A General Duality Theorem for the Monge–Kantorovich Transport Problem

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Abstract

The duality theory of the Monge–Kantorovich transport problem is analyzed in a general setting. The spaces X, Y are assumed to be polish and equipped with Borel probability measures μ and ν. The transport cost function $c : X \times Y \to [0, \infty]$ is assumed to be Borel. Our main result states that in this setting there is no duality gap, provided the optimal transport problem is formulated in a suitably relaxed way. The relaxed transport problem is defined as the limiting cost of the partial transport of masses $1-\varepsilon$ from (X, μ) to (Y, ν), as $\varepsilon > 0$ tends to zero. The classical duality theorems of H. Kellerer, where c is lower semi-continuous or uniformly bounded, quickly follow from these general results.

We also show that, in the present setting, a dual optimizer always exists, provided we interpret it as a projective limit of certain finitely additive measures. Several counter-examples indicate the limitations of these results.

Keywords: Monge-Kantorovich problem, duality, dual attainment

1 Introduction

We consider the *Monge-Kantorovich transport problem* for Borel probability measures μ, ν on polish spaces X, Y. See [Vil03, Vil09] for an excellent account of the theory of optimal transportation.

The set $\Pi(\mu, \nu)$ consists of all Monge-Kantorovich *transport plans*, that is, Borel probability measures on $X \times Y$ which have X-marginal μ and Y-marginal ν. The *transport costs* associated to a transport plan π are given by

$$\langle c, \pi \rangle = \int_{X \times Y} c(x, y)\, d\pi(x, y). \tag{1}$$

In most applications of the theory of optimal transport, the cost function $c : X \times Y \to [0, \infty]$ is lower semi-continuous and only takes values in $\mathbb{R}_+$. But equation (1) makes perfect sense if the $[0, \infty]$-valued cost function only is Borel measurable. We therefore assume throughout this paper that $c : X \times Y \to [0, \infty]$ is a Borel measurable function which may very well assume the value $+\infty$ for "many" $(x, y) \in X \times Y$.

An application where the value ∞ occurs in a natural way is transport between measures on Wiener space $X = (C[0, 1], \|.\|_\infty)$, where $c(x, y)$ is the squared norm of $x - y$ in the Cameron-Martin space, defined to be ∞ if $x - y$ does not belong to this space. Hence in this situation the set $\{y : c(x, y) < \infty\}$ has ν-measure 0, for every $x \in X$, if the measure ν

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is absolutely continuous with respect to the Wiener measure on $C[0,1]$. (See [FÜ02, FÜ04a, FÜ04b, FÜ06]).

Turning back to the general problem: the (primal) Monge-Kantorovich problem is to determine the primal value

$$P := \inf\{\langle c, \pi\rangle : \pi \in \Pi(\mu,\nu)\} \tag{2}$$

and to identify a primal optimizer $\hat{\pi} \in \Pi(\mu,\nu)$. To formulate the dual problem, we define

$$\Psi(\mu,\nu) = \left\{(\varphi,\psi) : \begin{array}{l} \varphi : X \to [-\infty,\infty), \psi : Y \to [-\infty,\infty) \text{ integrable,} \\ \varphi(x)+\psi(y) \le c(x,y) \text{ for all } (x,y) \in X \times Y. \end{array}\right\}. \tag{3}$$

The dual Monge-Kantorovich problem then consists in determining

$$D := \sup\left\{\int_X \varphi\, d\mu + \int_Y \psi\, d\nu\right\} \tag{4}$$

for $(\varphi,\psi) \in \Psi(\mu,\nu)$. We say that Monge-Kantorovich duality holds true, or that *there is no duality gap*, if the primal value P of the problem equals the dual value D, i.e. if we have

$$\inf\{\langle c,\pi\rangle : \pi \in \Pi(\mu,\nu)\} = \sup\left\{\int_X \varphi\, d\mu + \int_Y \psi\, d\nu : (\varphi,\psi) \in \Psi(\mu,\nu)\right\}. \tag{5}$$

There is a long line of research on these questions, initiated already by Kantorovich ([Kan42]) himself and continued by numerous others (we mention [KR58, Dud76, Dud02, dA82, GR81, Fer81, Szu82, Mik06, MT06], see also the bibliographical notes in [Vil09, p 86, 87]).

The validity of the above duality (5) was established in pleasant generality by H. Kellerer [Kel84]. He proved that there is no duality gap provided that c is lower semi-continuous (see [Kel84, Theorem 2.2]) or just Borel measurable and bounded by a constant[1] ([Kel84, Theorem 2.14]). In [RR95, RR96] the problem is investigated beyond the realm of polish spaces and a characterization is given for which spaces duality holds for all bounded measurable cost functions. We also refer to the seminal paper [GM96] by W. Gangbo and R. McCann.

We now present a rather trivial example[2] which shows that, in general, there is a duality gap.

Example 1.1. Consider $X = Y = [0,1]$ and $\mu = \nu$ the Lebesgue measure. Define c on $X \times Y$ to be 0 below the diagonal, 1 on the diagonal and ∞ else, i.e.

$$c(x,y) = \begin{cases} 0, & \text{for } 0 \le y < x \le 1, \\ 1, & \text{for } 0 \le x = y \le 1, \\ \infty, & \text{for } 0 \le x < y \le 1. \end{cases}$$

Then the only finite transport plan is concentrated on the diagonal and leads to costs of one so that $P = 1$. On the other hand, for admissible $(\varphi,\psi) \in \Psi(\mu,\nu)$, it is straightforward to check, that $\varphi(x) + \psi(x) > 0$ can hold true for at most countably many $x \in [0,1]$. Hence the dual value equals $D = 0$, so that there is a duality gap.

[1] or, more generally, by the sum $f(x) + g(y)$ of two integrable functions f, g.
[2] This is essentially [Kel84, Example 2.5].

1.1 A General Duality Theorem

A common technique in the duality theory of convex optimisation is to pass to a *relaxed* version of the problem, i.e., to enlarge the sets over which the primal and/or dual functionals are optimized.

We do so, for the primal problem (2), by requiring only the transport of a portion of mass $1-\varepsilon$ from μ to ν, for every $\varepsilon > 0$. Fix $0 \le \varepsilon \le 1$ and define

$$\Pi^\varepsilon(\mu,\nu) = \{\pi \in \mathcal{M}_+(X\times Y), \|\pi\| \ge 1-\varepsilon, p_X(\pi) \le \mu, p_Y(\pi) \le \nu\}.$$

Here $\mathcal{M}_+(X\times Y)$ denotes the non-negative Borel measures π on $X\times Y$ with norm $\|\pi\| = \pi(X\times Y)$; by $p_X(\pi) \le \mu$ (resp. $p_Y(\pi) \le \nu$) we mean that the projection of π onto X (resp. onto Y) is dominated by μ (resp. ν). We denote by P^ε the value of the $1-\varepsilon$ partial transportation problem

$$P^\varepsilon = \inf\left\{\langle c,\pi\rangle = \int_{X\times Y} c(x,y)d\pi(x,y) : \pi \in \Pi^\varepsilon(\mu,\nu)\right\}. \tag{6}$$

This partial transport problem has recently been studied by L. Cafarelli and R. McCann [CM06] as well as A. Figalli [Fig09]. In their work the emphasis is on a finer analysis of the Monge problem for the squared Euclidean distance on $\mathbb{R}^n$, and pertains to a fixed $\varepsilon > 0$. In the present paper, we do not deal with these more subtle issues of the Monge problem and always remain in the realm of the Kantorovich problem (2). Our emphasis is on the limiting behavior for $\varepsilon \to 0$: we call

$$P^{\mathrm{rel}} = \lim_{\varepsilon\to 0} P^\varepsilon \tag{7}$$

the *relaxed primal value* of the transport plan. Obviously the above limit exists (assuming possibly the value $+\infty$) and $P^{\mathrm{rel}} \le P$.

As a motivation for the subsequent theorem the reader may observe that, in Example 1.1 above, we have $P^{\mathrm{rel}} = 0$ (while $P = 1$). Indeed, it is possible to transport the measure $\mu\mathbb{1}_{[\varepsilon,1]}$ to the measure $\nu\mathbb{1}_{[0,1-\varepsilon]}$ with transport cost zero by the partial transport plan $\pi = (id, id-\varepsilon)_\#\,(\mu\mathbb{1}_{[\varepsilon,1]})$.

We now can formulate our main result.

Theorem 1.2. *Let X, Y be polish spaces, equipped with Borel probability measures μ,ν, and let $c : X\times Y \to [0,\infty]$ be Borel measurable.*

Then there is no duality gap, if the primal problem is defined in the relaxed form (7) while the dual problem is formulated in its usual form (4)*. In other words, we have*

$$P^{\mathrm{rel}} = D. \tag{8}$$

We observe that in (8) also the value $+\infty$ is possible.

The theorem gives a positive result on the issue of the duality in the Monge–Kantorovich problem. There remain further questions pertaining to the duality theory to be settled which are not covered by the theorem. Under which conditions do we have $P = P^{\mathrm{rel}}$, and when do primal and dual optimizers $\hat\pi$ and $(\hat\varphi,\hat\psi)$ exist?

As a by-product of the proof of Theorem 1.2 we obtain in Section 2 a criterion (Proposition 2.3) characterizing the validity of $P = P^{\mathrm{rel}}$, i.e., whether the value of the optimization problem (2) is equal to its relaxed value. Moreover we have $P = P^{\mathrm{rel}}$ in any of the following cases.

(a) c is lower semi-continuous,

(b) c is uniformly bounded or, more generally,

(c) c is $\mu \otimes \nu$-a.s. finitely valued.

Concerning (a) and (b), it is rather straight forward to check that these assumptions imply $P = P^{\text{rel}}$ (see Corollaries 2.4 and 2.6 below). In particular, the classical duality results of Kellerer quickly follow from Theorem 1.2. To achieve that also property (c) is sufficient seems to be more sophisticated and follows from [BS09, Theorem 1].

As regards the issue of the existence of a primal optimizer $\hat{\pi}$, the situation is quite simple: if $c : X \times Y \to [0, \infty]$ is lower semi-continuous and $(\pi_n)_{n=1}^\infty$ is a minimizing sequence for (2), then it follows from Prokhorov's Theorem that there is a weakly convergent subsequence $(\pi_{n_k})_{k=1}^\infty$; the limit $\hat{\pi} := \lim_{k\to\infty} \pi_{n_k}$ then does the job (compare Lemma 2.5 below). On the other hand, if c fails to be lower semi-continuous, there is little reason why a primal optimizer $\hat{\pi}$ should exist. This was already evident to H. Kellerer [Kel84, Example 2.20].

Concerning the existence of dual optimizers $(\hat{\varphi}, \hat{\psi})$, things are more tricky: firstly, in general one cannot expect to find them in $L^1(\mu)$ and $L^1(\nu)$ even in rather regular situations. Ideally, $\hat{\varphi}$ and $\hat{\psi}$ should be in $\Psi(\mu, \nu)$, i.e. integrable $[-\infty, \infty)$-valued Borel functions, satisfying $\hat{\varphi}(x) + \hat{\psi}(y) \le c(x, y)$, for all $(x, y) \in X \times Y$. In general, this is asking for too much, unless c satisfies additional moment conditions such as the ones obtained by Ambrosio and Pratelli ([AP03, Theorem 3.2]):

$$\begin{aligned} \mu \left\{ x : \textstyle\int_Y c(x, y)\, d\nu(y) < \infty \right\} > 0, \\ \nu \left\{ y : \textstyle\int_X c(x, y)\, d\mu(x) < \infty \right\} > 0. \end{aligned} \tag{9}$$

To see the reason why an assumption on the lines of (9) is indeed necessary in order to find *integrable* dual optimizers $\hat{\varphi}, \hat{\psi}$, consider the following example. Let μ be any probability measure on $X = \mathbb{R}$ with full support and ν be the measure μ shifted by one to the right on $Y = \mathbb{R}$, i.e.

$$\nu(A) = \mu(A - 1),$$

where $A - 1 = \{x - 1 : x \in A\}$. Let $c(x, y) = (x - y)^2/2$ which is a very regular choice. Rather obviously the unique optimal transport $\hat{\pi}$ from μ to ν is to shift each $x \in \mathbb{R}$ to $x + 1$, i.e. $\hat{\pi} = (id, id + 1)_\#(\mu)$, for which we get $\int_{X\times Y} c\, d\hat{\pi} = \frac{1}{2}$. Obvious candidates for the dual optimizer $(\hat{\varphi}, \hat{\psi})$ are $\hat{\varphi}(x) = -x$ and $\hat{\psi}(y) = y - \frac{1}{2}$. Indeed, we have $\hat{\varphi}(x) + \hat{\psi}(y) \le c(x, y)$ for all $(x, y) \in X \times Y$ and, if μ has finite first moment $\int_X |x|\, d\mu < \infty$, we indeed get

$$\int_X \hat{\varphi}\, d\mu + \int_Y \hat{\psi}\, d\nu = -m + \left(m + 1 - \tfrac{1}{2}\right) = \tfrac{1}{2}, \tag{10}$$

as it should be. There is, however, a problem if the first moment of μ fails to exist as then the left hand side of (10) does not make sense any more. But, of course, *morally speaking*, $(\hat{\varphi}, \hat{\psi})$ still are the obvious dual optimizers, also in the case when μ and ν fail to have finite first moments and one then has to find a way to give a proper meaning to (10).

