E[d\tilde{\mathbf{Q}}/d\mathbf{P}|\mathcal{F}_t]$.

It is straightforward to check that $(\tilde{S}, \mathbf{Q})$ satisfy the requirements. ■

The following corollary has been used in the proof of Lemma 3.5. The deterministic time t may be replaced by a stopping time without alteration in the arguments.

Corollary 4.7 *Under the conditions of Proposition 4.5, for any $\mathcal{F}_t$-measurable random variable f_t and any probability $\mathbf{Q} \sim \mathbf{P}|_{\mathcal{F}_t}$ on $\mathcal{F}_t$ such that*

$$(1-\varepsilon)S_t < f_t < (1+\varepsilon)S_t,$$

there exists an ε-strictly consistent price system $((\tilde{S}')_{t\le u\le T}, \mathbf{Q}')$ on the interval $[t,T]$ such that $\tilde{S}'_t = f_t$ and $\mathbf{Q}'|_{\mathcal{F}_t} = \mathbf{Q}$.

Proof The construction in the first half of the the proof of Lemma 4.6 yields $\tilde{W}$ and $\tilde{\mathbf{Q}}$ such that $\tilde{W}_t = f_t$. Define $\tilde{S}'_u := \tilde{W}_u$. Finally set

$$\frac{d\mathbf{Q}'}{d\mathbf{P}} := \frac{d\mathbf{Q}/d\mathbf{P}}{E(d\tilde{\mathbf{Q}}/d\mathbf{P}|\mathcal{F}_t)}\frac{d\tilde{\mathbf{Q}}}{d\mathbf{P}}.$$

■

Proof of Proposition 4.5 We prove that (iii) implies (iii'), the reverse implication being just like in Proposition 9 of [1].

By contradiction, if $\widehat{V}_t + M\mathbf{1} \notin \widehat{K}^{\varepsilon}_t$ on a set of positive measure, then also $\widehat{V}_t + M\mathbf{1} \notin \widehat{K}^{\tilde{\varepsilon}}_t$ for some $\tilde{\varepsilon} < \varepsilon$ on some (possibly smaller) set B of positive measure.

Hence (by the measurable selection theorem) there exists an $\mathcal{F}_t$-measurable, $\operatorname{int}(\widehat{K}^{\tilde{\varepsilon}}_t)^*$-valued (bounded) random variable m_t such that $\langle m_t, \widehat{V}_t + M\mathbf{1}\rangle < 0$ on B.

As $\tilde{\varepsilon} < \varepsilon$, by Lemma 4.6 and Proposition 3.11 we obtain a martingale Z such that $Z_s \in \operatorname{int}(\widehat{K}^{\varepsilon}_s)^*$ for all $0 \leq s \leq T$ and $m_t = Z_t$. It follows that

$$\langle Z_t, \widehat{V}_t + M\mathbf{1} \rangle < 0,$$

on B, which is absurd by *(iii)* in Definition 3.14. ∎

Remark 4.8 Carrying out the above arguments for stopping times instead of deterministic times it becomes clear that, under the conditions of Proposition 4.5, conditions (iii) and (iii') are equivalent to

(iii") there is a threshold $M > 0$ such that, for each stopping time $0 \leq \tau \leq T$:

$$\widehat{V}_\tau \succeq_\tau -M\mathbf{1}, \quad \text{a.s..}$$

Proposition 4.9 *If $\widehat{V}^1_T$ is the terminal value of an M-admissible (in the sense of Definition 3.14) c.s. ε-self -financing portfolio process then there is an $M(1+\varepsilon)$-admissible θ (in the sense of Definition 1.5) such that $V^{\varepsilon}_T(\theta) \geq \widehat{V}^1_T$.*

Proof Remember the c.s. portfolio process on the boundary $\widehat{W}$ as constructed in Corollary 4.4. Let θ be a trading strategy corresponding to $\widehat{W}$ (see Proposition 4.2). We need to establish that θ is $(1+\varepsilon)M$-admissible. From

$$V_t(\theta) = \widehat{W}^1_t + \widehat{W}^2_t S_t(1-\varepsilon)1_{\{\widehat{W}^2_t > 0\}} + \widehat{W}^2_t S_t(1+\varepsilon)1_{\{\widehat{W}^2_t < 0\}},$$

it follows that

$$V_t(\theta) \geq \widehat{W}^1_t - M(1+\varepsilon)S_t \geq \widehat{V}^1_t - M(1+\varepsilon)S_t \geq -M[1 + (1+\varepsilon)S_t].$$

∎

5 Local martingales and numéraires

In this paper, Definition 1.3 of an ε-consistent price system requires that $\widetilde{S}$ is a *true* (as opposed to *local*) martingale under $\mathbf{Q}$. On the other hand, the frictionless characterization of no arbitrage for continuous processes usually involves the notion of equivalent *local* martingale measures, i.e., probability measures $\mathbf{Q} \sim \mathbf{P}$ under which the price process S is a *local* martingale.

