The super-replication theorem under proportional transaction costs revisited

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dedicated to Ivar Ekeland on the occasion of his seventieth birthday

June 5, 2014

Abstract

We consider a financial market with one riskless and one risky asset. The super-replication theorem states that there is no duality gap in the problem of super-replicating a contingent claim under transaction costs and the associated dual problem. We give two versions of this theorem.

The first theorem relates a numéraire-based admissibility condition in the primal problem to the notion of a local martingale in the dual problem. The second theorem relates a numéraire-free admissibility condition in the primal problem to the notion of a uniformly integrable martingale in the dual problem.

1 Introduction

The essence of the Black-Scholes theory ([BS 73], [M 73]) goes as follows: in the framework of their model $S = (S_t)_{0 \leq t \leq T}$ of a financial market (with riskless interest rate r normalized to $r = 0$) the unique arbitrage-free price for a contingent claim X_T maturing at time T is given by

$$X_0 = \mathbb{E}_Q[X_T]. \tag{1}$$

Here Q is the "martingale measure" for the Black-Scholes model, i.e. the probability measure on $(\Omega, \mathcal{F}_T, \mathbb{P})$ under which S is a martingale. The paper of Harrison-Kreps [HK 79] marked the beginning of a deeper understanding of the notion of arbitrage and its relation to martingale theory. Today it is very well understood that the salient feature of the Black-Scholes model which causes (1) to yield the unique arbitrage-free price is the fact that the martingale measure Q is *unique* in this model.

Financial markets S admitting a unique martingale measure Q are called "complete financial markets". We remark in passing that in this informal introduction we leave

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technicalities aside, such as integrability assumptions or the requirement that this measure Q should be *equivalent* to the original measure $\mathbb{P}$, i.e. $Q[A] = 0$ if and only if $\mathbb{P}[A] = 0$.

In a complete market $S = (S_t)_{0\le t\le T}$ every contingent claim X_T can be *perfectly replicated*, i.e. there is a predictable process $H = (H_t)_{0\le t\le T}$ such that

$$X_T = X_0 + \int_0^T H_t dS_t. \tag{2}$$

We now pass to the more realistic setting of a possibly *incomplete* financial market $S = (S_t)_{0\le t\le T}$. By definition we assume that the set $\mathcal{M}^e(S)$ of equivalent martingale measures is non-empty, but (possibly) not reduced to a singleton. In this setting the valuation formula (1) is replaced by

$$X_0 = \sup_{Q\in\mathcal{M}^e(S)} \mathbb{E}_Q[X_T] \tag{3}$$

This real number X_0 is called the *super-replication price* of X_T. The reason for this name is that one may find a predictable strategy $H = (H_t)_{0\le t\le T}$ such that the equality (2) now is replaced by the inequality

$$X_T \le X_0 + \int_0^T H_t dS_t \tag{4}$$

and X_0 is the smallest number with this property. This is the message of the *super-replication theorem* which was established by N. El Karoui and M.-C. Quenez [EQ 95] in a Brownian framework and, in greater generality, by F. Delbaen and the author in [DS 94] (compare [DS 06] for a comprehensive account).

The theme of the present paper is to show (two versions of) a super-replication theorem in the presence of proportional transaction costs $\lambda > 0$. For a given financial market $S = (S_t)_{0\le t\le T}$ as above we now suppose that we can *buy* the stock at price S but can only *sell* it at price $(1-\lambda)S$. The higher price S is called the *ask price* while the lower price $(1-\lambda)S$ is called the *bid price*.

In this context the notion of *martingale measures* Q appearing in (3) is replaced by the following concept which goes back to the pioneering work of E. Jouini and H. Kallal [JK 95].

Definition 1.1. *Fix a price process $S = (S_t)_{0\le t\le T}$ and transaction costs $0 < \lambda < 1$ as above. A consistent price system (resp. a consistent local price system) is a pair $(\widetilde{S}, Q)$ such that Q is a probability measure equivalent to $\mathbb{P}$ and $\widetilde{S} = (\widetilde{S}_t)_{0\le t\le T}$ takes its values in the bid-ask spread $[(1-\lambda)S, S] = ([(1-\lambda)S_t, S_t])_{0\le t\le T}$ and $\widetilde{S}$ is a Q-martingale (resp. a local Q-martingale).*

To stress the difference of the two notions we shall sometimes call a consistent price system a consistent price system in the non-local sense.

The condition of the *existence of an equivalent martingale measure* in the frictionless setting corresponds to the following notion.

Definition 1.2. *For $0 < \lambda < 1$, we say that a price process $S = (S_t)_{0\le t\le T}$ satisfies (CPS^λ) (resp. (CPS^λ) in a local sense) if there exists a consistent price system (resp. a consistent local price system).*

It is the purpose of this article to identify the precise assumptions in order to establish an analogue to (3) and (4) above, after translating these statements into the context of financial markets under transaction costs. To make concrete what we have in mind, we formulate our program in terms of a not yet precisely formulated "meta-theorem".

Theorem 1.3. *(not yet precise version of super-hedging) Fix a financial market $S = (S_t)_{0\leq t\leq T}$, transaction costs $0 < \lambda < 1$, and a contingent claim which pays X_T many units of bond at time T. Assume that S satisfies an appropriate regularity condition (of no arbitrage type). For a number $X_0 \in \mathbb{R}$, the following assertions are equivalent.*

(i) X_T can be super-replicated by starting with an initial portfolio of X_0 many units of bond and subsequently trading in S under transaction costs λ. The trading strategy has to be admissible in an appropriate sense.

(ii) For every consistent price system $(\widetilde{S}, Q)$ (in an appropriate sense, i.e. local or global) we have

$$X_0 \geq \mathbb{E}_Q[X_T].$$

We shall formulate below two versions which turn the above "meta-theorem" into precise mathematical statements. Let us first comment on the history of the above result. E. Jouini and H. Kallal in their pioneering paper [JK 95] considered a Hilbert space setting and proved a version of the above theorem in this context. They have thus established a perfect equivalent to the paper [HK 79] of Harrison-Kreps, replacing the frictionless theory by a model involving proportional transaction costs.