The following solution was proposed in [BS09, Section 1.1]. If φ and ψ are integrable functions and $\pi \in \Pi(\mu, \nu)$ then

$$\int_X \varphi\, d\mu + \int_Y \psi\, d\nu = \int_{X\times Y} (\varphi(x) + \psi(y))\, d\pi(x, y). \tag{11}$$

It turns out that, if we drop the integrability condition on φ and ψ, but require that $\varphi(x) + \psi(y) \le c(x, y)$ and that π is a finite cost transport plan, i.e. $\int_{X\times Y} c\, d\pi < \infty$, the right hand side of (11) still makes good sense, while the left hand side need not. It was shown in [BS09, Lemma 1.1] that the right hand side of (11) does not depend on the choice of the finite cost transport plan π. Thus we make the following definition:

Definition 1.3. *Let X, Y be polish spaces, equipped with Borel probability measures μ, ν, and let $c : X \times Y \to [0, \infty]$ be Borel measurable. Under the assumption that there exists some finite transport plan $\pi \in \Pi(\mu, \nu)$ we define for Borel functions φ, ψ such that $\varphi(x) + \psi(y) \le c(x, y)$, for $(x, y) \in X \times Y$*

$$J_c(\varphi, \psi) = \int_{X \times Y} (\varphi(x) + \psi(y))\, d\pi(x, y).$$

We then say that we have dual attainment *in the optimization problem* (4) *if there exist Borel measurable functions $\hat{\varphi} : X \to [-\infty, \infty)$ and $\hat{\psi} : Y \to [-\infty, \infty)$ verifying $\varphi(x) + \psi(y) \le c(x, y)$, for $(x, y) \in X \times Y$, such that*

$$D = J_c(\hat{\varphi}, \hat{\psi}). \tag{12}$$

We remark that if $(\hat{\varphi}, \hat{\psi})$ is a pair of dual maximizers in the sense of (12) and c satisfies the moment conditions (9), then $\hat{\varphi}$ and $\hat{\psi}$ are automatically integrable so that $J_c(\hat{\varphi}, \hat{\psi}) = \int_X \varphi\, d\mu + \int_Y \hat{\psi}\, d\nu$.

The question dealt with in the next subsection is to determine under which assumptions we have dual attainment.

1.2 Dual attainment and robustness

To motivate the subsequently introduced notion of "robustness" we reinterpret the value P^ε defined in (6) above. While the above notion P^ε involves a transport of mass $1 - \varepsilon$ we shall now introduce a notion $\tilde{P}^\varepsilon$ involving a transport of mass $1 + \varepsilon$, which will turn out to amount to the same value. Here we go a small step beyond the usual setting of optimal transport in that we allow transports of masses bigger than one. But, of course, making the obvious changes, all concepts and result carry over to the present setup.

Lemma 1.4. *Let X, Y be polish spaces equipped with Borel probability measures μ, ν, let $c : X \times Y \to [0, \infty]$ be a Borel measurable function, and $\varepsilon > 0$. Let Z be a polish space disjoined from $X \cup Y$ and λ a Borel measure on Z such that $\lambda(Z) = \varepsilon$. Define an extension*[3] *$\tilde{c} : (X \cup Z) \times (Y \cup Z) \to [0, \infty]$ by setting*

$$\tilde{c}(x, y) = 0 \text{ for } (x, y) \in (X \cup Z) \times (Y \cup Z) \setminus X \times Y$$

and define

$$\tilde{P}^\varepsilon = \inf\{\langle \tilde{c}, \pi \rangle : \pi \in \Pi(\mu + \lambda, \nu + \lambda)\}.$$

Then we have

$$P^\varepsilon = \tilde{P}^\varepsilon.$$

Whence in particular $P^{\mathrm{rel}} = \lim_{\varepsilon \to 0} \tilde{P}^\varepsilon$.

Proof. To see that $P^\varepsilon \le \tilde{P}^\varepsilon$, it suffices to note that a transport plan $\tilde{\pi} \in \Pi(\mu + \lambda, \nu + \lambda)$ of the enlarged transport problem gives rise to transport plan of the relaxed problem if we restrict it to $X \times Y$, i.e. if we define $\pi^\varepsilon := \tilde{\pi}_{|(X \times Y)}$. We have $\tilde{\pi}(Z \times (Y \cup Z)) = \lambda(Z) = \varepsilon$ and $\tilde{\pi}(X \times Z) \le \tilde{\pi}((X \cup Z) \times Z) = \lambda(Z) = \varepsilon$, so that

$$\pi^\varepsilon(X \times Y) = \tilde{\pi}((X \cup Z) \times (Y \cup Z)) - \tilde{\pi}(Z \times (Y \cup Z)) - \tilde{\pi}(X \times Z) \in [1 - \varepsilon, 1].$$

[3] Here $X \cup Y$ is the disjoint union of the polish spaces X and Z, which again is a polish space which is naturally equipped with the measure $\mu + \lambda$. We denote by $\pi(\mu + \lambda, \nu + \lambda)$ the set of all positive Borel measures on $(X \cup Z) \times (Y \cup Z)$ having $\mu + \lambda$, resp. $\nu + \lambda$ as marginal.

Therefore $\pi^\varepsilon \in \Pi^\varepsilon(\mu, \nu)$.

To prove the converse inequality, we have to extend $\pi^\varepsilon \in \Pi^\varepsilon(\mu,\nu)$ to a transport plan in $\Pi(\mu+\lambda, \nu+\lambda)$. Denote by μ^ε (resp. ν^ε) the Borel measure $\mu^\varepsilon = \mu - p_X(\pi^\varepsilon)$ (resp. $\nu^\varepsilon = \nu - p_Y(\pi^\varepsilon)$). We may assume w.l.g. that $\|1-\pi^\varepsilon\| = \varepsilon$. Then define $\tilde{\pi}$ on $(X \cup Z) \times (Y \cup Z)$ by letting

$$\tilde{\pi} := \pi^\varepsilon + \tfrac{1}{\lambda(Z)}(\mu^\varepsilon \otimes \lambda) + \tfrac{1}{\lambda(Z)}(\lambda \otimes \nu^\varepsilon) \qquad \square$$

If $P = P^{\text{rel}} = \lim_{\varepsilon \to 0} \tilde{P}^\varepsilon$ then we say that the transport problem is *stable*. This property fits nicely with the colorful interpretation of optimal transport as shipments of croissants from Parisian bakeries to cafés given in [Vil09]. Our traditional consortium, transporting croissants from bakeries ($\in X$) to cafés ($\in Y$) is invaded by an evil tycoon. She builds a storage of size $\lambda(Z)$ where she buys up croissants and sends them to the cafés. Her intention is to disturb the traditional relations of bakeries and cafés by offering the transport for free, i.e. at cost zero. The notion of a *stable* cost function $c : X \times Y \to [0, \infty]$ describes the situation where the effect of her appearance can lower the total transport cost only slightly, provided the Parisian authorities only allow her to have a small storage size $\lambda(Z)$.

Another strategy of the Parisian administration might be to keep the size $\lambda(Z)$ fixed, but to impose (possibly very high, but still finite) tolls for all transports to and from the tycoon's storage, thus resulting in finite costs $\tilde{c}(a, b)$ for $(a, b) \in (X \times Z) \cup (Z \times Y)$. Remarkably, also this second strategy can successfully marginalize the tycoon's impact precisely if the system is stable (cf. Proposition 1.6 below).

Particularly convenient is the case where the traditional bonds are so *robust* (compare [BGMS09, Definition 1.6]) that its is possible to reduce the tycoon's effect to 0 (as opposed to an arbitrarily small $\delta > 0$) by choosing appropriately high (but finite) tolls. It turns out (Theorem 1.7 below) that this is intimately related to the attainment of the dual problem.

Definition 1.5. *Let X, Y be polish spaces equipped with Borel probability measures μ, ν, let $c : X \times Y \to [0, \infty]$ be a Borel measurable cost function. Let Z be a polish space and λ a Borel measure on Z such that $\lambda(Z) > 0$.*
We call $\tilde{c} : (X \cup Z) \times (Y \cup Z) \to [0, \infty]$ a viable extension *of c if $\tilde{c}$ is Borel measurable and satisfies*

$$\tilde{c}(x, y) = \begin{cases} 0 & \text{for } (x, y) \in Z \times Z, \\ c(x, y) & \text{for } (x, y) \in X \times Y, \\ < \infty & \text{for } (x, y) \in (X \times Z) \cup (Z \times Y). \end{cases} \tag{13}$$

For a given viable extension $\tilde{c}$ we shall consider the value of the associated primal problem

$$P_{\tilde{c}} := \inf\{\langle \tilde{c}, \pi \rangle : \pi \in \Pi(\mu + \lambda, \nu + \lambda)\}.$$

Proposition 1.6. *Let X, Y be polish spaces equipped with Borel probability measures μ, ν, let $c : X \times Y \to [0, \infty]$ be a Borel measurable cost function and assume that there exists a finite transport plan $\pi \in \Pi(\mu, \nu)$. Let Z be a polish space and λ a Borel measure on Z such that $\lambda(Z) > 0$. If $\tilde{c} : (X \cup Z) \times (Y \cup Z) \to [0, \infty]$ is a viable extension of c then*

$$P_{\tilde{c}} \le P^{\text{rel}}.$$

For $n \in \mathbb{N}$ let $\tilde{c}_n : (X \cup Z) \times (Y \cup Z) \to [0, \infty]$ be a viable extension of c which satisfies $\tilde{c}_n(x, y) \ge n$ for $(x, y) \in (X \times Z) \cup (Z \times Y)$. Then

$$\lim_{n \to \infty} P_{\tilde{c}_n} = P^{\text{rel}}.$$

The main result of this section is the following characterization of dual attainment.

Theorem 1.7. *Let X, Y be polish spaces equipped with Borel probability measures μ, ν, let $c : X \times Y \to [0, \infty]$ be a Borel measurable cost function and assume that there exists a finite transport plan $\pi \in \Pi(\mu, \nu)$. Let Z be a polish space and λ a finite Borel measure on Z such that $\lambda(Z) > 0$. Then the following are equivalent.*[4]

a. *We have dual attainment, i.e. there exist Borel measurable functions $\hat{\varphi} : X \to [-\infty, \infty)$ and $\hat{\psi} : Y \to [-\infty, \infty)$ such that $\hat{\varphi}(x) + \hat{\psi}(y) \leq c(x, y)$, for all $(x, y) \in X \times Y$ and*

$$J_c(\hat{\varphi}, \hat{\psi}) = D.$$

b. *There exists a viable extension $\tilde{c}$ of c such that*

$$P_{\tilde{c}} = P^{\mathrm{rel}}.$$

We note that lower semi-continuity of the cost function c (which implies the existence of a primal optimizer and that there is no duality gap) does *not* imply dual attainment. For instance, the cost function constructed in [BS09, Example 4.2] is continuous and satisfies the moment conditions (9) but does not have dual attainment as in Theorem 1.7 above.

A sufficient condition for dual attainment, is that the cost function c is $\mu \otimes \nu$-a.s. finitely valued ([BS09, Theorem 2]).

After providing the proof of Theorem 1.2 in Section 2 and the proofs of Proposition 1.6 and Theorem 1.7 in Section 3, we turn in Section 4 to the question whether a dual optimizer, possibly in a relaxed form, exists without imposing additional finiteness assumptions on the cost function. Recall from [BS09, Example 4.3] an example of a lower semi-continuous cost function $c : [0, 1) \times [0, 1) \to [0, \infty]$ (so that there is no duality gap) such that there is no dual attainment. Nevertheless, something weaker does hold true in this example which may be seen as a generalized dual optimizer: there is an optimizing sequence $(\varphi_n, \psi_n)_{n=1}^\infty$ for (4) such that, for every finite transport plan $\pi \in \Pi(\mu, \nu)$, the sequence $(\varphi_n(x) + \psi_n(y))_{n=1}^\infty$ converges π-a.s. and in the norm $\|\cdot\|_{L^1(\pi)}$ to a Borel function $\hat{h}(x, y)$ on $X \times Y$. We then have $\hat{h}(x, y) \leq c(x, y), \pi$-a.s., for each finite transport plan π. Hence, for a given finite transport plan $\pi \in \Pi(\mu, \nu)$, this function $\hat{h}$ may be viewed as some kind of a dual optimizer. However, $\hat{h}(x, y)$ is *not* of the form $\hat{h}(x, y) = \varphi(x) + \psi(y)$, for Borel functions φ, ψ, but only a limit of functions of this form.

In Theorems 4.2 and 5.2 we show that the phenomenon just described for the very special case of the above example, carries over to the case of a fully general Borel measurable cost function $c : X \times Y \to [0, \infty]$. These theorems need quite some functional analytic machinery, e.g. the use of the space $L^\infty(\pi)^*$ of finitely additive measures, as well as the notion of a projective limit.

2 The Proof of the Duality Theorem

The proof of Theorem 1.2 relies on Fenchel's perturbation technique. We refer to the accompanying paper [BLS09] for a didactic presentation of this technique: we there give an elementary version of this argument, where $X = Y = \{1, \ldots, N\}$ equipped with the uniform measure $\mu = \nu$, in which case the optimal transport problem reduces to a finite linear programming problem.

We start with an easy result showing that the relaxed version (6) of the optimal transport problem is not "too much relaxed", in the sense that the trivial implication of the minmax theorem still holds true.

[4] We emphasize that the polish space (Z, λ) does not appear in the formulation of (a.). Thus, if (b.) is valid for *some* polish space, then (a.) is true and hence yields (b.) for *every* polish space.

Proposition 2.1. *Under the assumptions of Theorem 1.2. we have*

$$P^{\mathrm{rel}} \geq D.$$

Proof. Let (φ, ψ) be integrable Borel functions such that

$$\varphi(x) + \psi(y) \leq c(x, y), \qquad \text{for every } (x, y) \in X \times Y. \tag{14}$$

Let $\pi_n \in \Pi(f_n\mu, g_n\nu)$ be an optimizing sequence for the relaxed problem, where $f_n \leq \mathbb{1}, g_n \leq \mathbb{1}$, and $\pi_n(X \times Y) = \|f_n\|_{L^1(\mu)} = \|g_n\|_{L^1(\nu)}$ tends to one. By passing to a subsequence we may assume that $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ converge a.s. to $\mathbb{1}$. We may estimate

$$\liminf_{n\to\infty} \int_{X\times Y} c\, d\pi_n \geq \liminf_{n\to\infty} \left[\int_X \varphi f_n \, d\mu + \int_Y \psi g_n \, d\nu \right] = \int_X \varphi\, d\mu + \int_Y \psi\, d\nu,$$

where in the last equality we have used Lebesgue's theorem on dominated convergence. □

The next lemma is a technical result which will be needed in the formalization of the proof of Theorem 1.2.