This subtle difference corresponds to the difference in the choice of the notion of admissibility of self-financing processes, as was made clear — in the frictionless case — by the work of Ji-An Yan and his co-authors ([28], [26], [27], see also [7], [25] and [24]).

Mathematically speaking, Definition 1.10 of an arbitrage as well as the admissibility condition (iii) in Definition 3.14 involve comparisons with scalar mulitples of the vector $\mathbf{1} = (1, 1)$ in $\mathbb{R}^2$. Economically speaking, these inequalities compare the portfolio with positions which may be short in each of the assets. This approach seems quite natural for the applications to financial markets with transaction costs when there is no natural numéraire (compare to [15]).

On the other hand, the classical approach in the frictionless theory is to consider the bond as numéraire and to formulate the inequalities controlling the portfolio only in units of the bond, not allowing for short positions in the risky asset. The corresponding notion of admissibility, which compares the portfolio to the vector $(1, 0)$ rather than to $(1, 1)$, goes back to Harrison and Pliska [12] (compare also to [6] and [8]).

We now proceed in the opposite direction, deriving the numéraire-based approach in the present transaction cost context where the asset $B \equiv 1$ is chosen as numéraire.

Definition 5.1 (numéraire-based variant of Definition 3.14) *In the setting of Definition 3.14 above we call an ε-self-financing portfolio process $\widehat{V}$* admissible in a numéraire-based sense *if there is a threshold, i.e., a constant $M > 0$ such that $\widehat{V}_T \succeq_T (-M, 0)$ and $\langle Z_\tau, \widehat{V}_\tau \rangle \geq -M \langle Z_\tau, (1, 0) \rangle$ a.s. for all $[0, T]$-valued stopping times τ and for every $Z \in \mathcal{Z}^s(\varepsilon)$.*

Definition 5.2 (numéraire-based variant of Definition 1.10) *A given continuous, adapted $\mathbb{R}_+$-valued price process S admits a* numéraire-based arbitrage for small transaction costs *if there is $\varepsilon > 0$ and an ε-self financing portfolio process $\widehat{V}$, admissible in the numéraire-based sense such that $\widehat{V}_T \in \mathbb{R}^2_+$ a.s. and $\mathbf{P}[\widehat{V}_T \neq 0] > 0$.*

The above numéraire-based notions clearly impose a stronger condition on arbitrage and the admissible portfolio processes than the ones given in sections 1 and 3 above. Dually, we have to *weaken* the concept of ε-consistent price systems in order to obtain an analogue of Theorem 1.11.

Definition 5.3 (numéraire-based variant of Definition 1.3) *Given an $\mathbb{R}_+$-valued (adapted, càdlàg) price process $S = (S_t)_{0 \leq t \leq T}$, a* numéraire-based ε-consistent price system *is a pair $(\widetilde{S}, \mathbf{Q})$ such that $\widetilde{S} = (\widetilde{S}_t)_{0 \leq t \leq T}$ is an* *adapted, càdlàg process satisfying*

$$1-\varepsilon \leq \frac{\widetilde{S}_t}{S_t} \leq 1+\varepsilon, \quad 0 \leq t \leq T,$$

and $\mathbf{Q}$ *is a probability measure equivalent to* $\mathbf{P}$ *such that* $\widetilde{S}$ *is a* local *martingale under* $\mathbf{Q}$.

We now formulate a numéraire-based version of Theorem 1.11.

Theorem 5.4 (numéraire-based Fundamental Theorem) *Let* $S = (S_t)_{0\leq t\leq T}$ *satisfy Assumption 1.1. The following assertions are equivalent.*

(i) For each $\varepsilon > 0$ *there exists a numéraire-based* ε*-consistent price system.*

(ii) There is no numéraire-based arbitrage for small transaction costs.