Y. Kabanov [K 99] proposed a numéraire-free setting of multi-currency markets (see [KS 09]) for more detailed information) which is much more general than the present setting. In [KS 02] Y. Kabanov and Ch. Stricker proved a version of the super-hedging theorem in Kabanov's model under the assumption of continuity of the exchange rate processes. This continuity assumption was removed by L. Campi and the author in [CS 06] thus establishing a general version of the super-hedging theorem in Kabanov's framework. However, due to the generality of the model considered in [K 99], [KS 02], and [CS 06], the precise definitions of e.g. self-financing portfolios and admissibility are sometimes difficult to check in applications.

We therefore change the focus in the present paper and concentrate on a more concrete setting with just one stock and one (normalised) bond, as well as fixed transaction costs $\lambda > 0$. Our aim is to establish clear-cut and easy-to-apply versions of the above super-hedging "meta-theorem" 1.3. Most importantly, we shall clarify the difference between a numéraire-free and a numéraire-based notion of admissible portfolios and its correspondence to the concepts of martingales and local martingales. This is somewhat analogue to the "numéraire-free" and "numéraire-based" versions of the Fundamental Theorem of Asset Pricing under Transaction Costs established in [GRS 10]. In the frictionless setting, analogous results are due to J. Yan [Y 05] (compare also [DS 95], and [Y 98]).

An extension of these results to the framework of multi-currency markets is left for future research.

We now state the two versions of the super-hedging theorem which we shall prove in this paper. The terms appearing in the statements will be carefully defined in the next section.

Theorem 1.4 (numéraire-based super-hedging). *Fix an $\mathbb{R}_+$-valued adapted càdlàg process $S = (S_t)_{0\le t\le T}$, transaction costs $0 < \lambda < 1$, and a contingent claim which pays X_T many units of bond at time T. The random variable X_T is assumed to be uniformly bounded from below. Assume that, for each $0 < \lambda' < 1$, the process S satisfies $(CPS^{\lambda'})$ in a local sense. For a number $X_0 \in \mathbb{R}$, the following assertions are equivalent.*

(i) There is a self-financing trading strategy $\varphi = (\varphi^0_t, \varphi^1_t)_{0\le t\le T}$ such that

$$\varphi_0 = (X_0, 0) \quad \textit{and} \quad \varphi_T = (X_T, 0)$$

which is admissible in the following numéraire-based sense: there is $M \geq 0$ such that, for every $[0, T]$-valued stopping time τ,

$$V_\tau(\varphi) \geq -M, \qquad a.s. \tag{5}$$

(ii) For every consistent local price system, i.e. for every probability measure Q, equivalent to $\mathbb{P}$, such that there is a local martingale $\widetilde{S} = (S_t)_{0\le t\le T}$ under Q, taking its values in the bid-ask spread $[(1-\lambda)S, S] = ([(1-\lambda)S_t, S_t])_{0\le t\le T}$, we have

$$X_0 \geq \mathbb{E}_Q[X_T]. \tag{6}$$

Theorem 1.5 (numéraire-free super-hedging). *Fix an $\mathbb{R}_+$-valued adapted càdlàg process $S = (S_t)_{0\le t\le T}$, transaction costs $0 < \lambda < 1$, and consider a non-negative contingent claim which pays X_T many units of bond at time T. The random variable X_T is assumed to be bounded from below by a multiple of $(1 + S_T)$. Assume that, for each $0 < \lambda' < 1$ the process S satisfies $(CPS^{\lambda'})$ in a non-local sense. For a number $X_0 \in \mathbb{R}$, the following assertions are equivalent.*

(i) There is a self-financing trading strategy $\varphi = (\varphi^0_t, \varphi^1_t)_{0\le t\le T}$ such that

$$\varphi_0 = (X_0, 0) \quad \textit{and} \quad \varphi_T = (X_T, 0)$$

which is admissible in the following sense: there is $M \geq 0$ such that, for every $[0, T]$-valued stopping time τ,

$$V_\tau(\varphi) \geq -M(1 + S_\tau), \quad a.s. \tag{7}$$

(ii) For every consistent price system, i.e. for every probability measure Q, equivalent to $\mathbb{P}$, such that there is a martingale $\widetilde{S} = (S_t)_{0\le t\le T}$ under Q, taking its values in the bid-ask spread $[(1-\lambda)S, S] = ([(1-\lambda)S_t, S_t])_{0\le t\le T}$ we have

$$X_0 \geq \mathbb{E}_Q[X_T] \tag{8}$$

Why do we speak about "numéraire-based" and "numéraire-free"? The admissibility condition of Theorem 1.4 refers to the bond as numéraire. Condition (5) means that an agent can cover the trading strategy φ by holding M units of bond. In contrast, condition (7) means that an agent can cover the trading strategy φ by holding M units of bond as well as M units of stock. The latter assumption is symmetric between stock and bond. It does not single out one asset as numéraire and is therefore called "numéraire-free".

2 Definitions and Notations

We consider a financial market consisting of one riskless asset and one risky asset. The riskless asset has constant price 1 and can be traded without transaction cost. The price of the risky asset is given by a strictly positive adapted càdlàg stochastic process $S = (S_t)_{0\le t\le T}$ on some underlying filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P\big)$ satisfying the usual assumptions of right continuity and completeness. In addition, we assume that $\mathcal{F}_0$ is trivial. For technical reasons (compare [CS 06]) we also assume (w.l.g.) that $\mathcal{F}_T = \mathcal{F}_{T-}$ and $S_T = S_{T-}$.