Lemma 2.2. *Let V be a normed vector space, $x_0 \in V$, and let $\Phi : V \to (-\infty, \infty]$ be a positively homogeneous*[5] *convex function such that*

$$\liminf_{\|x - x_0\| \to 0} \Phi(x) \geq \Phi(x_0).$$

If $\Phi(x_0) < \infty$ then, for each $\varepsilon > 0$, there exists a continuous linear functional $v : V \to \mathbb{R}$ such that

$$\Phi(x_0) - \varepsilon \leq v(x_0) \text{ and } \Phi(x) \geq v(x) \text{ for all } x \in V.$$

If $\Phi(x_0) = \infty$ then for each $M > 0$ there exists a continuous linear functional $v : V \to \mathbb{R}$ such that

$$M \leq v(x_0) \text{ and } \Phi(x) \geq v(x) \text{ for all } x \in V.$$

Proof. Assume first that $\Phi(x_0) < \infty$. Let $K = \{(x, t) : x \in V, t \geq \Phi(x)\}$ be the epigraph of Φ and $\overline{K}$ its closure in $V \times \mathbb{R}$. Since Φ is assumed to be lower semi continuous at x_0, we have $\inf\{t : (x_0, t) \in \overline{K}\} = \Phi(x_0)$, hence $(x_0, \Phi(x_0) - \varepsilon) \notin \overline{K}$. By Hahn-Banach, there is a continuous linear functional $w \in V^* \times \mathbb{R}$ given by $w(x, t) = u(x) + st$ (where $u \in V^*$ and $s \in \mathbb{R}$) and $\beta \in \mathbb{R}$ such that $w(x, t) > \beta$ for $(x, t) \in \overline{K}$ and $w(x_0, \Phi(x_0) - \varepsilon) < \beta$. By the positive homogeneity of Φ, we have $\beta < 0$, hence $s > 0$. Also $u(x) + s\Phi(x) \geq \beta$ and by applying positive homogeneity once more we see that β can be replaced by 0. Hence we have

$$u(x) + s\Phi(x) \geq 0 \quad u(x_0) + s(\Phi(x_0) - \varepsilon) < 0,$$

so just let $v(x) := -u(x)/s$. In the case $\Phi(x_0) = \infty$ the assertion is proved analogously. □

We now define the function Φ to which we shall apply the previous lemma.

Let $W = L^1(\mu) \times L^1(\nu)$ and V the subspace of co-dimension one, formed by the pairs (f, g) such that $\int_X f\, d\mu = \int_Y g\, d\nu$. By $V_+ = \{(f, g) \in V : \ f \geq 0, \ g \geq 0\}$ we denote the positive orthant of the Banach space V. For $(f, g) \in V_+$, we define, by slight abuse of notation, $\Pi(f, g)$ as the set of non-negative Borel measures π on $X \times Y$ with marginals $f\mu$

[5] By positively homogeneous we mean $\Phi(\lambda x) = \lambda\Phi(x)$, for $\lambda \geq 0$, with the convention $0 \cdot \infty = 0$.

and $g\nu$ respectively. With this notation $\Pi(\mathbb{1},\mathbb{1})$ is just the set $\Pi(\mu,\nu)$ introduced above. Define $\Phi : V_+ \longrightarrow [0,\infty]$ by

$$\Phi(f,g) = \inf\left\{\int_{X\times Y} c(x,y)\,d\pi(x,y) : \pi \in \Pi(f,g)\right\}, \quad (f,g) \in V_+$$

which is a convex function. By definition we have $\Phi(\mathbb{1},\mathbb{1}) = P$, where P is the primal value of (2). Our matter of concern will be the lower semi-continuity of the function Φ at the point $(\mathbb{1},\mathbb{1}) \in V_+$.

Proposition 2.3. *Denote by $\overline{\Phi} : V \to [0,\infty]$ the lower semi-continuous envelope of Φ, i.e., the largest lower semi-continuous function on V dominated by Φ on V_+. Then*

$$\overline{\Phi}(\mathbb{1},\mathbb{1}) = P^{\mathrm{rel}}. \tag{15}$$

Hence the function Φ is lower semi-continuous at $(\mathbb{1},\mathbb{1})$ if and only if $P = P^{\mathrm{rel}}$.

Proof. Let $(\pi_n)_{n=1}^\infty$ be an optimizing sequence for the relaxed problem (7), i.e., a sequence of non-negative measures on $X \times Y$ such that

$$\lim_{n\to\infty} \int_{X\times Y} c(x,y)\,d\pi_n(x,y) = P^{\mathrm{rel}},$$

$$\lim_{n\to\infty} \|\pi_n\| = \lim_{n\to\infty} \int_{X\times Y} 1\,d\pi_n(x,y) = 1,$$

and such that $p_X(\pi_n) \le \mu$ and $p_Y(\pi_n) \le \nu$. In particular $p_X(\pi_n) = f_n\mu$ and $p_Y(\pi_n) = g_n\mu$ with $(f_n)_{n=1}^\infty$ (resp. $(g_n)_{n=1}^\infty$) converging to $\mathbb{1}$ in the norm of $L^1(\mu)$ (resp. $L^1(\nu)$). It follows that

$$\overline{\Phi}(\mathbb{1},\mathbb{1}) \le \lim_{n\to\infty} \Phi(f_n,g_n) = P^{\mathrm{rel}}.$$

To prove the reverse inequality $\overline{\Phi}(\mathbb{1},\mathbb{1}) \ge P^{\mathrm{rel}}$, fix $\delta > 0$. We have to show that for each $\varepsilon > 0$ there is some $\tilde\pi \in \Pi^\varepsilon(\mu,\nu)$ such that

$$\overline{\Phi}(\mathbb{1},\mathbb{1}) + \delta \ge \int c\,d\tilde\pi. \tag{16}$$

Pick $\gamma \in (0,1)$ such that $(1-\gamma)^3 \ge 1-\varepsilon$. Pick f, g and $\pi \in \Pi(f,g)$ such that $\|f - \mathbb{1}\|_{L^1(\mu)}, \|g - \mathbb{1}\|_{L^1(\nu)} < \gamma$ and $\overline{\Phi}(\mathbb{1},\mathbb{1}) + \delta \ge \int c\,d\pi$. We note for later use that $\|\pi\| = \|f\|_{L^1(\mu)} = \|g\|_{L^1(\nu)} \in (1-\gamma, 1+\gamma)$. Define the Borel measure $\tilde\pi \ll \pi$ on $X \times Y$ by

$$\frac{d\tilde\pi}{d\pi}(x,y) := \frac{1}{(1+|f(x)-1|)(1+|g(y)-1|)},$$

and set $\tilde\mu := p_X(\tilde\pi), \tilde\nu := p_Y(\tilde\pi)$. As $\frac{d\tilde\pi}{d\pi} \le 1$, we have $\tilde\pi \le \pi$ so that (16) is satisfied. Also $\tilde\mu \le \mu$ and $\tilde\nu \le \nu$. Thus it remains to check that $\|\tilde\pi\| \ge 1-\varepsilon$.

The function $F(a,b) = \frac{1}{(1+a)(1+b)}$ is convex on $[0,\infty)^2$ and by Jensen's inequality we have

$$\|\tilde\pi\| = \|\pi\| \int F(|f(x)-1|, |g(y)-1|)\,\tfrac{d\pi(x,y)}{\|\pi\|} \ge \tag{17}$$

$$\ge \|\pi\| F\left(\tfrac{\|f-\mathbb{1}\|_{L^1(\mu)}}{\|\pi\|}, \tfrac{\|g-\mathbb{1}\|_{L^1(\nu)}}{\|\pi\|}\right) = (1-\gamma)\tfrac{1}{(1+\gamma/(1-\gamma))^2} \ge 1-\varepsilon, \tag{18}$$

as required.

The final assertion of the proposition is now obvious. $\square$

Proof of Theorem 1.2. By the preceding proposition we have to show that

$$\overline{\Phi}(\mathbb{1},\mathbb{1}) = D,$$

where the dual value D of the optimal transport problem is defined in (4).

By Lemma 2.2 we know that there are sequences[6] $(\varphi_n,\psi_n)_{n=1}^{\infty} \in W^* = L^\infty(\mu)\times L^\infty(\nu)$ such that

$$\lim_{n\to\infty}\langle(\varphi_n,\psi_n),(\mathbb{1},\mathbb{1})\rangle = \lim_{n\to\infty}\left[\int_X \varphi_n\,d\mu + \int_Y \psi_n\,d\nu\right] = \overline{\Phi}(\mathbb{1},\mathbb{1}) \in [0,\infty],$$

and such that

$$\langle(\varphi_n,\psi_n),(f,g)\rangle = \langle\varphi_n,f\rangle + \langle\psi_n,g\rangle \le \Phi(f,g), \quad \text{for all } (f,g)\in V \tag{19}$$

We shall show that (19) implies that, for each fixed $n\in\mathbb{N}$ there are representants[7] $(\tilde{\varphi}_n,\tilde{\psi}_n)$ of (φ_n,ψ_n) such that

$$\tilde{\varphi}_n(x) + \tilde{\psi}_n(y) \le c(x,y) \tag{20}$$

for *all* $(x,y)\in X\times Y$. Indeed, choose any $\mathbb{R}$-valued representants $(\check{\varphi}_n,\check{\psi}_n)$ of (φ_n,ψ_n) and consider the set

$$C = \{(x,y)\in X\times Y : \check{\varphi}_n(x) + \check{\psi}_n(y) > c(x,y)\}. \tag{21}$$

Claim: For every $\pi\in\Pi(\mu,\nu)$ we have that $\pi(C)=0$.

Indeed, fix $\pi\in\Pi(\mu,\nu)$ and denote by (f,g) the density functions of the projections $p_X(\pi_{|C})$ and $p_Y(\pi_{|C})$. By (19) we have, for $n\ge 1$,

$$\int_{X\times Y} c\mathbb{1}_C\,d\pi \ge \Phi(f,g) \ge \langle\varphi_n,f\rangle + \langle\psi_n,g\rangle = \int_{X\times Y}(\check{\varphi}_n(x)+\check{\psi}_n(y))\mathbb{1}_C\,d\pi(x,y)$$

By the definition of D the first term above can only be greater than or equal to the last term if $\pi(C)=0$, which readily shows the above claim.

Now we are in a position to apply an innocent looking, but deep result due to H. Kellerer [Kel84, Lemma 1.8][8]: a Borel set $C = X\times Y$ satisfies $\pi(C)=0$, for each $\pi\in\Pi(\mu,\nu)$, if and only if there are Borel sets $M\subseteq X, N\subseteq Y$ with $\mu(M)=\nu(N)=0$ such that $C\subseteq (M\times Y)\cup(X\times N)$. Choosing such sets M and N for the set C in (21), define the representants $(\tilde{\varphi}_n,\tilde{\psi}_n)$ by $\tilde{\varphi}_n = \check{\varphi}_n\mathbb{1}_{X\setminus M} - \infty\mathbb{1}_M$ and $\tilde{\psi}_n = \check{\psi}_n\mathbb{1}_{Y\setminus N} - \infty\mathbb{1}_N$. We then have $\tilde{\varphi}_n(x)+\tilde{\psi}_n(y)\le c(x,y)$, for *every* $(x,y)\in X\times Y$. As

$$\lim_{n\to\infty}\int_X \tilde{\varphi}_n\,d\mu + \int_Y \tilde{\psi}_n\,d\nu = \overline{\Phi}(\mathbb{1},\mathbb{1}) = P^{\mathrm{rel}},$$

the proof is complete. □

After finishing the proof of Theorem 1.2 it is time to harvest some corollaries.

[6] The dual space V^* of the subspace V of $W = L^1(\mu)\times L^1(\nu)$ equals the quotient of the dual $L^\infty(\mu)\times L^\infty(\nu)$, modulo the annihilator of V, i.e. the one dimensional subspace formed by the $(\varphi,\psi)\in L^\infty(\mu)\times L^\infty(\nu)$ of the form $(\varphi,\psi)=(a,-a)$, for $a\in\mathbb{R}$.

[7] Strictly speaking, (φ_n,ψ_n) are elements of $L^\infty(\mu)\times L^\infty(\nu)$, i.e. *equivalence classes* of functions. The $[-\infty,\infty[$-valued Borel measurable functions $(\tilde{\varphi}_n,\tilde{\psi}_n)$ will be properly chosen representants of these equivalence classes.

[8] For the convenience of the reader and in order to keep the present paper self-contained, we provide in the appendix (Lemma A.1) a proof of Kellerer's lemma, which is not relying on duality arguments.

Assume first that the Borel measurable cost function $c : X \times Y \to [0, \infty]$ is $\mu \otimes \nu$-almost surely bounded by some constant[9] M. We then may estimate

$$P \leq P^\varepsilon + \varepsilon M.$$

Indeed, for $\varepsilon > 0$, every partial transport plan π^ε with marginals $\mu^\varepsilon \leq \mu, \nu^\varepsilon \leq \nu$ and mass $\|\pi^\varepsilon\| = 1 - \varepsilon$ may be completed to a full transport plan π by letting, e.g.,

$$\pi = \pi^\varepsilon + \varepsilon^{-1}(\mu - \mu^\varepsilon) \otimes (\nu - \nu^\varepsilon).$$

As $c \leq M$ we have $\int c\, d\pi \leq \int c\, d\pi^\varepsilon + \varepsilon M$. This yields the following corollary due to H. Kellerer [Kel84, Theorem 2.2].

Corollary 2.4. *Let X, Y be polish spaces equipped with Borel probability measures μ, ν, and let $c : X \times Y \to [0, \infty]$ be a Borel measurable cost function which is uniformly bounded. Then there is no duality gap, i.e. $P = D$.*

To establish duality in the setup of a lower semi-continuous cost function c, it suffices to note that in this setting also the cost functional Φ is lower semi-continuous:

Lemma 2.5. *[Vil09, Lemma 4.3] Let $c : X \times Y \to [0, \infty]$ be lower semi-continuous and assume that a sequence of measures π_n on $X \times Y$ converges to a transport plan $\pi \in \Pi(\mu, \nu)$ weakly, i.e. in the topology induced by the bounded continuous functions on $X \times Y$. Then*

$$\int c\, d\pi \leq \liminf_{n \to \infty} \int c\, d\pi_n.$$

Corollary 2.6. *[Kel84, Theorem 2.6] Let X, Y be polish spaces equipped with Borel probability measures μ, ν, and let $c : X \times Y \to [0, \infty]$ be a lower semi-continuous cost function. Then there is no duality gap, i.e. $P = D$.*

Proof. It follows from Prokhorov's theorem and Lemma 2.5 that the function $\Phi : V_+ \to [0, \infty]$ is lower semi-continuous with respect to the norm topology of V. $\square$

Finally we recall a result established in [BS09], generalizing Corollary 2.4. We remark that we do not know how to directly deduce it from Theorem 1.2.