Proof The technical difference is that we do *not* consider $G(\sigma, \varepsilon)$, but only $F(\sigma, \varepsilon)$ of Definition 3.4 this time. The rest follows the proof of Theorem 1.11. ∎

Remark 5.5 All this parallels what happens in the frictionless case: when admitting strategies whose value process is bounded below by a constant times the numéraire, no free lunch with vanishing risk (NFLVR) implies the existence of $\mathbf{Q} \sim \mathbf{P}$ such that S is a $\mathbf{Q}$-local martingale, see [6]. Allowing value processes bounded below by constant times the prices of the risky assets, (NFLVR) provides $\mathbf{Q}$ such that S is a true $\mathbf{Q}$-martingale, see [28].

A Appendix: Some examples

Let $C^+[u,v]$ denote the set of continuous positive functions on $[u,v]$ and let $C^+_x[u,v]$ denote the family of continuous positive functions taking the value x at u. In [10] the following result has been shown:

Theorem A.1 *Let* $(S_t)_{t\in[0,T]}$ *be a positive continuous adapted process satisfying the* conditional full support *(CFS) condition, i.e. for all* $t \in [0,T)$*:*

$$\operatorname{supp} P(S|_{[t,T]}|\mathcal{F}_t) = C^+_{S_t}[t,T] \qquad a.s. \tag{CFS}$$

where $P(S|_{[t,T]}|\mathcal{F}_t)$ *denotes the* $\mathcal{F}_t$*-conditional distribution of the* $C^+[t,T]$*-valued random variable* $S|_{[t,T]}$*. Then* S *admits an* ε*-consistent pricing system for all* $\varepsilon > 0$*.*

We start observing the rather obvious fact that there are continuous martingales S failing the conditional full support (CFS) condition, and therefore such condition is not necessary for the existence of consistent price systems.

For example, consider a standard Brownian motion $(W_t)_{0\le t\le T}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$ and define S by $S_T = \exp(\operatorname{sign}(W_T))$ and

$$S_t = \mathbf{E}[S_T \mid \mathcal{F}_t].$$

A second easy example shows that the condition (CFS) does not imply that the law of S_T is absolutely continuous with respect to the Lebesgue measure. Let $(q_n)_{n=1}^{\infty}$ be an enumeration of the rationals in $\mathbb{R}_{++}$ and $(p_n)_{n=1}^{\infty}$ strictly positive numbers such that $\sum_{n=1}^{\infty} p_n = 1$. Find a function $\varphi : \mathbb{R} \to \mathbb{Q}$ such that $\mathbf{P}[\varphi(W_T) = q_n] = p_n$, for each $n \ge 1$. Letting $S_T = \varphi(W_T)$ and

$$S_t = \mathbf{E}[S_T \mid \mathcal{F}_t], \quad 0 \le t \le T$$

we have a martingale satisfying (CFS), but such that the law of S_T is concentrated on the rationals.

We now pass to a less trivial example which shows the limitations of the "local" Theorem 1.4: Proposition A.2 will show that the assumption (NOA) is not sufficient to insure the existence of ε-consistent price systems.

Proposition A.2 *There is an $\mathbb{R}_{++}$-valued continuous process $(S_t)_{0\le t\le T}$ which satisfies the condition of "no obvious arbitrage" (NOA), and yet has no ε-consistent price system for all $0 < \varepsilon < 1$.*

Proof Let $(W_t)_{t\ge 0}$ be a standard Brownian motion with respect to the stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{Q}^0)$, and define $X_t = \exp[W_t - t/2]$ so that $(X_t)_{t\ge 0}$ is a geometric Brownian motion with respect to $\mathbf{Q}^0$.

Define a sequence of stopping times $(\rho_n)_{n=1}^{\infty}$ by $\rho_0 := 0$, $\rho_1 := \inf\{t \mid X_t = 2^{-2} \text{ or } 2\}$ and, for $n \ge 1$ let

$$\rho_{n+1} := \rho_n \cdot 1_{\{X_{\rho_n} \ne 2^{-2^n}\}} + \sigma_{n+1} \cdot 1_{\{X_{\rho_n} = 2^{-2^n}\}}$$

where

$$\sigma_{n+1} := \inf\left\{t \ge 0 \mid X_t = 2^{-2^{n+1}} \text{ or } 2^{-n+1}\right\}.$$