Trading strategies are modeled by $\mathbb{R}^2$-valued, predictable processes $\varphi = (\varphi^0_t, \varphi^1_t)_{0\le t\le T}$ of finite variation, where φ^0_t and φ^1_t denote the holdings in units of the riskless and the risky asset, respectively, after rebalancing the portfolio at time t. For any process X of finite variation we denote by $X = X_0 + X^\uparrow - X^\downarrow$ its Jordan-Hahn decomposition into two non-decreasing processes $X^\uparrow$ and $X^\downarrow$ both null at zero. The total variation $\mathrm{Var}_t(X)$ of X on $[0,t]$ is then given by $\mathrm{Var}_t(X) = X^\uparrow_t + X^\downarrow_t$ and the continuous part X^c of X by

$$X^c_t := X_t - \sum_{s<t} \Delta_+ X_s - \sum_{s\le t} \Delta X_s,$$

where $\Delta_+ X_t := X_{t+} - X_t$ and $\Delta X_t := X_t - X_{t-}$. Trading in the risky asset incurs proportional transaction costs of size $\lambda \in (0,1)$. This means that one has to pay a (higher) ask price S_t when buying risky shares at time t but only receives a (lower) bid price $(1-\lambda)S_t$ when selling them.

A strategy $\varphi = (\varphi^0_t, \varphi^1_t)_{0\le t\le T}$ is called *self-financing under transaction costs* λ if

$$\int_s^t d\varphi^0_u \le -\int_s^t S_u d\varphi^{1,\uparrow}_u + \int_s^t (1-\lambda) S_u d\varphi^{1,\downarrow}_u \tag{9}$$

a.s. for all $0 \le s < t \le T$, where

$$\int_s^t S_u d\varphi^{1,\uparrow}_u := \int_s^t S_u d\varphi^{1,\uparrow,c}_u + \sum_{s<u\le t} S_{u-}\Delta\varphi^{1,\uparrow}_u + \sum_{s\le u<t} S_u \Delta_+\varphi^{1,\uparrow}_u,$$

$$\int_s^t (1-\lambda) S_u d\varphi^{1,\downarrow}_u := \int_s^t (1-\lambda) S_u d\varphi^{1,\downarrow,c}_u + \sum_{s<u\le t} (1-\lambda) S_{u-}\Delta\varphi^{1,\downarrow}_u + \sum_{s\le u<t} (1-\lambda) S_u \Delta_+\varphi^{1,\downarrow}_u$$

can be defined by using Riemann-Stieltjes integrals, as S is càdlàg. The self-financing condition (9) then states that purchases and sales of the risky asset are accounted for in the riskless position:

$$d\varphi^{0,c}_t \le -S_t d\varphi^{1,\uparrow,c}_t + (1-\lambda) S_t d\varphi^{1,\downarrow,c}_t, \qquad 0 \le t \le T, \tag{10}$$

$$\Delta\varphi^0_t \le -S_{t-}\Delta\varphi^{1,\uparrow}_t + (1-\lambda) S_{t-}\Delta\varphi^{1,\downarrow}_t, \qquad 0 \le t \le T, \tag{11}$$

$$\Delta_+\varphi^0_t \le -S_t\Delta_+\varphi^{1,\uparrow}_t + (1-\lambda) S_t\Delta_+\varphi^{1,\downarrow}_t, \qquad 0 \le t \le T. \tag{12}$$

We define the *liquidation value* at time t by

$$V_t(\varphi) := \varphi^0_t + (\varphi^1_t)^+ (1-\lambda) S_t - (\varphi^1_t)^- S_t. \tag{13}$$

We have the following two notions of admissibility

Definition 2.1. *(a) A self-financing trading strategy φ is called admissible in a numéraire-based sense if there is $M > 0$ such that, for every $[0, T]$-valued stopping time τ,*

$$V_\tau(\varphi) \geq -M, \quad a.s., \tag{14}$$

(b) A self-financing trading strategy φ is called admissible in a numéraire-free sense if there is $M > 0$ such that, for every $[0, T]$-valued stopping time τ,

$$V_t(\varphi) \geq -M(1 + S_t), \quad a.s. \tag{15}$$

Here are typical examples of self-financing trading strategies.

Definition 2.2. *Fix S and $\lambda > 0$, as above, let $\tau : \Omega \to [0, T] \cup \{\infty\}$ be a stopping time, and let f_τ, g_τ be $\mathcal{F}_\tau$-measurable $\mathbb{R}_+$-valued functions. We define the corresponding* ask *and* bid *processes as*

$$a_t = (-S_\tau, 1) f_\tau \ \mathbb{1}_{]\!]\tau,T]\!]}(t), \qquad 0 \leq t \leq T, \tag{16}$$

$$b_t = ((1-\lambda)S_\tau, -1) g_\tau \ \mathbb{1}_{]\!]\tau,T]\!]}(t), \qquad 0 \leq t \leq T. \tag{17}$$

Similarly, let $\tau : \Omega \to [0, T] \cup \{\infty\}$ be a predictable stopping time, and let f_τ, g_τ be $\mathcal{F}_{\tau-}$-measurable $\mathbb{R}_+$-valued functions. We define

$$a_t = (-S_{\tau-}, 1) f_\tau \ \mathbb{1}_{[\![\tau,T]\!]}(t), \qquad 0 \leq t \leq T, \tag{18}$$

$$b_t = ((1-\lambda)S_{\tau-}, -1) g_\tau \ \mathbb{1}_{[\![\tau,T]\!]}(t), \qquad 0 \leq t \leq T. \tag{19}$$

We call a process $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ a predictable, simple, self-financing *process, if it is a finite sum of ask and bid processes as above.*

We note that $a = (a_t^0, a_t^1)_{0 \leq t \leq T}$ defined in (16) is admissible (in either sense of the above definitions) if the random variable $f_\tau S_\tau$ is bounded from above. As regards $b = (b_t^0, b_t^1)_{0 \leq t \leq T}$ defined in (17) it is admissible in the numéraire-free sense if g_τ is bounded; it is admissible in the numéraire-based sense if the process $(g_\tau S_t)_{\tau < t \leq T}$ is uniformly bounded.

Analogous remarks apply to (18) and (19).

3 Closedness in measure

The following lemma was proved by L. Campi and the author ([CS 06], Lemma 3.2) in the general framework of Kabanov's modeling of d-dimensional currency markets. Here we adapt the proof for a single risky asset model.