Theorem 2.7. *[BS09, Theorem 1] Let X, Y be polish spaces equipped with Borel probability measures μ, ν, let $c : X \times Y \to [0, \infty]$ be a Borel measurable cost function such that $\mu \otimes \nu(\{(x, y) : c(x, y) = \infty\}) = 0$. Then there is no duality gap, i.e. $P = D$.*

3 Robustness and Dual Attainment

This section is devoted to the proofs of Proposition 1.6 and Theorem 1.7. In both cases, the argument is based on the following proposition.

Proposition 3.1. *Let X, Y be polish spaces equipped with Borel probability measures μ, ν, let $c : X \times Y \to [0, \infty]$ be a Borel measurable cost function and assume that there exists a finite transport plan $\pi \in \Pi(\mu, \nu)$. Let Z be a polish space and λ a Borel measure on Z such that $\lambda(Z) > 0$. Let $\tilde{c}$ be a viable extension of the cost function c (in the sense of Definition 1.5).*

Then there exist dual maximizers, i.e. Borel measurable functions $\tilde{\varphi} : (X \cup Z) \to [-\infty, \infty)$ and $\tilde{\psi} : (Y \cup Z) \to [-\infty, \infty)$ such that

$$\begin{aligned} J_{\tilde{c}}(\tilde{\varphi}, \tilde{\psi}) &= P_{\tilde{c}} \quad \text{and} \\ \tilde{\varphi}(x) + \tilde{\psi}(y) &\leq \tilde{c}(x, y), \text{ for all } (x, y) \in (X \cup Z) \times (Y \cup Z). \end{aligned}$$

[9] In fact, this argument works provided that $c(x, y) \leq f(x) + g(y)$ for integrable functions f, g.

We postpone the proof of Proposition 3.1 to the end of this section and draw some conclusions first.

Proof of Theorem 1.7. **(b.) implies (a.):** Fix a viable extension $\tilde{c}$ such that $P_{\tilde{c}} = P^{\mathrm{rel}}$ and apply Proposition 3.1 to find dual maximizers $\tilde{\varphi} : (X \cup Z) \to [-\infty, \infty)$ and $\tilde{\psi} : (Y \cup Z) \to [-\infty, \infty)$ such that $J_{\tilde{c}}(\tilde{\varphi}, \tilde{\psi}) = P_{\tilde{c}}$. Define $\hat{\varphi} := \tilde{\varphi}_{|X}$ and $\hat{\psi} := \tilde{\psi}_{|Y}$. We want to prove that $\hat{\varphi}$ and $\hat{\psi}$ are maximizers for the (original) dual problem. Since $P_{\tilde{c}} = P^{\mathrm{rel}} = D$ it suffices to show that $J_c(\hat{\varphi}, \hat{\psi}) \geq J_{\tilde{c}}(\tilde{\varphi}, \tilde{\psi})$. To see this, recall that we assume that $\pi \in \Pi(\mu, \nu)$ is a finite transport plan and consider

$$\tilde{\pi} := \pi + \tfrac{1}{\lambda(Z)}(\lambda \otimes \lambda) \in \Pi(\mu + \lambda, \nu + \lambda).$$

Since $\tilde{\varphi}(x) + \tilde{\psi}(y) \leq \tilde{c}(x, y) = 0$ for $(x, y) \in Z \times Z$ we obtain

$$\begin{aligned} J_c(\hat{\varphi}, \hat{\psi}) &= \int_{X \times Y} \hat{\varphi}(x) + \hat{\psi}(y)\, d\pi(x, y) = \int_{X \times Y} \tilde{\varphi}(x) + \tilde{\psi}(y)\, d\pi(x, y) \\ &\geq \int_{(X \cup Z) \times (Y \cup Z)} \tilde{\varphi}(x) + \tilde{\psi}(y)\, d\tilde{\pi}(x, y) = J_{\tilde{c}}(\tilde{\varphi}, \tilde{\psi}). \end{aligned} \tag{22}$$

(a.) implies (b.): Pick dual maximizers $\hat{\varphi}, \hat{\psi}$ such that $J_c(\hat{\varphi}, \hat{\psi}) = D = P^{\mathrm{rel}}$. We define $\tilde{c} : (X \cup Z) \times (Y \cup Z) \to [0, \infty]$ and $\tilde{\varphi} : X \cup Z \to [-\infty, \infty), \tilde{\psi} : Y \cup Z \to [-\infty, \infty)$

$$\tilde{c}(a, b) = \begin{cases} c(a, b) & \text{for } (a, b) \in X \times Y, \\ \max(\hat{\varphi}(a), 0) & \text{for } (a, b) \in X \times Z, \\ \max(\hat{\psi}(b), 0) & \text{for } (a, b) \in Z \times Y, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{\varphi}(a) := \begin{cases} \hat{\varphi}(a) & \text{for} \quad a \in X, \\ 0 & \text{for} \quad a \in Z, \end{cases} \qquad \text{and} \qquad \tilde{\psi}(b) := \begin{cases} \hat{\psi}(b) & \text{for} \quad b \in Y, \\ 0 & \text{for} \quad b \in Z. \end{cases}$$

Then $\tilde{\varphi}$ and $\tilde{\psi}$ satisfy $\tilde{\varphi}(a) + \tilde{\psi}(b) \leq \tilde{c}(a, b)$ and $J_{\tilde{c}}(\tilde{\varphi}, \tilde{\psi}) = J_c(\hat{\varphi}, \hat{\psi}) = P^{\mathrm{rel}}$, hence $P_{\tilde{c}} = P^{\mathrm{rel}}$. ☐

Proof of Proposition 1.6. Given a viable extension $\tilde{c}$, find by Proposition 3.1 dual maximizers $\tilde{\varphi}$ and $\tilde{\psi}$ such that $J_{\tilde{c}}(\tilde{\varphi}, \tilde{\psi}) = P_{\tilde{c}}$. Letting $\hat{\varphi} := \tilde{\varphi}_{|X}$ and $\hat{\psi} := \tilde{\psi}_{|Y}$ we have, using Theorem 1.2, that

$$P_{\tilde{c}} = J_{\tilde{c}}(\tilde{\varphi}, \tilde{\psi}) \leq J_c(\hat{\varphi}, \hat{\psi}) \leq D = P^{\mathrm{rel}}$$

where the first inequality follows from $\tilde{\varphi}(a) + \tilde{\psi}(b) \leq \tilde{c}(a, b) = 0$, for all $a, b \in Z$. To prove the second assertion, we observe first that there exists $(\varphi_n, \psi_n) \in \Psi(\mu, \nu)$ which are bounded and such that

$$\int_X \varphi_n\, d\mu + \int_Y \psi_n\, d\nu \to P^{\mathrm{rel}}. \tag{23}$$

Indeed given $(\varphi, \psi) \in \Psi(\mu, \nu)$ define $\varphi^{(k)} := \max(\min(\varphi, k), -k)$, and $\psi^{(k)} := \max(\min(\psi, k), -k)$ for each k, then

$$\int \varphi^{(k)}\, d\mu + \int \psi^{(k)}\, d\nu \quad \longrightarrow \quad \int \varphi\, d\mu + \int \psi\, d\nu.$$

Fix a maximizing (in the sense of (23)) sequence $(\varphi_n, \psi_n) \in \Psi(\mu, \nu)$ such that $|\varphi_n| \leq n$ and $|\psi_n| \leq n$. Define

$$\tilde{\varphi}_n(a) := \begin{cases} \varphi_n(a) & \text{for} \quad a \in X, \\ 0 & \text{for} \quad a \in Z, \end{cases} \qquad \text{and} \qquad \tilde{\psi}_n(b) := \begin{cases} \psi_n(b) & \text{for} \quad b \in Y, \\ 0 & \text{for} \quad b \in Z. \end{cases}$$

Then $\tilde{\varphi}_n(x) + \tilde{\psi}_n(y) \le \tilde{c}_n(x,y)$ for all $(x,y) \in (X \cup Z) \times (Y \cup Z)$ so that

$$\int \varphi_n \, d\mu + \int \psi_n \, d\nu = \int \tilde{\varphi}_n \, d(\mu + \lambda) + \int \tilde{\psi}_n \, d(\nu + \lambda) \le P_{\tilde{c}_n}.$$

As the left side tends to P^{rel} and the right side is bounded by P^{rel}, the right side tends to P^{rel} too. $\square$

The proof of Proposition 3.1 is based on [BS09, Proposition 3.2] which we recast below.

Proposition 3.2. *Let X, Y be polish spaces equipped with Borel probability measures μ, ν. Let $c : X \times Y \to [0, \infty]$ be Borel measurable and assume that π is a finite transport plan. Then there exists a Borel measurable function $f : X \times Y \to [0, \infty]$ such that $\int f \, d\pi = \langle c, \pi \rangle - P$ and, for all $(x_1, y_1), \dots, (x_n, y_n) \in X \times Y$,*

$$\sum_{i=1}^{n} c(x_{i+1}, y_i) + f(x_i, y_i) - c(x_i, y_i) \ge 0.^{10} \tag{24}$$

We also cite here a simple result ([BS09, Lemma 3.7]) which is useful to deal with certain measurability issues:

Lemma 3.3. *Let X, Z be polish spaces, μ a finite Borel measure on X and $g : X \times Z \to [-\infty, \infty]$ a Borel measurable function. Assume that $\varphi : X \to [-\infty, \infty)$ is defined by*

$$\varphi(x) = \inf_{z \in Z} g(x, z). \tag{25}$$

Then there exists a Borel measurable function $\bar{\varphi} : X \to [-\infty, \infty]$ and a μ-null set N such that $\bar{\varphi} = \varphi$ holds on $X \setminus N$ and $\bar{\varphi}(x) \le \varphi(x)$ for all $x \in X$.

Proof of Proposition 3.1. For notational convenience we set $X_1 = X \cup Z, Y_1 = Y \cup Z, \mu_1 := \mu + \lambda$ and $\nu_1 := \nu + \lambda$. Pick a finite transport plan $\pi \in \Pi(\mu, \nu)$ and consider the associated transport plan $\pi_1 := \pi + \frac{1}{\lambda(Z)} \lambda \otimes \lambda \in \Pi(\mu_1, \nu_1)$.

It is sufficient to define $\tilde{\varphi}$ and $\tilde{\psi}$ on Borel sets $X_1' \subseteq X_1$ and $Y_1' \subseteq Y_1$ which have full measure since they can then be extended to X_1 and Y_1 by setting them $-\infty$ on the null-sets $X_1 \setminus X_1', Y_1 \setminus Y_1'$. We will use this several times subsequently.

By Proposition 3.2 there is a Borel measurable function $f : X_1 \times Y_1 \to [0, \infty]$ satisfying (24) such that $\int f \, d\pi_1 = \langle \tilde{c}, \pi_1 \rangle - P_{\tilde{c}}$. Then

$$\Gamma := \{(x, y) \in X_1 \times Y_1 : \tilde{c}(x, y) < \infty, f(x, y) < \infty\}$$

has full measure with respect to π_1. It follows that, after shrinking X_1 by a null set if necessary, we may assume that $p_{X_1}[\Gamma] = X_1$ and that $p_{X_1}[\Gamma \cap (Z \times Z)] = Z$. Take $x_1 \in Z$ and define[11]

$$\begin{aligned} \tilde{\varphi}_n(x) &= \inf \Big\{ \sum_{i=1}^{n} \tilde{c}(x_{i+1}, y_i) + f(x_i, y_i) - \tilde{c}(x_i, y_i) : x_{n+1} = x \Big\}, \\ \tilde{\varphi}(x) &= \inf_{n \ge 1} \tilde{\varphi}_n(x) \end{aligned}$$

where the infimum in the definition of $\tilde{\varphi}_n$ runs through all $\{(x_i, y_i); 1 \le i \le n\} \subset \Gamma$ with x_1 fixed.

Consider $x \in X_1$. To see that $\tilde{\varphi}(x) < \infty$, choose $y_1 \in Z$ such that $(x, y_1) \in \Gamma$. Then

$$\tilde{\varphi}(x) \le \tilde{\varphi}_1(x) \le \tilde{c}(x, y_1) + f(x_1, y_1) - \tilde{c}(x_1, y_1) < \infty.$$

[10] Here we use the conventions $x_{n+1} := x_1$ and $\infty - \infty = \infty$.

[11] This argument is similar to the construction given in [Rüs96], see also [AP03, Theorem 3.2] and [Vil09, Chapter 2]

Next we prove that also $\tilde{\varphi}(x) > -\infty$. Pick $y \in Y_1$ such that $(x_1, y) \in \Gamma$. Given $n \geq 1$ and $x_2, \dots, x_n \in X_1, y_1, \dots, y_n \in Y_1$ set $x_{n+1} = x, y_{n+1} = y, x_{n+2} = x_1$. Then

$$\sum_{i=1}^{n+1} \tilde{c}(x_{i+1}, y_i) + f(x_i, y_i) - \tilde{c}(x_i, y_i) \geq 0,$$

by the virtue of (24). We may rewrite this as

$$\sum_{i=1}^{n} \tilde{c}(x_{i+1}, y_i) + f(x_i, y_i) - \tilde{c}(x_i, y_i) \geq -\big[\tilde{c}(x_1, y) + f(x, y) - \tilde{c}(x, y)\big].$$

Taking the infimum over all possible choices of $n \geq 1$ and $x_2, \dots, x_n \in X_1, y_1, \dots, y_n \in Y_1$, we achieve that $\tilde{\varphi}(x) \geq -\big[\tilde{c}(x_1, y) + f(x, y) - \tilde{c}(x, y)\big] > -\infty$.

Fix $x, x' \in X_1, y \in Y_1$ and $n \geq 1$. Then

$$\begin{aligned} \tilde{\varphi}_{n+1}(x) &\leq \inf \Big\{ \sum_{i=1}^{n+1} \tilde{c}(x_{i+1}, y_i) + f(x_i, y_i) - \tilde{c}(x_i, y_i) : x_{n+2} = x, x_{n+1} = x', y_{n+1} = y \Big\} \\ &= \tilde{\varphi}_n(x') + \big[\tilde{c}(x, y) + f(x', y) - \tilde{c}(x', y)\big] \end{aligned}$$

Taking the infimum over all $n \geq 1$ we obtain

$$\inf_{n \geq 1} \tilde{\varphi}_{n+1}(x) \leq \tilde{\varphi}(x') + \big[\tilde{c}(x, y) + f(x', y) - \tilde{c}(x', y)\big].$$

Trivially $\tilde{\varphi} \leq \inf_{n\geq 1} \tilde{\varphi}_{n+1}$, thus we have shown

$$-f(x', y) + \tilde{c}(x', y) - \tilde{\varphi}(x') \leq \tilde{c}(x, y) - \tilde{\varphi}(x) \tag{26}$$

for all $x, x' \in X_1$ and $y \in Y_1$.