The stopping time τ is defined as

$$\tau = \min\left\{\rho_n \mid X_{\rho_n} = 2^{-n+2}\right\}$$

and the process S is the process X^{τ} stopped at time τ, i.e.,

$$S_t = X_{t\wedge\tau}, \quad 0 \le t < \infty.$$

Note that $\mathbf{Q}^0[\tau = \infty] > 0$. Indeed, since

$$\{\tau > \rho_n\} = \{X_{\rho_n} = 2^{-2^n}\} \text{ and } \{\tau > \rho_{n+1}\} = \{X_{\rho_{n+1}} = 2^{-2^{n+1}}\}$$

the martingale property of X implies that

$$\mathbf{Q}^0[\tau > \rho_{n+1} \mid \tau > \rho_n] = 1 - \frac{2^{-2^n} - 2^{-2^{n+1}}}{2^{-n+1} - 2^{-2^{n+1}}}$$

and hence $\mathbf{Q}^0[\tau = \infty] > 0$ as

$$\sum_{n=1}^{\infty} \frac{2^{-2^n} - 2^{-2^{n+1}}}{2^{-n+1} - 2^{-2^{n+1}}} < \infty.$$

Next, define a probability measure $\mathbf{P}$ on $\mathcal{F}$ such that $\mathbf{P} \ll \mathbf{Q}^0$, the restrictions of $\mathbf{P}$ and $\mathbf{Q}^0$ to $\mathcal{F}_{\rho_n}$ are equivalent probability measures on $\mathcal{F}_{\rho_n}$, for each n, and such that

$$\mathbf{P}[\tau = \infty] = 0.$$

For example, define

$$\frac{d\mathbf{P}}{d\mathbf{Q}^0} = \sum_{n=1}^{\infty} \frac{2^{-n}}{\mathbf{Q}^0[\tau = \rho_n]} 1_{\{\tau = \rho_n\}},$$

so that $\mathbf{P}[\tau = \rho_n] = 2^{-n}$, for $n \geq 1$.

The market model consists of the price process S under the probability $\mathbf{P}$. First note that $\mathbf{P}$-a.s. the trajectories of $(S_t)_{t\geq 0}$ become eventually constant so that we may define $S_\infty = \lim_{t\to\infty} S_t$ and consider the process $S = (S_t)_{0\leq t\leq\infty}$ on the closed interval $[0,\infty]$, which is, of course, isomorphic to $[0,T]$, for any $T \in \mathbb{R}_{++}$. For notational convenience we consider the closed time interval $[0,\infty]$ rather than performing a (cosmetic) deterministic time change to $[0,T]$.

We now show that S allows no obvious arbitrage. Indeed, supppose that there are $\alpha > 0$ and stopping times $0 \leq \kappa \leq \lambda \leq \infty$ such that $\mathbf{P}[\kappa < \infty] > 0$, $\{\kappa < \infty\} = \{\lambda < \infty\}$ and

$$\begin{array}{lrcll} & S_\lambda/S_\kappa & \geq & (1+\alpha), & \text{a.s. on } \{\kappa < \infty\} \\ \text{or} & & & & \\ & S_\lambda/S_\kappa & \leq & (1+\alpha)^{-1}, & \text{a.s. on } \{\kappa < \infty\}. \end{array} \tag{40}$$

Find n large enough such that $\mathbf{P}[\kappa < \rho_n] > 0$. Note that on the set $A_n = \{\kappa < \rho_n\}$ we have $S_\kappa \geq 2^{-2^n}$. Find $m \geq n$ such that $2^{-m+1} < 2^{-2^n}$ and note that $(S_t)_{\rho_m < t \leq \infty}$ is a.s. bounded from above by 2^{-m+1}. Recall that the restrictions of $\mathbf{P}$ and $\mathbf{Q}^0$ to $\mathcal{F}_{\rho_m}$ are equivalent. As $(S_t)_{0\le t\le\rho_m}$ is a (uniformly integrable) $\mathbf{Q}^0$-martingale we have either

$$\text{(i):} \qquad S_\kappa = S_{\lambda\wedge\rho_m}, \quad \mathbf{P}\text{-a.s. on } A_n$$

or

$$\text{(ii):} \qquad \mathbf{P}[\{S_\kappa > S_{\lambda\wedge\rho_m}\}\cap A_n] > 0 \ \text{ and } \ \mathbf{P}[\{S_\kappa < S_{\lambda\wedge\rho_m}\}\cap A_n] > 0.$$

In case (i) we have $S_{\lambda\wedge\rho_m} = S_\lambda$ $\mathbf{P}$-a.s. on A_n which gives a contradiction to (40). Similarly, in the first subcase in case (ii) we have that $\{S_\kappa > S_{\lambda\wedge\rho_m}\}\cap A_n = \{S_\kappa > S_\lambda\}\cap A_n$ so that we again find a contradiction to the first equation of (40). The second subcase leads to a contradiction to the second equation in (40).