In section 2 we postulated as a qualitative — a priori — assumption that the strategies $\varphi = (\varphi^0, \varphi^1)$ have *finite variation.* The next lemma provides an automatic — a posteriori — quantitative control on the size of the finite variation. Note that we make a combination of the weaker versions of our hypotheses: as regards the no-arbitrage type assumption we only suppose $(CPS^{\lambda'})$ in the local sense and as regards admissibility we only require it in the numéraire-free sense.

Lemma 3.1. *Let S and $0 < \lambda < 1$ be as above, and suppose that $(CPS^{\lambda'})$ is satisfied in the local sense, for some $0 < \lambda' < \lambda$. Fix $M > 0$. Then the total variation of the process $(\varphi^0_t, \varphi^1_t)_{0 \le t \le T}$ remains bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$, when $\varphi = (\varphi^0, \varphi^1)$ runs through all M-admissible λ-self-financing strategies (in the numéraire-free sense (15)).*

More explicitly: for $M > 0$ and $\varepsilon > 0$, there is $C > 0$ such that, for all M-admissible, λ-self-financing strategies (φ^0, φ^1), starting at $(\varphi^0_0, \varphi^1_0) = (0, 0)$, and all increasing sequences $0 = \tau_0 < \tau_1 < \ldots < \tau_K = T$ of stopping times we have

$$\mathbb{P}\left[\sum_{k=1}^{K} |\varphi^0_{\tau_k} - \varphi^0_{\tau_{k-1}}| \ \geq C\right] < \varepsilon, \tag{20}$$

$$\mathbb{P}\left[\sum_{k=1}^{K} |\varphi^1_{\tau_k} - \varphi^1_{\tau_{k-1}}| \ \geq C\right] < \varepsilon. \tag{21}$$

Proof: Fix $0 < \lambda' < \lambda$ as above. By hypothesis there is a probability measure $Q \sim \mathbb{P}$, and a local Q-martingale $(\widetilde{S}_t)_{0 \le t \le T}$ such that $\widetilde{S}_t \in [(1 - \lambda')S_t, S_t]$. As the assertion of the lemma is of local type we may assume, by stopping, that $\widetilde{S}$ is a true martingale. We also may assume w.l.g. that $\varphi^1_T = 0$, i.e., that the position in stock is liquidated at time T.

Fix $M > 0$ and a λ-self-financing, M-admissible (in the sense of (15)) process $(\varphi^0_t, \varphi^1_t)_{t \ge 0}$, starting at $(\varphi^0_0, \varphi^1_0) = (0, 0)$. Write $\varphi^0 = \varphi^{0,\uparrow} - \varphi^{0,\downarrow}$ and $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ as the canonical differences of increasing processes. We shall show that

$$\mathbb{E}_Q[\varphi^{0,\uparrow}_T] \le \frac{M(1 + \mathbb{E}_Q[S_T])}{\lambda - \lambda'} \tag{22}$$

Define the process $\varphi' = ((\varphi^0)', (\varphi^1)')$ by

$$\varphi'_t = \big((\varphi^0)'_t, (\varphi^1)'_t\big) = \left(\varphi^0_t + \frac{\lambda - \lambda'}{1 - \lambda}\varphi^{0,\uparrow}_t, \varphi^1_t\right), \qquad 0 \le t \le T.$$

This is a self-financing process under transaction costs λ': indeed, whenever $d\varphi^0_t > 0$ so that $d\varphi^0_t = d\varphi^{0,\uparrow}_t$, the agent sells stock and receives $d\varphi^{0,\uparrow}_t = (1 - \lambda)S_t d\varphi^{1,\downarrow}_t$ (resp. $(1 - \lambda')S_t d\varphi^{1,\downarrow}_t = \frac{1-\lambda'}{1-\lambda} d\varphi^{0,\uparrow}_t$) many bonds under transaction costs λ (resp. λ'). The difference between these two terms is $\frac{\lambda-\lambda'}{1-\lambda} d\varphi^{0,\uparrow}_t$; this is the amount by which the λ'-agent does better than the λ-agent. It is also clear that $((\varphi^0)', (\varphi^1)')$ under transaction costs λ' still is a M-admissible strategy (in the numéraire-free sense of (15)).

By Proposition 2.3 of [S 13] the process

$$((\varphi^0)'_t + (\varphi^1)'_t \widetilde{S}_t)_{0 \le t \le T} = ((\varphi^0)'_t + \varphi^1_t \tilde{S}_t)_{0 \le t \le T} = (\varphi^0_t + \tfrac{\lambda-\lambda'}{1-\lambda}\varphi^{0,\uparrow}_t + \varphi^1_t \tilde{S}_t)_{0 \le t \le T}$$

is an optional strong Q-super-martingale. The reader might recall, that this notion pertains to optional, not necessary càdlàg, processes for which the super-martingale inequality holds true, for each pair of stopping times $0 \le \sigma \le \tau \le T$. Hence in particular

$$\mathbb{E}_Q[\varphi^0_T + \varphi^1_T \tilde{S}_T] + \tfrac{\lambda-\lambda'}{1-\lambda}\mathbb{E}_Q[\varphi^{0,\uparrow}_T] \le 0. \tag{23}$$

As

$$\varphi^0_T = \varphi^0_T + \varphi^1_T \widetilde{S}_T \ge -M(1 + S_T). \tag{24}$$

we have shown (22).