At this point we take care about measurability of $\tilde{\varphi}$. Apply Lemma 3.3 to the spaces X_1 and $X^\omega = \bigcup_{n=1}^\infty (Y_1 \times X_1)^n$ to achieve that there exists a Borel measurable function $\tilde{\tilde{\varphi}} : X_1 \to [-\infty, \infty]$ which is equal to $\tilde{\varphi}$ on a Borel subset of X_1 of full μ_1-measure. Shrinking X_1 by a null-set, we thus may assume that $\tilde{\varphi}$ is Borel measurable.

For $x \in Z, y \in Y_1$ we have $\tilde{c}(x, y) < \infty$, hence

$$\tilde{\psi}(y) := \inf_{x \in X_1} \tilde{c}(x, y) - \tilde{\varphi}(x) < \infty. \tag{27}$$

As above, Lemma 3.3 implies that we may assume that $\tilde{\psi}$ is Borel measurable after shrinking Y_1 by a null-set if necessary. By (26) and (27),

$$-f(x', y) + \tilde{c}(x', y) \leq \tilde{\varphi}(x') + \tilde{\psi}(y) \tag{28}$$

holds for all $x' \in X_1, y \in Y_1$. Integrating against π_1 we obtain

$$-(\langle \tilde{c}, \pi \rangle - \tilde{P}) + \langle \tilde{c}, \pi \rangle \leq J(\tilde{\varphi}, \tilde{\psi}),$$

i.e. $\tilde{P} \leq J(\tilde{\varphi}, \tilde{\psi})$. By definition of $\tilde{\psi}$ we have $\tilde{\varphi}(x) + \tilde{\psi}(y) \leq \tilde{c}(x, y)$ for all $(x, y) \in X_1 \times Y_1$, thus $\tilde{\varphi}$ and $\tilde{\psi}$ are dual maximizers. $\square$

We point out that Proposition 3.1 yields an alternative proof of our main result under the assumption that there exists some finite transport plan in $\Pi(\mu, \nu)$.

Alternative proof of Theorem 1.2. For fixed $\varepsilon > 0$, let Z, λ and $\tilde{c}$ be as in Lemma 1.4. By Proposition 3.1 there exist dual maximizers $\tilde{\varphi}$, $\tilde{\psi}$ such that $J_{\tilde{c}}(\tilde{\varphi}, \tilde{\psi}) = P_{\tilde{c}} = \tilde{P}^\varepsilon = P^\varepsilon$. But then $\hat{\varphi} := \tilde{\varphi}_{|X}$ and $\hat{\psi} := \tilde{\psi}_{|Y}$ witness that $D \geq P_{\tilde{c}} = P^\varepsilon$; for $\varepsilon \to 0$ we obtain $D \geq \lim_{\varepsilon \to 0} P^\varepsilon = P^{\mathrm{rel}}$. □

We close this section with some comments concerning a possible relaxed version of the dual problem.

Remark 3.4. Define

$$P^{\mathrm{rel}} = \sup \left\{ \int \varphi \, d\mu + \int \psi \, d\nu : \begin{array}{l} \varphi, \psi \text{ integrable,} \\ \varphi(x) + \psi(y) \leq c(x,y) \ \pi\text{-a.e.} \\ \text{for every finite cost } \pi \in \Pi(\mu, \nu) \end{array} \right\} \tag{29}$$

where $\pi \in \Pi(\mu, \nu)$ has finite cost if $\int_{X \times Y} c \, d\pi < \infty$. It is straightforward to verify that we still have $P^{\mathrm{rel}} \leq P$. One might conjecture (and the present authors did so for some time) that, similarly to the situation in Theorem 1.2, duality in the form $P^{\mathrm{rel}} = P$ holds without any additional assumption. For instance this is the case in Example 1.1 and combining the methods of [BGMS09] and [BS09] one may prove that $P^{\mathrm{rel}} = P$ provided that the Borel measurable cost function $c : X \times Y \to [0, \infty]$ satisfies that the set $\{c = \infty\}$ is *closed* in the product topology of $X \times Y$. However an example constructed in [BLS09, Section 4] shows that under the assumptions of Theorem 1.2 it may happen that P^{rel} is strictly smaller then P, i.e. that there still is a duality gap.

4 The Existence of the Dual Optimizer

The aim of this section is to discuss the existence of an optimizer in the dual optimization problem (4), without imposing any further conditions on the Borel measurable cost function $c : X \times Y \to [0, \infty]$. To develop a feeling for what we are after, we consider a specific example.

Example 4.1. [BS09, Example 4.3] Let $X = Y = [0, 1)$, equipped with Lebesgue measure $\lambda = \mu = \nu$. Pick $\alpha \in [0, 1)$ irrational. Set

$$\Gamma_0 = \{(x, x) : x \in X\} \quad \Gamma_1 = \{(x, x \oplus \alpha) : x \in X\},$$

where $\oplus$ is addition modulo 1. Define $c : X \times Y \to [0, \infty]$ by

$$c(x,y) = \begin{cases} 1 & \text{for } (x,y) \in \Gamma_0 \\ 2 & \text{for } (x,y) \in \Gamma_1, x \in [0, 1/2) \\ 0 & \text{for } (x,y) \in \Gamma_1, x \in [1/2, 1) \\ \infty & \text{else} \end{cases}.$$

For $i = 0, 1$, let π_i be the obvious transport plan supported by Γ_i. Following the arguments of [AP03], it is easy to see that all finite transport plans are given by convex combinations of the form $\rho\pi_0 + (1 - \rho)\pi_1, \rho \in [0, 1]$ and each of these transport plans leads to costs of 1.

Note that c is lower semi-continuous provided we refine the topology on $[0, 1)$ by adding the open sets $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$. Hence duality holds true (Corollary 2.6). Thus, for each $\varepsilon > 0$, there are integrable functions $\varphi, \psi : [0, 1) \to [-\infty, \infty)$ such that $\varphi(x) + \psi(y) \leq c(x,y)$ for all $(x, y) \in X \times Y$ and $\int (c(x, y) - (\varphi(x) + \psi(y))) \, d\pi_i(x, y) \leq \varepsilon$ for $i = 0, 1$.[12]

On the other hand, it is shown in [BS09] that there do not exist measurable functions $\varphi, \psi : [0, 1) \to [-\infty, \infty)$ satisfying $\varphi(x) + \psi(y) \leq c(x, y)$ for all $(x, y) \in X \times Y$ such that $\varphi(x) + \psi(y) = c(x, y)$ holds π_0- as well as π_1-almost surely.

[12] In the accompanying paper [BLS09] such functions φ, ψ are constructed explicitly and rather elaborate extensions of the above example are analyzed.

Let us have a closer look at the previous example: while it is *not possible* to find Borel measurable limits $\hat{\varphi}, \hat{\psi}$ of an optimizing sequence $(\varphi_n, \psi_n)_{n=1}^{\infty}$, it *is possible* to find a limiting Borel function $\hat{h}(x, y)$ of the sequence of functions $(\varphi_n(x) + \psi_n(y))_{n=1}^{\infty}$ on the set $\{(x, y) \in X \times Y : c(x, y) < \infty\}$. Indeed, on this set, which simply equals $\Gamma_0 \cup \Gamma_1$, any optimizing sequence $(\varphi_n(x) + \psi_n(y))_{n=1}^{\infty}$ for (4) has a subsequence which converges π-a.s. to $\hat{h}(x, y) := c(x, y)$, for any finite cost transport plan π.

Summing up: in the context of the previous example, there is a Borel function $\hat{h}(x, y)$ on $X \times Y$, which equals $c(x, y)$ on $\Gamma_0 \cup \Gamma_1$; it may take any value on $(X \times Y) \setminus (\Gamma_0 \cup \Gamma_1)$, e.g. the value $+\infty$. This function $\hat{h}(x, y)$ may be considered as a kind of dual optimizer: it is, for any finite cost transport plan π, the limit of an optimizing sequence $(\varphi_n(x) + \psi_n(y))_{n=1}^{\infty}$ with respect to the norm $\| \cdot \|_{L^1(\pi)}$.

The remainder of this section and Section 5 are devoted to developing a theory which makes this idea precise in the general setting of Borel measurable cost functions $c : X \times Y \to [0, \infty]$. To do so we shall again apply Fenchel's perturbation method. In addition, we need some functional analytic machinery, in particular we shall use the space $(L^1)^{**} = (L^\infty)^*$ of finitely additive measures.

Denote by $\Pi(\mu, \nu, c)$ the set of finite cost transport plans

$$\Pi(\mu, \nu, c) := \left\{ \pi \in \Pi(\mu, \nu) : \int_{X \times Y} c \, d\pi < \infty \right\},$$

and assume $\Pi(\mu, \nu, c) \neq \emptyset$ to avoid the trivial case.

We fix $\pi_0 \in \Pi(\mu, \nu, c)$ and stress that we do *not* assume that π_0 has minimal transport cost. In fact, there is little reason in the present setting (where c is not assumed to be lower semi-continuous) why a primal optimizer $\widehat{\pi}$ should exist. We denote by $\Pi^{(\pi_0)}(\mu, \nu)$ the set of elements $\pi \in \Pi(\mu, \nu)$ such that $\pi \ll \pi_0$ and $\left\| \frac{d\pi}{d\pi_0} \right\|_{L^\infty(\pi_0)} < \infty$. Note that $\Pi^{(\pi_0)}(\mu, \nu) = \Pi(\mu, \nu) \cap L^\infty(\pi_0) \subseteq \Pi(\mu, \nu, c)$.

We shall replace the usual Kantorovich optimization problem over the set $\Pi(\mu, \nu, c)$ by the optimization over the smaller set $\Pi^{(\pi_0)}(\mu, \nu)$.

$$P^{(\pi_0)} = \inf\{\langle c, \pi \rangle = \textstyle\int c \, d\pi : \pi \in \Pi^{(\pi_0)}(\mu, \nu)\}. \tag{30}$$

As regards the dual problem, we define, for $\varepsilon > 0$,

$$\begin{aligned} D^{(\pi_0, \varepsilon)} = \sup \Big\{ \int \varphi \, d\mu + \int \psi \, d\nu : \varphi \in L^1(\mu), \psi \in L^1(\nu), \\ \int_{X \times Y} (\varphi(x) + \psi(y) - c(x, y))_+ \, d\pi_0 \leq \varepsilon \Big\} \quad \text{and} \\ D^{(\pi_0)} = \lim_{\varepsilon \to 0} D^{(\pi_0, \varepsilon)}. \end{aligned} \tag{31}$$

Define the "summing" map S by

$$\begin{aligned} S : L^1(X, \mu) \times L^1(Y, \nu) &\to L^1(X \times Y, \pi_0) \\ (\varphi, \psi) &\mapsto \varphi \oplus \psi, \end{aligned}$$

where $\varphi \oplus \psi$ denotes the function $\varphi(x) + \psi(y)$ on $X \times Y$. Denote by $L^1_S(X \times Y, \pi_0)$ the $\|.\|_1$-closed linear subspace of $L^1(X \times Y, \pi_0)$ spanned by $S(L^1(X, \mu) \times L^1(Y, \nu))$. Clearly $L^1_S(X \times Y, \pi_0)$ is a Banach space under the norm $\|.\|_1$ induced by $L^1(X \times Y, \pi_0)$.

We shall also need the bi-dual $L^1_S(X \times Y, \pi_0)^{**}$ which may be identified with a subspace of $L^1(X \times Y, \pi_0)^{**}$. In particular, an element $h \in L^1_S(X \times Y, \pi_0)^{**}$ can be decomposed into $h = h^r + h^s$, where $h^r \in L^1(X \times Y, \pi_0)$ is the regular part of the finitely additive measure h and h^s its purely singular part.

Theorem 4.2. *Let $c : X \times Y \to [0, \infty]$ be Borel measurable, and let $\pi_0 \in \Pi(\mu, \nu, c)$ be a finite transport plan. We have*

$$P^{(\pi_0)} = D^{(\pi_0)}. \tag{32}$$

There is an element $\hat{h} \in L^1_S(X \times Y, \pi_0)^{**}$ such that $\hat{h} \leq c$[13] *and*

$$D^{(\pi_0)} = \langle \hat{h}, \pi_0 \rangle.$$

If $\pi \in \Pi^{(\pi_0)}(\mu, \nu)$ (identifying π with $\frac{d\pi}{d\pi_0}$) satisfies $\int c \, d\pi \leq P^{(\pi_0)} + \alpha$ for some $\alpha \geq 0$, then

$$|\langle \hat{h}^s, \pi \rangle| \leq \alpha. \tag{33}$$

In particular, if π is an optimizer of (30), then $\hat{h}^s$ vanishes on the set $\{\frac{d\pi}{d\pi_0} > 0\}$.