The final part of the proof consists in showing that, there is no ε-consistent price system for any ε. Fix $\varepsilon > 0$ (and note that here we do not exclude that ε is large, e.g. $\varepsilon = 10^{10}$) and suppose there is a process $(\widetilde{S}_t)_{0\le t\le\infty}$ and a probability measure $\mathbf{Q}$ on $\mathcal{F}$, equivalent to $\mathbf{P}$, such that $(\widetilde{S}_t)_{0\le t\le\infty}$ is a $\mathbf{Q}$-martingale and such that

$$(1+\varepsilon)^{-1} \le \frac{\widetilde{S}_t}{S_t} \le 1+\varepsilon, \quad \mathbf{P}\text{-a.s., for } 0\le t\le\infty.$$

We shall estimate

$$q_n := \mathbf{Q}[\tau = \rho_{n+1} \mid \tau > \rho_n],$$

for sufficiently large n. We have

$$\widetilde{S}_{\rho_n} \le (1+\varepsilon)S_{\rho_n} = (1+\varepsilon)2^{-2^n}, \qquad \text{a.s. on } \{\tau > \rho_n\},$$

and

$$\widetilde{S}_{\rho_{n+1}} \ge (1+\varepsilon)^{-1}S_{\rho_{n+1}} = \begin{cases} (1+\varepsilon)^{-1}2^{-n+1} & \text{on } \{\tau = \rho_{n+1}\} \\ (1+\varepsilon)^{-1}2^{-2^{n+1}} & \text{on } \{\tau > \rho_{n+1}\}. \end{cases}$$

We obtain from the martingale property of $\widetilde{S}$ under $\mathbf{Q}$ that

$$\mathbf{E}_\mathbf{Q}\big[\widetilde{S}_{\rho_{n+1}}1_{\{\tau=\rho_{n+1}\}} \mid \tau > \rho_n\big] + \mathbf{E}_\mathbf{Q}\big[\widetilde{S}_{\rho_{n+1}}1_{\{\tau>\rho_{n+1}\}} \mid \tau > \rho_n\big] = \mathbf{E}\big[\widetilde{S}_{\rho_n} \mid \tau > \rho_n\big]$$

which yields

$$q_n(1+\varepsilon)^{-1}2^{-n+1} + (1-q_n)(1+\varepsilon)^{-1}2^{-2^{n+1}} \le (1+\varepsilon)2^{-2^n}.$$

Solving for q_n yields

$$q_n\big[(1+\varepsilon)^{-1}\big(2^{-n+1} - 2^{-2^{n+1}}\big)\big] \le (1+\varepsilon)2^{-2^n} - (1+\varepsilon)^{-1}2^{-2^{n+1}},$$

so that

$$q_n \le (1+\varepsilon)^2 2^{-2^n+n}.$$

Hence $\sum_{n=1}^{\infty} q_n < \infty$. As $\mathbf{Q} \sim \mathbf{P}$ we also have that $0 < q_n < 1$ for all $n \ge 1$, and therefore:

$$\mathbf{Q}[\tau = \infty] = \prod_{n=1}^{\infty} (1 - q_n) > 0.$$

This gives the desired contradiction to $\mathbf{Q} \sim \mathbf{P}$ as $\mathbf{P}[\tau = \infty] = 0$. ■

References

[1] Campi, L., Schachermayer, W.: A Super-Replication Theorem in Kabanov's Model of Transaction Costs, Finance and Stochastics **10**, 579–596 (2006)

[2] Cheridito, P.: Arbitrage in fractional Brownian motion models, Finance and Stochastics **7**, 533–553 (2003)

[3] Cherny, A.: General arbitrage pricing model: transaction costs, In: Séminaire de Probabilités XL, Lecture Notes in Mathematics **1899**, 447-462, Berlin: Springer 2007