To obtain a control on $\varphi_T^{0,\downarrow}$ too, note that $\varphi_T^0 = \varphi_T^0 + \varphi_T^1 \widetilde{S}_T \geq -M(1+S_T)$ as $\varphi_T^1 = 0$ so that $\varphi_T^{0,\downarrow} \leq \varphi_T^{0,\uparrow} + M(1+S_T)$. Therefore we obtain the following estimate for the total variation $\varphi_T^{0,\uparrow} + \varphi_T^{0,\downarrow}$ of φ^0

$$\mathbb{E}_Q \left[\varphi_T^{0,\uparrow} + \varphi_T^{0,\downarrow} \right] \leq M \left(\frac{2}{\lambda - \lambda'} + 1 \right) \left(1 + \mathbb{E}_Q[S_T] \right). \tag{25}$$

The passage from the $L^1(Q)$-estimate (25) to the $L^0(\mathbb{P})$-estimate (20) is standard: for $\varepsilon > 0$ there is $\delta > 0$ such that for subsets $A \in \mathcal{F}$ with $Q[A] < \delta$ we have $\mathbb{P}[A] < \varepsilon$. Letting $C = \frac{M}{\delta}(\frac{2}{\lambda-\lambda'} + 1)(1 + \mathbb{E}_Q[S_T])$ and applying Tschebyscheff to (25) we get

$$\mathbb{P} \left[\varphi_T^{0,\uparrow} + \varphi_T^{0,\downarrow} \geq C \right] < \varepsilon, \tag{26}$$

which implies (20).

As regards (21) it follows from (9) that

$$d\varphi_t^{1,\uparrow} \leq \frac{d\varphi_t^{0,\downarrow}}{S_t}, \tag{27}$$

or, more precisely, by (10), (11), and (12),

$$d\varphi_t^{1,\uparrow,c} \leq \frac{d\varphi_t^{0,\downarrow,c}}{S_t}, \tag{28}$$

$$\Delta\varphi_t^{1,\uparrow} \leq \frac{\Delta\varphi_t^{0,\downarrow}}{S_{t-}}, \tag{29}$$

$$\Delta_+\varphi_t^{1,\uparrow} \leq \frac{\Delta_+\varphi_t^{0,\downarrow}}{S_t}. \tag{30}$$

By assumption the trajectories of $(S_t)_{0 \leq t \leq T}$ are strictly positive. In fact, we even have, for almost all trajectories $(S_t(\omega))_{0 \leq t \leq T}$, that $\inf_{0 \leq t \leq T} S_t(\omega)$ is strictly positive. Indeed, $\widetilde{S}$ being a Q-martingale with $\widetilde{S}_T > 0$ a.s. satisfies that $\inf_{0 \leq t \leq T} \widetilde{S}_t(\omega)$ is Q-a.s. and therefore $\mathbb{P}$-a.s. strictly positive.

Summing up, for $\varepsilon > 0$, we may find $\delta > 0$ such that

$$\mathbb{P} \left[\inf_{0 \leq t \leq T} S_t < \delta \right] < \frac{\varepsilon}{2}.$$

Hence we may control $\varphi_T^{1,\uparrow}$ by using (27) and estimating $\varphi^{0,\downarrow}$ by (26). Finally, we can control $\varphi_T^{1,\downarrow}$ by simply observing that $\varphi_T^{1,\uparrow} - \varphi_T^{1,\downarrow} = \varphi_T^1 - \varphi_0^1 = 0$. ■

Remark 3.2. In the above proof we have shown that the elements $\varphi_T^{0,\uparrow}, \varphi_T^{0,\downarrow}, \varphi_T^{1,\uparrow}, \varphi_T^{1,\downarrow}$ remain bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$, when (φ^0, φ^1) runs through the M-admissible (in the numéraire-free sense (15)) self-financing processes and $\varphi^0 = \varphi^{0,\uparrow} - \varphi^{0,\downarrow}$ and $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ denote the canonical decompositions. For later use we remark that the proof shows, in fact, that also the convex combinations of the functions $\varphi_T^{0,\uparrow}$ etc. remain bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$. Indeed the estimate (22) shows that the convex hull of the functions $\varphi_T^{0,\uparrow}$ is bounded in $L^1(Q)$ and (25) yields the same for $\varphi_T^{0,\downarrow}$. For $\varphi_T^{1,\uparrow}$ and $\varphi_T^{1,\downarrow}$ the argument is similar.

We can now formulate the main result of this section, in a numéraire-based as well as a numéraire-free version (Theorem 3.4 and Theorem 3.6)

Definition 3.3. *For $M > 0$ we denote by $\mathcal{A}_{nb}^M$ (resp. $\mathcal{A}_{nf}^M$) the set of pairs $(\varphi_T^0, \varphi_T^1) \in L^0(\mathbb{R}^2)$ of terminal values of self-financing trading strategies φ, starting at $\varphi_0 = (0,0)$, which are M-admissible in the numéraire-based sense* (14) *(resp. in the numéraire-free sense* (15)*).*

We denote by $\mathcal{C}_{nb}^M$ (resp. $\mathcal{C}_{nf}^M$) the set of random variables $\varphi_T^0 \in L^0$ such that $(\varphi_T^0, 0)$ is in $\mathcal{A}_{nb}^M$ (resp. in $\mathcal{A}_{nf}^M$).

We shall occasionally drop the sub-scripts nb (resp. nf) when it is clear from the context that we are in the numéraire-based (resp. numéraire-free) setting.

Theorem 3.4. *(numéraire-based version) Fix $S = (S_t)_{0 \le t \le T}$ and $0 < \lambda < 1$ as above, and suppose that $(CPS^{\lambda'})$ is satisfied in a local sense, for each $0 < \lambda' < \lambda$. For $M > 0$, the convex set $\mathcal{A}_{nb}^M \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ as well as the convex set $\mathcal{C}_{nb}^M \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ are closed with respect to the topology of convergence in measure.*