In addition, we may find a sequence of elements $(\varphi_n, \psi_n) \in L^1(\mu) \times L^1(\nu)$ such that

$$\varphi_n(x) + \psi_n(y) \to \hat{h}^r(x, y), \ \pi_0\text{-a.s.}, \ \|(\varphi_n(x) + \psi_n(y) - \hat{h}^r(x, y))_+\|_{L_1(\pi_0)} \to 0 \tag{34}$$

and

$$\lim_{\delta \to 0} \sup_{A \subseteq X \times Y, \pi_0(A) < \delta} \lim_{n \to \infty} -\langle (\varphi_n \oplus \psi_n) \mathbb{1}_A, \pi_0 \rangle = \|\hat{h}^s\|_{L_1(\pi_0)^{**}}. \tag{35}$$

Proof. It is straightforward to verify the trivial duality relation $D^{(\pi_0)} \leq P^{(\pi_0)}$. To show the reverse inequality and to find the dual optimizer $\hat{h} \in L^1(X \times Y, \pi_0)^{**}$, as in the proof of Theorem 1.2, we apply W. Fenchel's perturbation argument. (For an elementary treatment, compare also [BLS09].) The summing map S factors through $L^1_S(\pi_0)$ as indicated in the subsequent diagram:

$$\begin{tikzcd}
L^1(\mu) \times L^1(\nu) \arrow[rr, "S"] \arrow[dr, "S_1"'] & & L^1(\pi_0) \\
& L^1_S(\pi_0) \arrow[ur, "S_2"'] &
\end{tikzcd}$$

Then S_1 has dense range and S_2 is an isometric embedding. Denote by $\left(L^1_S(\pi_0)^*, \|.\|_{L^1_S(\pi_0)^*}\right)$ the dual of $L^1_S(\pi_0)$ which is a quotient space of $L^\infty(\pi_0)$. Transposing the above diagram we get

$$\begin{tikzcd}
L^\infty(\mu) \times L^\infty(\nu) & & L^\infty(\pi_0) \arrow[ll, "T"'] \arrow[dl, "T_2"] \\
& L^1_S(\pi_0)^* \arrow[ul, "T_1"] &
\end{tikzcd}$$

where T, T_1, T_2 are the transposed maps of S, S_1, resp. S_2. Clearly $T(\pi) = (p_X(\pi), p_Y(\pi))$ for $\pi \in L^\infty(\pi_0)$, where p_X, p_Y are the projections of a measure π (identified with the Radon-Nikodym-derivative $\frac{d\pi}{d\pi_0}$) onto its marginals. By elementary duality relations we have that T_2 is a quotient map and T_1 is injective; the latter fact allows us to identify the space $L^1_S(\pi_0)^*$ with a subspace of $L^\infty(\mu) \times L^\infty(\nu)$.
For example, consider the element $\mathbb{1} \in L^\infty(\pi_0)$, which corresponds to the measure π_0 on $X \times Y$. The element $T_2(\mathbb{1}) \in L^1_S(\pi_0)^*$ may then be identified with the element $(\mathbb{1}, \mathbb{1}) = T(\mathbb{1})$

[13]The inequality $\hat{h} \leq c$ pertains to the lattice order of $L^1(X \times Y)^{**}$, where we identify the π_0-integrable function c with an element of $L^1(X \times Y, \pi_0)^{**}$. If $\hat{h}$ decomposes into $\hat{h} = \hat{h}^r + \hat{h}^s$, the inequality $\hat{h} \leq c$ holds true iff $\hat{h}^r(x, y) \leq c(x, y)$, π_0-a.s. and $\hat{h}^s \leq 0$ (compare the discussion after (39))

in $L^\infty(\mu) \times L^\infty(\nu)$ which corresponds to the pair (μ, ν). We take the liberty to henceforth denote this element simply by $\mathbb{1}$, independently of whether we consider it as an element of $L^\infty(\pi_0)$, $L^1_S(\pi_0)^*$ or $L^\infty(\mu) \times L^\infty(\nu)$.

We may now rephrase the primal problem (30) as

$$\langle c, \pi \rangle = \int_{X \times Y} c(x,y)\, d\pi(x,y) \to \min, \quad \pi \in L^\infty_+(\pi_0),$$

under the constraint

$$T(\pi) = \mathbb{1}. \tag{36}$$

The decisive trick is to replace (36) by the, trivially equivalent, constraint

$$T_2(\pi) = \mathbb{1},$$

and to perform the Fenchel perturbation argument *not* in the space $L^\infty(\mu) \times L^\infty(\nu)$ but rather in the subspace $L^1_S(\pi_0)^*$ which is endowed with a *stronger norm*. The map Φ: $L^1_S(\pi_0)^* \to [0, \infty]$,

$$\Phi(p) := \inf\{\langle c, \pi\rangle : \pi \in L^\infty_+(\pi_0), T_2(\pi) = p\}, \quad p \in L^1_S(\pi_0)^*,$$

is convex, positively homogeneous and $\Phi(\mathbb{1}) = I_c(\pi_0)$.

Claim: There is a neighbourhood V of $\mathbb{1}$ in $L^1_S(\pi_0)^*$ on which Φ is bounded.

Indeed, let $U = \{\pi \in L^\infty(\pi_0) \mid \|\pi - \mathbb{1}\|_{L^\infty(\pi_0)} < \frac{1}{2}\}$. Then U is contained in the positive orthant $L^\infty_+(\pi_0)$ of $L^\infty(\pi_0)$ and

$$\Phi(T_2(\pi)) \leq \langle c, \pi \rangle \leq \tfrac{3}{2}\|c\|_{L^1(\pi_0)} \text{ for all } \pi \in U.$$

Hence on $T_2(U)$, which simply is the ball of radius $\frac{1}{2}$ around $\mathbb{1}$ in the Banach space $L^1_S(\pi_0)^*$, we have that Φ is bounded by $\frac{3}{2}\|c\|_{L^1(\pi_0)}$.

It follows from elementary geometric facts that the convex function Φ is continuous on $T_2(U)$ with respect to the norm of $L^1_S(\pi_0)^*$. By Hahn-Banach there exists $f \in L^1_S(\pi_0)^{**}$ such that

$$\begin{aligned} &\langle f, \mathbb{1}\rangle = \Phi(\mathbb{1}), \\ &\langle f, p \rangle \leq \Phi(p) \text{ for all } p \in L^1_S(\pi_0)^*. \end{aligned}$$

The adjoint T_2^* of T_2 maps $L^1_S(\pi_0)^{**}$ isometrically onto a subspace E of $L^1(\pi_0)^{**} = L^\infty(\pi_0)^*$. The space E consists of those elements of $L^1(\pi_0)^{**}$ which are σ^*-limits of nets $(\varphi_\alpha \oplus \psi_\alpha)_{\alpha \in I}$ with $\varphi_\alpha \in L^1(\mu)$, $\psi_\alpha \in L^1(\nu)$. Write $\hat{h} := T_2^*(f)$. Then for all $\pi \in L^\infty_+(\pi_0)$,

$$\langle \hat{h}, \pi \rangle = \langle T_2^*(f), \pi \rangle = \langle f, T_2(\pi)\rangle \;\leq\; \Phi(T_2(\pi)) \;\leq\; \langle c, \pi \rangle, \tag{37}$$

and if $\pi \in L^\infty_+(\pi_0), T_2(\pi) = \mathbb{1}$ then

$$\langle \hat{h}, \pi \rangle = \langle T_2^*(f), \pi \rangle = \langle f, T_2(\pi)\rangle = \langle f, \mathbb{1}\rangle = \Phi(\mathbb{1}) = P^{(\pi_0)}. \tag{38}$$

By (37), the inequality $\hat{h} \leq c$ holds true in the Banach-lattice $L^\infty(\pi_0)^*$. Combining this with (38) we obtain that $\hat{h}$ is a dual optimizer in the sense of

$$\begin{aligned} D^{(\pi_0)}_{**} := \sup\big\{\langle g, \pi_0\rangle : g \in L^1_S(\pi_0)^{**},\, g \leq c \\ \text{in the Banach lattice } L^1(\pi_0)^{**}\big\} \end{aligned} \tag{39}$$

(where we identify π_0 with the element $\mathbb{1}$ of $L^\infty(\pi_0)$) and that there is no duality gap in this sense, i.e. $D^{(\pi_0)}_{**} = P^{(\pi_0)}$.

As mentioned above, every element $g \in L^\infty(\pi_0)^*$ splits in a regular part g^r lying in $L^1(\pi_0)$ and a purely singular part g^s. Given $g_1, g_2 \in L^\infty(\pi_0)^*$, we have $g_1 \leq g_2$ if and only if $g_1^r \leq g_2^r$ and $g_1^s \leq g_2^s$. Since $c \in L^1(\pi_0)$ we have $c^s = 0$. The inequality $\hat{h} \leq c$ implies that $\hat{h}^s \leq c^s = 0$ and $\hat{h}^r \leq c^r = c$. It follows that

$$\langle \hat{h}^r, \pi\rangle \leq \langle c, \pi\rangle \text{ for each } \pi \in L^\infty_+(\pi_0). \tag{40}$$

Assume additionally that π satisfies $T_2(\pi) = \mathbb{1}$ and choose $\alpha \geq 0$ such that $\langle c, \pi\rangle \leq P^{(\pi_0)} + \alpha$. Then $\langle \hat{h}, \pi\rangle = P^{(\pi_0)}$ and subtracting this quantity from (40) we get

$$\langle -\hat{h}^s, \pi\rangle = \langle \hat{h}^r - \hat{h}, \pi\rangle \leq \langle c, \pi\rangle - P^{(\pi_0)} \leq \alpha$$

showing (33).

We still have to show the existence of a *sequence* $(\varphi_n, \psi_n)_{n=1}^\infty$ satisfying the above assertions about convergence. So far we know that there is a net $(\varphi_\alpha, \psi_\alpha)_{\alpha \in I}$ such that $\varphi_\alpha \oplus \psi_\alpha$ weak-star converges to $\hat{h}$. First we claim that there exists a net $(f_\alpha)_{\alpha\in I}$ of elements of $L^1(\pi_0)$, such that $\|f_\alpha\|_1 \leq \|\hat{h}^s\|$, $\hat{h}^r + f_\alpha \in L^1_S(\pi_0)$ and $\hat{h}^r + f_\alpha \to \hat{h}$ in the σ^*-topology. To see this, note that Alaoglu's theorem [RS80, Theorem IV.21] implies that in a Banach space V, the unit ball $B_1(V)$ is σ^*-dense in the unit ball $B_1(V^{**})$ of the bidual. Thus $\hat{h}^r + \|\hat{h}^s\| B_1(L^1_S(\pi_0))$ is σ^*-dense in $\hat{h}^r + \|\hat{h}^s\| B_1(L^1_S(\pi_0)^{**})$ which yields the existence of a net $(f_\alpha)_{\alpha\in I}$ as required.

As $\hat{h}^s$ is purely singular, we may find a sequence $(\alpha_n)_{n=1}^\infty$ in I such that $\|f_{\alpha_n}\| \leq \|\hat{h}^s\|$ and $\int f_{\alpha_n}\, d\pi_0 = -\|\hat{h}^s\| + 2^{-n}$, and that $\int (|f_{\alpha_n}| \wedge 2^n)\, d\pi_0 \leq 2^{-n}$, which implies that the sequence $(f_{\alpha_n})_{n=1}^\infty$ converges π_0-a.s. to zero.

As $\hat{h}^r + f_{\alpha_n} \in L^1_S(\pi_0)$ we may find $(\varphi_n, \psi_n) \in L^1(\mu) \times L^1(\nu)$ such that

$$\|\varphi_n \oplus \psi_n - (\hat{h}^r + f_{\alpha_n})\|_{L^1(\pi_0)} < 2^{-n}.$$

We then have that $(\varphi_n(x) + \psi_n(y))_{n=1}^\infty$ converges π_0-a.s. to $\hat{h}^r(x,y)$ and that $\|(\varphi_n \oplus \psi_n - \hat{h}^r)_+\|_{L^1(\pi_0)} \to 0$.

As regards assertion (35) we note that, for $A_m = \bigcup_{n=m+1}^\infty \{|f_{\alpha_n}| > 2^{-n}\}$ we have $\pi_0(A_m) \leq 2^{-m}$ and

$$\begin{aligned}
\liminf_{n\to\infty}(-\langle(\varphi_n \oplus \psi_n)\mathbb{1}_{A_m}, \pi_0\rangle) &= -\limsup_{n\to\infty}\langle(\hat{h}^r + f_{\alpha_n})\mathbb{1}_{A_m}, \pi_0\rangle \\
&= -\langle \hat{h}^r \mathbb{1}_{A_m}, \pi_0\rangle - \lim_{n\to\infty}\langle f_{\alpha_n}\mathbb{1}_{A_m}, \pi_0\rangle \\
&= -\langle \hat{h}^r \mathbb{1}_{A_m}, \pi_0\rangle + \|\hat{h}^s\|_{L^1(\pi_0)}**.
\end{aligned}$$

Letting m tend to infinity we obtain that the left hand side of (35) is greater than or equal to the right hand side. As regards the reverse inequality it suffices to note that $\|f_{\alpha_n}\|_{L^1(\pi_0)} \leq \|\hat{h}^s\|_{L^1(\pi_0)^{**}}$.

As $\hat{h}^r \leq c$, π_0-a.s., we obtain in particular that $\|(\varphi_n \oplus \psi_n - c)_+\|_{L^1(\pi_0)} \to 0$ showing that $D^{(\pi_0)} \geq P^{(\pi_0)}$ and therefore (32), the reverse inequality being straightforward. ☐

As a by-product of this proof, we have shown at (39) that

$$D^{(\pi_0)}_{**} = D^{(\pi_0)} = P^{(\pi_0)}.$$

Admittedly, Theorem 4.2 is rather abstract. However, we believe that it may be useful in applications to have the possibility to pass to *some kind of limit* $\hat{h}$ of an optimizing sequence

$(\varphi_n, \psi_n)_{n=1}^{\infty}$ in the dual optimization problem, even if this limit is somewhat awkward. To develop some intuition for the message of Theorem 4.2, we shall illustrate the situation at the hand of some examples.

Let us start with Example 4.1. In this case we may apply Theorem 4.2 to the finite transport plan $\pi_{\frac{1}{2}} = \frac{1}{2}(\pi_0 + \pi_1)$, (we apologize for using $\pi_{\frac{1}{2}}$ instead of π_0 in Theorem 4.2 as the notation π_0 is already taken). As we have seen above, there are sequences $(\varphi_n \oplus \psi_n)_{n=1}^{\infty}$ converging $\pi_{\frac{1}{2}}$-a.s. as well as in the norm of $L^1(\pi_{\frac{1}{2}})$ to $\hat{h} = c$, as defined in Example 4.1 above. In particular we do not have to bother about the singular part $\hat{h}^s$ of $\hat{h}$, as we have $\hat{h} = \hat{h}^r$ in this example. We find again that h represents the limit of $(\varphi_n \oplus \psi_n)_{n=1}^{\infty}$, considered as a Borel function on $\{c < \infty\}$ which is the support of $\pi_{\frac{1}{2}}$.

We now make the example a bit more interesting and challenging. (See Example 4.3 below.)