[4] Cvitanić, J., Karatzas, I.: Hedging and portfolio optimization under transaction costs: a martingale approach, Mathematical Finance **6**, 133–165 (1996)

[5] Davis, M. H. A., Norman, A. R.: Portfolio selection with transaction costs. Mathematics of Operations Research, **15**, 676–713 (1990)

[6] Delbaen, F., Schachermayer, W.: A General Version of the Fundamental Theorem of Asset Pricing, Mathematische Annalen **300**, 463–520 (1994)

[7] Delbaen, F., Schachermayer, W.: The no-arbitrage property under a change of numéraire, Stochastics and Stochastics Reports **53**, 926–945 (1995)

[8] Delbaen, F., Schachermayer, W.: The Mathematics of Arbitrage. Springer Finance Series, Berline: Springer 2006

[9] Guasoni, P.: No Arbitrage with Transaction Costs, with Fractional Brownian Motion and Beyond, Mathematical Finance **16**, 569–582 (2006)

[10] Guasoni, P., Rásonyi, M., Schachermayer, W.: Consistent Price Systems and Face-Lifting Pricing under Transaction Costs, Annals of Applied Probability **18**, 491–520 (2008)

[11] Harrison, J. M., Kreps, D. M.: Martingales and arbitrage in multiperiod securities markets, Journal of Economic Theory **20**, 381–408 (1979)

[12] Harrison, J. M., Pliska, S. R.: Martingales and stochastic integrals in the theory of continuous trading, Stochastic Processes and their Applications **11**, 215–260 (1981)

[13] Jacod, J., Shiryaev, A.: Limit Theorems for Stochastic Processes. Berlin: Springer 1987

[14] Jouini, E., Kallal, H.: Martingales and arbitrage in securities markets with transaction costs, Journal of Economic Theory **66**, 178–197 (1995)

[15] Kabanov, Yu. M.: Hedging and liquidation under transaction costs in currency markets, Finance and Stochastics **3**, 237–248 (1999)

[16] Kabanov, Yu. M., Last, G.: Hedging under transaction costs in currency markets: a continuous-time model, Mathematical Finance **12**, 63–70 (2002)

[17] Kabanov, Yu. M., Stricker, Ch.: Hedging of contingent claims under transaction costs, In: Advances in Finance and Stochastics. Essays in honour of Dieter Sondermann, ed. K. Sandman and Ph. Schönbucher, 125–136, Berlin: Springer 2002

[18] Kabanov, Yu. M., Rásonyi, M., Stricker, Ch.: No-arbitrage criteria for financial markets with efficient friction, Finance and Stochastics **6**, 371–382 (2002)

[19] Kreps, D. M.L: Arbitrage and equilibrium in economies with infinitely many commodities, Journal of Mathematical Economics **8**, 15–35 (1981)

[20] Protter, Ph. E.: Stochastic integration and differential equations. (Second edition.) Applications of Mathematics **21**, Berlin: Springer 2004

[21] Rogers, L. C. G.: Arbitrage with Fractional Brownian Motion, Mathematical Finance **7**, 95–105 (1997)

[22] Schachermayer, W.: Martingale Measures for discrete time processes with infinite horizon, Methematical Finance **4**, 25–55 (1994)

[23] Schachermayer, W.: The Fundamental Theorem of Asset Pricing Under Proportional Transaction Costs in Finite Discrete Time, Mathematical Finance **14**, 19–48 (2004)

[24] Sin, C. A.: Strictly local martingales and hedge rations on stochastic volatility models, PhD Thesis, Cornell University (1996)

[25] Stricker, Ch., Yan, J.-A.: Some remarks on the optional decomposition theorem. Séminaire de Probabilités, XXXII, Lecture Notes in Mathematics **1686**, 56–66, Berlin: Springer 1998

[26] Xia, J. M., Yan, J. A-.: Some remarks on arbitrage pricing theory. In: J.M. Yong, eds, Recent Developments in Mathematical Finance (Shanghai 2001), 218–227, River Edge, NJ: World Scientific 2002

[27] Yan, J. -A.: A Numeraire-free and Original Probability Based Framework for Financial Markets. In: Proceedings of the International Congress of Mathematicians Vol. III (Beijing 2002), 861–871, Beijing: Higher Education Press 2002

[28] Yan, J. -A.: A new look at the fundamental theorem of asset pricing. Journal of Korean Mathematics Society **33**, 659–673 (1998)