Proof: Fix $M > 0$ and let $(\varphi_T^n)_{n=1}^\infty = (\varphi_T^{0,n}, \varphi_T^{1,n})_{n=1}^\infty$ be a sequence in $\mathcal{A}^M = \mathcal{A}_{nb}^M$ converging a.s. to some $\varphi_T = (\varphi_T^0, \varphi_T^1) \in L^0(\mathbb{R}^2)$. We have to show that $\varphi_T \in \mathcal{A}^M$. We may find self-financing, admissible (in the numéraire-based sense) strategies $\varphi^n = (\varphi_t^{0,n}, \varphi_t^{1,n})_{0 \le t \le T}$, starting at $(\varphi_0^{0,n}, \varphi_0^{1,n}) = (0,0)$, and ending with terminal values $(\varphi_T^{0,n}, \varphi_T^{1,n})$. By the assumption $(CPS^{\lambda'})$, for *each* $0 < \lambda' < \lambda$, we may conclude that these processes are M-admissible in the numéraire-based sense ([S 13], Th. 1.7). As above, decompose canonically these processes as $\varphi_T^{0,n} = \varphi_T^{0,n,\uparrow} - \varphi_T^{0,n,\downarrow}$, and $\varphi_T^{1,n} = \varphi_T^{1,n,\uparrow} - \varphi_T^{1,n,\downarrow}$. By Lemma 3.1 and the subsequent remark we know that $(\varphi_T^{0,n,\uparrow})_{n=1}^\infty, (\varphi_T^{0,n,\downarrow})_{n=1}^\infty, (\varphi_T^{1,n,\uparrow})_{n=1}^\infty$, and $(\varphi_T^{1,n,\downarrow})_{n=1}^\infty$ as well as their convex combinations are bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$, so that by Lemma A1.1a in [DS 94] we may find convex combinations converging a.s. to elements $\varphi_T^{0,\uparrow}, \varphi_T^{0,\downarrow}, \varphi_T^{1,\uparrow}$, and $\varphi_T^{1,\downarrow} \in L^0(\Omega, \mathcal{F}, \mathbb{P})$. To alleviate notation we denote these sequences of convex combinations still by the original sequences. We claim that $(\varphi_T^0, \varphi_T^1) = (\varphi_T^{0,\uparrow} - \varphi_T^{0,\downarrow}, \varphi_T^{1,\uparrow} - \varphi_T^{1,\downarrow})$ is in $\mathcal{A}^M$ which will readily show the closedness of $\mathcal{A}^M$ with respect to the topology of convergence in measure.

By inductively passing to convex combinations, still denoted by the original sequences, we may, for each rational number $r \in [0, T[$, assume that $(\varphi_r^{0,n,\uparrow})_{n=1}^\infty, (\varphi_r^{0,n,\downarrow})_{n=1}^\infty, (\varphi_r^{1,n,\uparrow})_{n=1}^\infty$, and $(\varphi_r^{1,n,\downarrow})_{n=1}^\infty$ converge to some elements $\bar{\varphi}_r^{0,\uparrow}, \bar{\varphi}_r^{0,\downarrow}, \bar{\varphi}_r^{1,\uparrow}$, and $\bar{\varphi}_r^{1,\downarrow}$ in $L^0(\Omega, \mathcal{F}, \mathbb{P})$. By passing to a diagonal subsequence, we may suppose that this convergence holds true for all rationals $r \in [0, T[$.

Clearly the four processes $\bar{\varphi}_{r \in \mathbb{Q} \cap [0,T[}^{0,\uparrow}$ etc, indexed by the rationals r in $[0, T[$, still are a.s. increasing and define an M-admissible process in the numéraire-based sense of (14), indexed by $[0, T[\cap \mathbb{Q}$.

We have to extend these processes to all real numbers $t \in [0, T]$. This is done by first letting

$$\widehat{\varphi}_t^{0,\uparrow} = \lim_{\substack{r \searrow t \\ r \in \mathbb{Q}}} \bar{\varphi}_r^{0,\uparrow}, \qquad 0 \le t < T, \tag{31}$$

and $\widehat{\varphi}_0^{0,\uparrow} = 0$. The terminal value $\widehat{\varphi}_T^{0,\uparrow} = \varphi_T^{0,\uparrow}$ is still given by the first step of the construction. The càdlàg process $\widehat{\varphi}^{0,\uparrow}$ is not yet the desired limit as we still have to take special care of the jumps of $\widehat{\varphi}^{0,\uparrow}$. The jumps of the process $\widehat{\varphi}^{0,\uparrow}$ can be exhausted by a sequence $(\tau_k)_{k=1}^\infty$ of stopping times. By passing once more to a sequence of convex combinations, still denoted by $(\widehat{\varphi}^{0,n,\uparrow})_{n=1}^\infty$, we may also assume that $(\varphi_{\tau_k}^{0,n,\uparrow})_{n=1}^\infty$ converges almost surely, for each $k \in \mathbb{N}$. Define

$$\varphi_t^{0,\uparrow} = \begin{cases} \lim_{n\to\infty} \varphi_{\tau_k}^{0,n,\uparrow} & \text{if} \quad t = \tau_k, \quad \text{for some } k \in \mathbb{N} \\ \widehat{\varphi}_t^{0,\uparrow} & \text{otherwise.} \end{cases}$$

This process is predictable. Indeed, there is a subset $\Omega' \subseteq \Omega$ of full measure $\mathbb{P}[\Omega'] = 1$, such that $(\varphi^{0,n,\uparrow})_{n=1}^\infty$ converges pointwise to $\varphi^{0,\uparrow}$ *everywhere* on $\Omega' \times [0,T]$. The process $\varphi^{0,\uparrow}$ also is a.s. non-decreasing in $t \in [0,T]$. We thus have found a predictable process $\varphi^{0,\uparrow} = (\varphi_t^{0,\uparrow})_{0\leq t\leq T}$ such that a.s. the sequence $(\varphi_t^{0,n,\uparrow})_{0\leq t\leq T}$ converges to $(\varphi_t^{0,\uparrow})_{0\leq t\leq T}$ for *all* $t \in T$.

The three other cases, $\varphi^{0,\downarrow}, \varphi^{1,\uparrow}$, and $\varphi^{1,\downarrow}$ are treated in an analogous way. These processes are predictable, increasing, and satisfy condition (9).