Fix in the context of Example 4.1 (where we now write $\tilde{c}$ instead of c to keep the letter c free for a new function to be constructed) a sequence $(\varphi_n, \psi_n)_{n=1}^{\infty}$ such that $\|\tilde{c} - \varphi_n \oplus \psi_n\|_{L^1(\pi_i)} \to 0$ for $i = 0, 1$. We claim that $(\varphi_n \oplus \psi_n)_{n=1}^{\infty}$ converges in $\|.\|_{L^1(\pi_k)}$ where, for each $k \in \mathbb{N}$, π_k is the measure which is uniformly distributed on

$$\Gamma_k = \{(x, x \oplus k\alpha) : x \in [0, 1)\}. \tag{41}$$

Let us prove this convergence whose precise statement is given below at (46) and (47). We know that[14]

$$\begin{aligned} \varphi_n(x) + \psi_n(x) \quad &\to \quad \tilde{c}(x, x) \text{ and} && (42)\\ \varphi_n(x) + \psi_n(x \oplus \alpha) \quad &\to \quad \tilde{c}(x, x \oplus \alpha), \text{whence} \\ \psi_n(x \oplus \alpha) - \psi_n(x) \quad &\to \quad \underbrace{\tilde{c}(x, x \oplus \alpha) - \tilde{c}(x, x)}_{=:g(x)} = \begin{cases} +1 & \text{for } x \in [0, \frac{1}{2}), \\ -1 & \text{for } x \in [\frac{1}{2}, 1). \end{cases} && (43) \end{aligned}$$

Replacing x by $x \oplus i\alpha$, $i = 1, \ldots, k-1$ in (43) this yields

$$\psi_n(x \oplus \alpha) - \psi_n(x) \to \sum_{i=0}^{k-1} g(x \oplus i\alpha).$$

Combined with (42) we have

$$\begin{aligned} &\lim_{n\to\infty} [\varphi_n(x) + \psi_n(x \oplus k\alpha)] = 1 + \sum_{i=0}^{k-1} g(x \oplus i\alpha) && (44)\\ &= \ 1 + \ \#\left\{0 \le i < k : x \oplus i\alpha \in [0, \tfrac{1}{2})\right\} - \ \#\left\{0 \le i < k : x \oplus i\alpha \in [\tfrac{1}{2}, 1)\right\} \\ &=: \ \rho_k(x). && (45) \end{aligned}$$

Define the function h on $X \times Y$

$$h(x, y) = \begin{cases} \rho_k(x) & \text{for } (x, y) \in \Gamma_k, k \in \mathbb{N}, \\ \infty & \text{else.} \end{cases} \tag{46}$$

By (44), we have, for each $k \in \mathbb{N}$, $\lim_n \|h - \varphi_n \oplus \psi_n\|_{L^1(\pi_k)} = 0$. Somewhat more precisely, one obtains that

$$\|h - \varphi_n \oplus \psi_n\|_{L^1(\pi_k)} \le k\|\tilde{c} - \varphi_n \oplus \psi_n\|_{L^1(\pi_0+\pi_1)}. \tag{47}$$

Now we shall modify the cost function $\tilde{c}$ of Example 4.1 by defining it to be finite not only on $\Gamma_0 \cup \Gamma_1$, but rather on $\bigcup_{k\in\mathbb{N}} \Gamma_k$. We then obtain the following situation.

[14] The equations (42) to (45) refer to integrable functions on $[0, 1)$ and convergence is understood to be with respect to $\|.\|_{L^1(\mu)}$.

Example 4.3. Using (46) define $c : [0,1) \times [0,1) \to [0,\infty]$ by

$$c(x,y) = h(x,y)_+,$$

so that $\{c < \infty\} = \bigcup_{k \in \mathbb{N}} \Gamma_k$. For the resulting optimal transport problem we then find:

(i) The primal value P of the problem (2) equals zero and $\hat{\varphi} = \hat{\psi} = 0$ are (trivial) optimizers of the dual problem (4).

(ii) For strictly positive scalars $(a_k)_{k \geq 0}$, normalized by $\sum_{k \geq 0} a_k = 1$ apply Theorem 4.2 to the transport plan $\pi := \sum_{k \geq 0} a_k \pi_k$. (Again we apologize for using the notation π for the measure π_0 in Theorem 4.2, as all the letters π_k are already taken.) If $(a_k)_{\geq 0}$ tends sufficiently fast to zero, as $|k| \to \infty$, the following facts are verified.

- The primal value is

$$P^{(\pi)} = \inf \left\{ \int_{X \times Y} c \, d\bar{\pi} : \bar{\pi} \in \Pi(\mu,\nu), \|\tfrac{d\bar{\pi}}{d\pi}\|_{L^\infty} < \infty \right\} = 1.$$

- The Borel function $h \in L^1(\pi)$ defined in (46) is a dual optimizer in the sense of Theorem 4.2, i.e.

$$D^{(\pi)} = \int_{X \times Y} h \, d\pi = 1.$$

- There is a sequence $(\varphi_n, \psi_n)_{n=1}^\infty$ in $L^1(\mu) \times L^1(\nu)$ such that $(\varphi_n \oplus \psi_n)_{n=1}^\infty$ converges to h in the norm of $L^1(\pi)$.

Before proving the above assertions let us draw one conclusion: in (ii) we *can not assert* that the functions $(\varphi_n, \psi_n)_{n=1}^\infty$ satisfy – in addition to the properties above – the inequality $\varphi_n(x) + \psi_n(y) \leq c(x,y)$, for all $(x,y) \in X \times Y$. Indeed, if this were possible then, because of $\lim_{n \to \infty} (\int_X \varphi_n \, d\mu + \int_Y \psi_n \, d\nu) = D^{(\pi)} = 1$, we would have that the dual value D of the original dual problem (4) would equal $D = 1$, in contradiction to (i).

Proof of the assertions of Example 4.3. We start with assertion (ii). Fix an optimizing sequence $(\varphi_n, \psi_n)_{n=1}^\infty$ in the context of Example 4.1 such that

$$\|\tilde{c} - \varphi_n \oplus \psi_n\|_{L^1(\pi_0 + \pi_1)} \leq 1/n^3. \tag{48}$$

Pick a sequence $(a_k)_{k \in \mathbb{N}}$ of positive numbers such that

(a) $a_k \|h\|_{L^1(\pi_k)} \leq C 2^{-k}$ for all $k \in \mathbb{N}$,

(b) $a_k (\|\varphi_n\|_1 + \|\psi_n\|_1) \leq C 2^{-k}$ for all $k \in \mathbb{N}$ with $n \leq k$,

for some real constant C. After re-normalizing, if necessary, we may assume that $\sum_{k=1}^\infty a_k = 1$. Set $\pi := \sum_{k=1}^\infty a_k \pi_k$. From (a) we obtain $h \in L^1(\pi) \subseteq L^1(\pi)^{**}$ thus h is viable for the problem $D_{**}^{(\pi)}$ and hence $D_{**}^{(\pi)} \geq 1$. Clearly $P^{(\pi)} \leq 1$, hence $P^{(\pi)} = D_{**}^{(\pi)} = 1$ and h is a dual maximizer. Combining (48) with (47) we obtain

$$\|h - \varphi_n \oplus \psi_n\|_{L^1(\pi_k)} \leq k/n^3.$$

Therefore

$$\begin{aligned} \|h - \varphi_n \oplus \psi_n\|_{L^1(\pi)} &\leq \sum_{k \leq n} \|h - \varphi_n \oplus \psi_n\|_{L^1(\pi_k)} + \sum_{k > n} a_k (\|h\|_{L^1(\pi_k)} + \|\varphi_n\|_1 + \|\psi_n\|_1) \\ &\leq 1/n + 2C \sum_{k > n} 2^{-k}. \end{aligned}$$

Hence $\varphi_n \oplus \psi_n$ converges to h in $\|.\|_{L^1(\pi)}$. This shows assertion (ii) above.

To obtain (i) we construct a transport plan $\pi_\beta \in \Pi(\mu, \nu)$ such that $\int_{X\times Y} c\, d\pi_\beta = 0$. Note in passing that in view of (ii) we must have $\|\frac{d\pi_\beta}{d\pi}\|_{L^\infty(\pi)} = \infty$ for the π constructed above. On the other hand, we must have $\frac{d\pi_\beta}{d\pi} \in L^1(\pi)$, if $a_k > 0$ for all $k \in \mathbb{N}$, as every finite cost transport plan must be absolutely continuous with respect to π.

The idea is to concentrate π_β on the set

$$\begin{aligned}\Gamma &:= \{(x, y) : c(x, y) = 0\}\\ &= \{(x, x \oplus k\alpha) : k \geq 1, \textstyle\sum_{i=0}^{k-1}(\mathbb{1}_{[0,\frac{1}{2})}(x \oplus i\alpha) - \mathbb{1}_{[\frac{1}{2},1)}(x \oplus i\alpha)) \leq -1\}.\end{aligned}$$

To prove that this can be done it is sufficient to show that whenever $A \subseteq X$, $B \subseteq Y$, $\mu(A), \nu(B) > 0$, a subset A' of A can be transported to a subset B' of B with $\nu(B') = \mu(A') > 0$ via Γ. Then an exhaustion argument applies.

At this stage we encounter an interesting connection to the theory of measure preserving systems. For $x \in X$ and $m \in \mathbb{N}$ set

$$S(x, m) := \Big(x \oplus \alpha, m + \mathbb{1}_{[0,\frac{1}{2})}(x) - \mathbb{1}_{[\frac{1}{2},1)}(x)\Big).$$

Then S is a measure preserving transformation of the space $([0, 1] \times \mathbb{Z}, \lambda \times \#)$. (See [Aar97] for an introduction to infinite ergodic theory and the basic definitions in this field.) It is not hard to see that the ergodic theorem, applied to the rotation by α on the torus, shows that S is non wandering. Much less trivial is the fact that S is also ergodic. This was shown by K. Schmidt [Sch78] for a certain class of irrational numbers $\alpha \in [0, 1)$, and in full generality by M. Keane and J.-P. Conze [CK76], see also [AK82].
The relevance of these facts to our situation is that for $k \geq 1$, the pair $(x, x \oplus k\alpha)$ is an element of Γ if and only if $S^k(x, 0) \in [0, 1) \times \{-1, -2, \ldots\}$. By ergodicity of S, there exists k such that

$$(\lambda \times \#)\big((S^k[A \times \{0\}]) \cap (B \times \{-1, -2, \ldots\})\big) > 0,$$

thus it is possible to shift a positive portion of A to B as required. By exhaustion, there indeed exists a transport π_β such that $\langle c, \pi_\beta\rangle = 0$. $\square$

The above example illustrates some of the subtleties of Theorem 4.2. However, it does not yet provide evidence for the necessity of allowing for the singular part $\hat{h}^s$ of the optimizer $\hat{h}$ in Theorem 4.2. We have constructed yet a more refined – and rather longish – variant of the Ambrosio–Pratelli example above, which shows that, in general, there is no way of avoiding this complications in the statement of Theorem 4.2. We refer to the accompanying paper [BLS09, Section 3] for a presentation of this example, where it is shown that it can indeed occur that the singular part $\hat{h}^s$ in Theorem 4.2 does not vanish.

5 The Projective Limit Theorem

We again consider the general setting of Theorem 1.2, i.e. a $[0, \infty]$-valued, Borel measurable cost functions c. To avoid trivialities we shall always assume that $\Pi(\mu, \nu, c) = \{\pi \in \Pi(\mu, \nu) : \int c\, d\pi < \infty\}$ is non-empty.

Theorem 4.2 only pertains to the situation of a *fixed* element $\pi_0 \in \Pi(\mu, \nu, c)$: one then optimizes the transport problem of all $\pi \in \Pi(\mu, \nu)$ with $\|\frac{d\pi}{d\pi_0}\|_{L^\infty(\pi_0)} < \infty$.

The purpose of this section is to find an optimizer h which does work simultaneously, *for all* $\pi_0 \in \Pi(\mu, \nu, c)$. We are not able to provide a result showing that a *function* h – plus possibly some singular part h^s – exists which fulfills this duty, for all $\pi_0 \in \Pi(\mu, \nu, c)$. We leave the question whether this is always possible as an open problem. But we can show that a projective limit $\hat{H} = (\hat{h}_\pi)_{\pi \in \Pi(\mu,\nu,c)}$ exists which does the job.

We introduce an order relation on $\Pi(\mu,\nu,c)$: we say that $\pi_1 \preceq \pi_2$ if $\pi_1 \ll \pi_2$ and $\| \frac{d\pi_1}{d\pi_2} \|_{L^\infty(\pi_2)} < \infty$. For $\pi_1 \preceq \pi_2$ there is a natural, continuous projection $P_{\pi_1,\pi_2} : L^1(\pi_2) \to L^1(\pi_1)$ associating to each $h_{\pi_2} \in L^1(\pi_2)$, which is an equivalence class modulo π_2-null functions, the equivalence class modulo π_1-null functions which contains the equivalence class h_{π_2} (and where this inclusion of equivalence classes may be strict, in general). We may define the locally convex vector space E as the projective limit

$$E = \varprojlim_{\pi \in \Pi(\mu,\nu,c)} L^1(X \times Y, \pi).$$

The elements of E are families $H = (h_\pi)_{\pi \in \Pi(\mu,\nu,c)}$ such that, for $\pi_1 \preceq \pi_2$, we have $P_{\pi_1,\pi_2}(h_{\pi_2}) = h_{\pi_1}$.

A net $(H^\alpha)_{\alpha \in I} \in E$ converges to $H \in E$ if,

$$\lim_{\alpha \in I} \| h^\alpha_\pi - h_\pi \|_{L^1(\pi)} = 0, \quad \text{for each } \pi \in \Pi(\mu,\nu,c).$$

We may also define the projective limit

$$E_S = \varprojlim_{\pi \in \Pi(\mu,\nu,c)} L^1_S(X \times Y, \pi),$$

which is a closed subspace of E.

We start with an easy result.

Proposition 5.1. *Let $(X,\mu), (Y,\nu)$ be polish spaces equipped with Borel probability measures μ,ν, and let $c : X \times Y \to [0,\infty]$ be Borel measurable. Assume that $\Pi(\mu,\nu,c) \neq \emptyset$.*

There is $\pi_0 \in \Pi(\mu,\nu,c)$ such that

$$P^{(\pi_0)} = \inf_{\pi \in \Pi(\mu,\nu,c)} P^{(\pi)}$$

Proof. Let $(\pi_n)_{n=1}^\infty$ be a sequence in $\Pi(\mu,\nu,c)$ such that

$$\lim_{n\to\infty} P^{(\pi_n)} = \inf_{\pi \in \Pi(\mu,\nu,c)} P^{(\pi)}.$$

It suffices to define π_0 as

$$\pi_0 = \sum_{n=1}^\infty 2^{-n}\ \pi_n$$

as we then have $\pi_n \preceq \pi_0$, for each $n \in \mathbb{N}$. □

The above proposition allows us to suppose w.l.o.g. in our considerations on the projective limit E that the π appearing in the definition are all bigger than π_0:

$$E = \varprojlim_{\pi \in \Pi(\mu,\nu,c)} L^1(\pi) = \varprojlim_{\pi \in \Pi(\mu,\nu,c), \pi \succeq \pi_0} L^1(\pi).$$

Clearly, we then have that the optimal transport cost $P^{(\pi)}$ is equal to $P^{(\pi_0)}$, for all $\pi \succeq \pi_0$.