Finally, define the process $(\varphi_t^0, \varphi_t^1)_{0\leq t\leq T}$ as $(\varphi_t^{0,\uparrow} - \varphi_t^{0,\downarrow}, \varphi_t^{1,\uparrow} - \varphi_t^{1,\downarrow})_{0\leq t\leq T}$. It is predictable and M-admissible in the numéraire-based sense (14) as this condition passes from the process $(\varphi^n)_{n=1}^\infty$ to the limit φ. Similarly, the process φ satisfies the self-financing condition (9) as the convergence of the processes $(\varphi^n)_{n=1}^\infty$ takes place, for *all* $t \in [0,T]$. We thus have shown that $\mathcal{A}^M = \mathcal{A}_{nb}^M$ is closed in $L^0(\mathbb{R}^2)$.

The closedness of $\mathcal{C}^M = \mathcal{C}_{nb}^M$ in L^0 is an immediate consequence. ■

Remark 3.5. We have not only proved a *closedness* property of $\mathcal{A}^M$ with respect to the topology of convergence in measure. Rather we have shown a *convex compactness* property (compare [KZ 11], [Z 09]). Indeed, we have shown that, for any sequence $(\varphi_T^n)_{n=1}^\infty \in \mathcal{A}^M$, we can find a sequence of convex combinations which converges a.s. to an element $\varphi_T \in \mathcal{A}^M$.

For later use (proof of Theorem 1.4) we also remark that the above proof yields the following technical variant of Theorem 3.4. Let $0 < \lambda_n < \lambda$ be a sequence of reals increasing to λ and $(\varphi_T^n)_{n=1}^\infty$ be in $\mathcal{A}_{nb}^{M,\lambda_n}$, where the super-script λ_n indicates that φ_T^n is the terminal value of an M-admissible λ_n-self-financing trading strategy starting at $(0,0)$. If $(\varphi_T^n)_{n=1}^\infty$ converges a.s. to φ_T^0 we may conclude that φ_T^0 is the terminal value of a strategy $\varphi^0 = (\varphi_t^{0,0}, \varphi_t^{1,0})_{0\leq t\leq T}$ which is M-admissible and λ_n-self-financing, for each $n \in \mathbb{N}$. From (9) we conclude that φ^0 is λ-self-financing.

Theorem 3.6. *(numéraire-free version) Fix $S = (S_t)_{0\leq t\leq T}$ and $0 < \lambda < 1$ as above, and suppose that $(CPS^{\lambda'})$ is satisfied, in the non-local sense, for each $0 < \lambda' < \lambda$. For $M > 0$, the convex set $\mathcal{A}_{nf}^M \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ as well as the convex set $\mathcal{C}_{nf}^M \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ are closed with respect to the topology of convergence in measure.*

Proof. As in the previous proof fix $M > 0$ and let $(\varphi_T^n)_{n=1}^\infty = (\varphi_T^{0,n}, \varphi_T^{1,n})_{n=1}^\infty$ be a sequence which we now assume to be in $\mathcal{A}^M = \mathcal{A}_{nf}^M$, converging a.s to some $\varphi_T = (\varphi_T^0, \varphi_T^1) \in L^0(\mathbb{R}^2)$. We have to show that $\varphi_T \in \mathcal{A}^M$. Again we may find self-financing, admissible (in the numéraire-free sense) strategies $(\varphi_T^{0,n}, \varphi_t^{1,n})_{0\leq t\leq T}$ starting at $(\varphi_0^{0,n}, \varphi_0^{1,n}) = (0,0)$, with terminal values $(\varphi_T^{0,n}, \varphi_T^{1,n})$. We now apply Th 2.4 of [S 13] to conclude that these processes are M-admissible in the numéraire-free sense (15).

We then may proceed verbatim as in the above proof to construct a limiting process $\varphi = (\varphi_t^0, \varphi_t^1)_{0\leq t\leq T}$ which is predictable, M-admissible (in the numéraire-free sense) and has the prescribed terminal value. This again shows that $\mathcal{A}^M = \mathcal{A}_{nf}^M$ and $\mathcal{C}^M = \mathcal{C}_{nf}^m$ are closed in L^0. □

4 The proof of Theorem 1.5

We now apply duality theory to the sets $\mathcal{A}^M$ and $\mathcal{C}^M$. We first deal with the numéraire-free case where we follow the lines of [K 99], [KS 02], [CS 06] and [KS 09]. As above fix a càdlàg adapted price process $S = (S_t)_{0\le t\le T}$ and transaction costs $0 < \lambda < 1$. We use the notation $\mathcal{A}_{nf} = \cup_{M=1}^{\infty}\mathcal{A}_{nf}^M$ and $\mathcal{C}_{nf} = \cup_{M=1}^{\infty}\mathcal{C}_{nf}^M$.

Definition 4.1. *We define $\mathcal{B}_{nf}$ as the set of all pairs $Z_T = (Z_T^0, Z_T^1) \in L^1(\mathbb{R}_+^2)$ such that $\mathbb{E}[Z_T^0] = 1$ and such that $\mathcal{B}_{nf}$ is polar to $\mathcal{A}_{nf}$, i.e.*

$$\mathbb{E}[\varphi_T^0 Z_T^0 + \varphi_T^1 Z_T^1] \le 0, \tag{32}$$

for all $\varphi_T = (\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{nf}$.

We associate to $Z_T \in \mathcal{B}_{nf}$ the martingale Z defined by

$$Z_t = \mathbb{E}[Z_T|\mathcal{F}_t], \qquad 0 \le t \le T. \tag{33}$$

In (32) we define the expectation by requiring that the negative part of $(\varphi_T^0 Z_T^0 + \varphi_T^1 Z_T^1)$ has to be integrable. Then (32) well-defines a number in $]-\infty, +\infty]$.

We shall identify the elements $(Z_T^0, Z_T^1) \in \mathcal{B}_{nf}$ with pairs $(\widetilde{S}, Q)$ by letting

$$\widetilde{S}_t = \frac{Z_t^1}{Z_t^0}, \quad \text{and} \quad \frac{dQ}{d\mathbb{P}} = Z_T^0. \tag{34}$$

The random variable Z_T^0 may vanish on a set of positive measure. This corresponds to the fact that the probability measure Q only is absolutely continuous w.r. to $\mathbb{P}$ and not necessarily equivalent. In this case we define $\widetilde{S}_t = S_t$ where Z_t^0 vanishes.