Theorem 5.2. *Let $(X,\mu), (Y,\nu)$ be polish spaces equipped with Borel probability measures μ,ν. Let $c : X \times Y \to [0,\infty]$ be Borel measurable, assume that $\Pi(\mu,\nu,c)$ is non-empty, and let π_0 be as in Proposition 5.1*

There is an element $\hat{H} = (\hat{h}_\pi)_{\pi \in \Pi(\mu,\nu,c), \pi \succeq \pi_0} \in E$ such that, for each $\pi \in \Pi(\mu,\nu,c), \pi \geq \pi_0$, the element $\hat{h}_\pi \in L^1_S(\pi)^{**}$ satisfies $\hat{h}_\pi \leq c$ in the order of $L^1(\pi)^{**}$ and $\hat{h}_\pi$ is an optimizer of the dual problem (39)

$$\langle \hat{h}_\pi, \pi \rangle = \sup\{\langle h, \pi \rangle : h \in L^1_S(\pi)^{**},\ h \leq c\}.$$

We then have that, for each $\pi \in \Pi(\mu,\nu,c), \pi \succeq \pi_0$, the decomposition $\hat{h}_\pi = \hat{h}^r_\pi + \hat{h}^s_\pi$ of $\hat{h}_\pi$ into its regular and singular part verifies $\hat{h}^s_\pi \leq 0$.

Proof. Fix $\pi \in \Pi(\mu, \nu, c), \pi \succeq \pi_0$. We have seen in Theorem 4.2 that the set

$$K_\pi = \{h_\pi \in L^1_S(\pi)^{**} : h_\pi \leq c, \langle h, \pi \rangle = \langle c, \pi \rangle\}$$

is non-empty. In addition K_π is closed and bounded in $L^1(\pi)^{**}$ and hence compact with respect to the $\sigma(L^1_S(\pi)^{**}, L^1_S(\pi)^*)$-topology.

For $\pi, \pi' \in \Pi(\mu, \nu, c)$ with $\pi \preceq \pi'$ the set

$$K_{\pi,\pi'} = P_{\pi,\pi'}(K_{\pi'})$$

is contained in K_π and still a non-empty σ^*-compact convex subset of $L^1(\pi)^{**}$.By compactness the following set is σ^*-compact and non-empty too:

$$K_{\pi,\infty} = \bigcap_{\pi' \succeq \pi} K_{\pi,\pi'}.$$

We have $K_{\pi,\infty} = P_{\pi,\pi'}(K_{\pi',\infty})$ for $\pi \preceq \pi'$. Hence by Tychonoff's theorem the projective limit

$$\varprojlim_{\pi \in \Pi(\mu,\nu,c), \pi \succeq \pi_0} K_{\pi,\infty}$$

of the compact sets $(K_{\pi,\infty})_{\pi \succeq \pi_0}$ is non-empty, which is precisely the assertion of the present theorem. $\square$

A Appendix

In our proof of Theorem 1.2 we made use of the following innocent looking result due H. Kellerer:

Lemma A.1. *Let X, Y be polish spaces equipped with Borel probability measures μ, ν, let $L \subseteq X \times Y$ be a Borel set and assume that $\pi(L) = 0$ for any $\pi \in \Pi(\mu, \nu)$. Then there exist sets $M \subseteq X, N \subseteq Y$ such that $\mu(M) = \nu(N) = 0$ and $L \subseteq M \times Y \cup X \times N$.*

Lemma A.1 seems quite intuitive and, as we will see subsequently, its proof is quite natural provided that the set L is *compact*. However the general case is delicate and relies on relatively involved results from measure theory. H. Kellerer proceeded as follows. First he established various sophisticated duality results. Lemma A.1 is then a consequence of the fact that there is no duality gap in the case that the Borel measurable cost function c is uniformly bounded (Corollary 2.4). To make the present paper more self-contained, we provide a direct proof of Lemma A.1. Still, most ideas of the subsequent proof are, at least implicitly, contained in the work of H. Kellerer.

Some steps in the proof of Lemma A.1 are (notationally) simpler in the case when $(X, \mu) = (Y, \nu) = ([0, 1], \lambda)$, therefore we bring a short argument which shows that it is legitimate to make this additional assumption.

Indeed it is rather obvious that one may reduce to the case that the measure spaces X and Y are free of atoms. A well known result of measure theory (see for instance [Kec95, Theorem 17.41]) asserts that for any polish space Z equipped with a continuous Borel probability measure σ, there exists a measure preserving Borel isomorphism between the spaces (Z, σ) and $([0, 1], \lambda)$. Thus there exist bijections $f : X \to [0, 1], g : Y \to [0, 1]$ which are measurable with measurable inverse, such that $f_\# \mu = g_\# \nu = \lambda$. Hence it is sufficient to consider the case $(X, \mu) = (Y, \nu) = ([0, 1], \lambda)$ and we will do so from now on.

For a measurable set $L \subseteq [0, 1]^2$ we define the functional

$$m(L) := \inf\{\lambda(A) + \lambda(B) : L \subseteq A \times Y \cup X \times B\}.$$

Our strategy is to show that under the assumptions of Lemma A.1, we have that $m(L) = 0$. This implies Lemma A.1 since we have the following result.

Lemma A.2. *Let $L \subseteq X \times Y$ be a Borel set with $m(L) = 0$. Then there exist sets $M \subseteq X, N \subseteq Y$ such that $\mu(M) = \nu(N) = 0$ and $L \subseteq M \times Y \cup X \times N$.*

Proof. Fix $\varepsilon > 0$. Since $m(L) = 0$, there exists sets A_n, B_n such that $\mu(A_n) < 1/n$ and $\nu(B_n) < \varepsilon 2^{-n}$ and $L \subseteq A_n \times Y \cup X \times B_n$. Set $A := \bigcap_{n \geq 1} A_n, B := \bigcup_{n \geq 1} B_n$. Then $\mu(A) = 0, \nu(B) < \varepsilon$ and

$$L \subseteq \bigcap_{n \geq 1} (A_n \times Y \cup X \times B) = A \times Y \cup X \times B.$$

Iterating this arguments with the roles of X and Y exchanged we get the desired conclusion. □

The next step proves Lemma A.1 in the case where L is compact.

Lemma A.3. *Assume that $K \subseteq [0, 1]^2$ is compact and satisfies $\pi(K) = 0$ for every $\pi \in \Pi(\lambda, \lambda)$. Then $m(K) = 0$.*

Proof. Assume that $\alpha := m(K) > 0$. We have to show that there exists a non-trivial measure π on $X \times Y$, i.e. $\pi(K) > 0$ such that $\operatorname{supp} \pi \subseteq K$ and the marginals of π satisfy $P_X(\pi) \leq \mu, P_Y(\pi) \leq \nu$. We aim to construct increasingly good approximations π_n of a such a measure.

Fix n large enough and choose $k \geq 1$ such that $\alpha/3 \leq k/n \leq \alpha/2$. Since K is non-empty, there exist $i_1, j_1 \in \{0, \dots, n-1\}$ such that

$$\Big(\big(\tfrac{i_1}{n}, \tfrac{j_1}{n}\big) + [0, \tfrac{1}{n}]^2 \Big) \cap K \neq \emptyset.$$

After $m < k$ steps, assume that we have already chosen $(i_1, j_1), \dots, (i_m, j_m)$. Since $(2 * m)/n < \alpha$, we have that K is not covered by

$$\Big(\bigcup_{l=1}^{m} \big[\tfrac{i_l}{n}, \tfrac{i_l+1}{n}\big] \Big) \times Y \cup X \times \Big(\bigcup_{l=1}^{m} \big[\tfrac{j_l}{n}, \tfrac{j_l+1}{n}\big] \Big).$$

Thus there exist

$$i_{m+1} \in \{0, \dots, n-1\} \setminus \{i_1, \dots, i_m\}, j_{m+1} \in \{0, \dots, n-1\} \setminus \{j_1, \dots, j_m\}$$

such that $\Big(\big(\tfrac{i_{m+1}}{n}, \tfrac{j_{m+1}}{n}\big) + [0, \tfrac{1}{n}]^2 \Big) \cap K \neq \emptyset$. After k steps we define the measure π_n to be the restriction of $n \cdot \lambda^2$, (i.e. the Lebesgue measure on $[0, 1]^2$ multplied with the constant n) to the set $\bigcup_{l=1}^{k} \big(\tfrac{i_l}{n}, \tfrac{j_l}{n}\big) + [0, \tfrac{1}{n}]^2$. Then the total mass of π_n is bounded from below by $k/n \geq \alpha/3$ and the marginals of π_n satisfy $P_X(\pi_n) \leq \mu, P_Y(\pi_n) \leq \nu$. These properties carry over to every weak-star limit point of the sequence (π_n) and each such limit point π satisfies $\operatorname{supp} \pi \subseteq K$ since K is closed. □

The next lemma will enable us to reduce the case of a Borel set L to the case where the set L is compact.

Lemma A.4. *Suppose that a Borel set $L \subseteq [0, 1]^2$ satisfies $m(L) > 0$. Then there exists a compact set $K \subseteq L$ such that $m(K) > 0$.*

Lemma A.4 will be deduced from *Choquet's capacitability Theorem.*[15] Before we formulate this result we introduce some notation. Given a compact metric space Z, a capacity on Z is a map $\gamma : \mathcal{P}(Z) \to \mathbb{R}_+$ such that:

[15] It seems worth noting that Kellerer also employs the Choquet capacitability Theorem.

1. $A \subseteq B \Rightarrow \gamma(A) \leq \gamma(B)$.

2. $A_1 \subseteq A_2 \subseteq \ldots \Rightarrow \sup_{n\geq 1} \gamma(A_n) = \gamma(\bigcup_{n\geq 1} A_n)$.

3. For every sequence $K_1 \supseteq K_2 \supseteq \ldots$ of compact sets we have $\inf_{n\geq 1} \gamma(K_n) = \gamma(\bigcap_{n\geq 1} K_n)$.

The typical example of a capacity is the *outer measure* associated to a finite Borel measure.

Theorem A.5 (Choquet capacitability Theorem). *See [Cho59] and also [Kec95, Theorem 30.13]. Assume that γ is a capacity on a polish space Z. Then*

$$\gamma(A) = \sup\{\gamma(K) : K \subseteq A,\ K \textit{ compact}\}$$

for every Borel[16] *set $A \subseteq Z$.*

Proof of Lemma A.4. We cannot apply Theorem A.5 directly to the functional m since m fails to be a capacity, even if it is extended in a proper way to all subsets of $[0,1]^2$. A clever trick[17] is to replace m by the mapping $\gamma : \mathcal{P}([0,1]^2) \to [0,2]$, defined by

$$\gamma(L) := \inf\Big\{\textstyle\int f\, d\lambda : f : [0,1] \to [0,1], f(x) + f(y) \geq \mathbb{1}_L(x,y) \text{ for } (x,y) \in [0,1]\Big\}.$$

We then have:

a. For any Borel set $A \subseteq [0,1]^2$ we have $\gamma(L) \leq m(L) \leq 4\gamma(L)$.

b. γ is a capacity.

To see that (a) holds true notice that $f(x) + f(y) \geq \mathbb{1}_L(x,y)$ implies $L \subseteq \{f \geq 1/2\} \times Y \cup X \times \{f \geq 1/2\}$ and that $L \subseteq A \times Y \cup X \times B$ yields $\mathbb{1}_{A\cup B}(x) + \mathbb{1}_{A\cup B}(y) \geq \mathbb{1}_L(x,y)$.

To prove (b) it remains to check that γ satisfies properties (2) and (3) of the capacity definition. To see continuity from below, consider a sequence of sets $A_1 \subseteq A_2 \subseteq \ldots$ increasing to A. Pick a sequence of functions f_n such that $f_n(x) + f_n(y) \geq \mathbb{1}_{A_n}(x,y)$ point-wise and $\int f\, d\lambda < \gamma(A_n) + 1/n$ for each $n \geq 1$. By Komlos' Lemma there exist functions $g_n \in \mathrm{conv}\{f_n, f_{n+1}, \ldots\}$ such that the sequence (g_n) converges λ-a.s. to a function $g : [0,1] \to [0,1]$. After changing g on a λ-null set if necessary, we have that $g(x)+g(y) \geq \mathbb{1}_A(x,y)$ point-wise. By dominated convergence, $\int g\, d\lambda = \lim_{n\to\infty} \int g_n\, d\lambda \leq \lim_{n\to\infty} \gamma(A_n) + 1/n = \gamma(A)$. Thus γ satisfies property 2. The proof of (3) follows precisely the same scheme.

An application of Choquet's Theorem A.5 now finishes the proof of Lemma A.4. ☐

We have done all preparations to prove Lemma A.1; the necessary steps are summarized below.

Proof of Lemma A.1. As discussed above, we may assume w.l.g. that $(X,\mu) = (Y,\nu) = ([0,1],\lambda)$. Suppose that the Borel set $L \subseteq [0,1]^2$ satisfies $\pi(L) = 0$ for all $\pi \in \Pi(\mu,\nu)$. Striving for a contradiction, we assume that $m(L) > 0$. By Lemma A.4, we find that there exists a compact set $K \subseteq L$ such that $m(K) > 0$. By Lemma A.3, there is a measure $\pi \in \Pi(\mu,\nu)$ such that $\pi(K) > 0$, hence also $\pi(L) > 0$ in contradiction to our assumption. Thus $m(L) = 0$. By Lemma A.2 we may conclude that there exist sets $M \subseteq X, N \subseteq Y, \mu(X) = \nu(N) = 0$ such that $L \subseteq M \times Y \cup X \times N$ hence we are done. ☐

[16] In fact, the assertion of the Choquet capacitability Theorem is true for the strictly larger class of *analytic sets*.

[17] We thank Richárd Balka and Márton Elekes for showing us this argument (private communication).

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