We now show that $\mathcal{B}_{nf}$ equals precisely the set of consistent price systems $(\widetilde{S}, Q)$ (in the non-local sense) where we allow Q to be only absolutely continuous to $\mathbb{P}$ (in Definition 1.1 we have required that Q is equivalent to $\mathbb{P}$).

Proposition 4.2. *In the setting of Definition 4.1 let $Z_T \in \mathcal{B}_{nf}$. Then the martingale $Z = (Z_t)_{0\le t\le T}$ in (33) satisfies*

$$\widetilde{S}_t := \frac{Z_t^1}{Z_t^0} \in [(1-\lambda)S_t, S_t], \quad 0 \le t \le T, \quad a.s. \tag{35}$$

Conversely, suppose that $Z = (Z_t^0, Z_t^1)_{0\le t\le T}$ is an $\mathbb{R}_+^2$-valued $\mathbb{P}$-martingale such that $Z_0^0 = 1$ and $\widetilde{S}_t := \frac{Z_t^1}{Z_t^0}$ takes a.s. on $\{Z_t^0 > 0\}$ its values in $[(1-\lambda)S_t, S_t]$. Then $Z_T = (Z_T^0, Z_T^1) \in \mathcal{B}_{nf}$.

Proof. To show (35) suppose that there is a $[0, T[$-valued stopping time τ such that $Q[\widetilde{S}_\tau > S_\tau] > 0$. Consider as in (16)

$$a_t = (-1, \frac{1}{S_\tau})\mathbb{1}_{\{\widetilde{S}_\tau > S_\tau\}}\mathbb{1}_{]\!]\tau,T]\!]}(t), \quad 0 \le t \le T.$$

This is a self-financing strategy which is admissible in the numéraire-free sense (in fact, also in the numéraire-based sense) for which (32) yields.

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[(-Z_T^0+\frac{Z_T^1}{S_\tau})\mathbb{1}_{\{\widetilde{S}_\tau>S_\tau\}}] &= \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[(-Z_T^0+\frac{Z_T^1}{S_\tau})\mathbb{1}_{\{\widetilde{S}_\tau>S_\tau\}}|\mathcal{F}_\tau]] \\ &= \mathbb{E}_{\mathbb{P}}[Z_\tau^0(-1+\frac{\widetilde{S}_\tau}{S_\tau})\mathbb{1}_{\{\widetilde{S}_\tau>S_\tau\}}] \\ &= \mathbb{E}_Q[(-1+\frac{\widetilde{S}_\tau}{S_\tau})\mathbb{1}_{\{\widetilde{S}_\tau>S_\tau\}}]>0,\end{aligned}$$

a contradiction. In the remaining case that $Q[\widetilde{S}_T>S_T]>0$ we consider the strategy $a_t=(-1,\frac{1}{S_T})\mathbb{1}_{\{\widetilde{S}_T>S_T\}}\mathbb{1}_{[\![T]\!]}(t)$ as in (18).

We still have to show that the case, $Q[\widetilde{S}_\tau<(1-\lambda)S_\tau]>0$, for some stopping time $0\le\tau<T$, leads to a contradiction too. As in (17) define

$$b_t=((1-\lambda)S_\tau,-1)\mathbb{1}_{\{\widetilde{S}_\tau<(1-\lambda)S_\tau\}}\mathbb{1}_{]\!]\tau,T]\!]}(t),\quad 0\le t\le T.$$

Again this strategy is self-financing and admissible (this time only in the numéraire-free sense) and we arrive at a contradiction

$$\mathbb{E}_{\mathbb{P}}[((1-\lambda)S_\tau Z_T^0-Z_T^1)\mathbb{1}_{\{\widetilde{S}_\tau<(1-\lambda)S_\tau\}}]=\mathbb{E}_Q[((1-\lambda)S_\tau-\widetilde{S}_\tau)\mathbb{1}_{\{\widetilde{S}_\tau<(1-\lambda)S_\tau\}}]>0.$$

The case $Q[\widetilde{S}_T<(1-\lambda)S_T]$ is dealt by considering (19) similarly as above. This shows the first part of the proposition.

As regards the second part, fix a martingale $Z=(Z_t^0,Z_t^1)_{0\le t\le T}$ with the properties stated there and let $(\widetilde{S},Q)$ be defined by (34). For every self-financing trading strategy $\varphi=(\varphi_t^0,\varphi_t^1)_{0\le t\le T}$, starting at $(\varphi_0^0,\varphi_0^1)=(0,0)$ and being M-admissible in the numéraire-free sense we deduce from Proposition 2.3 and the subsequent remark in [S 13] that $\widetilde{V}_t:=\varphi_t^0+\varphi_t^1\widetilde{S}_t$ is an optional strong super-martingale under Q (see [S 13], Def. 1.5, for a definition). This gives the desired inequality

$$0=\widetilde{V}_0\ge\mathbb{E}_Q[\widetilde{V}_T]=\mathbb{E}_{\mathbb{P}}[\varphi_T^0Z_T^0+\varphi_T^1Z_T^1].$$

The proof of Proposition 4.2 is now complete. □

In order to obtain a proof of Th. 1.5 we still need a version of the bipolar theorem for L^0. We first recall the bipolar theorem in the one-dimensional setting as obtained in [BS 99]. For a subset $A\subseteq L^0(\mathbb{R}_+)$ we define its polar in $L^1(\mathbb{R}_+)$ by

$$A^0=\{g\in L^1(\mathbb{R}_+):\mathbb{E}[fg]\le 1\}.$$

The bipolar theorem in [BS 99] states that $f\in L^0(\mathbb{R}_+)$ belongs to the closed (w.r. to convergence in measure), convex, solid hull of A if and only if

$$\mathbb{E}[fg]\le 1,\quad\text{for all}\quad g\in A^0.$$

We need the multi-dimensional version of this result established in ([KS 09], Th. 5.5.3) which applies to the cone $\mathcal{A}_{nf}$ in $L^0(\mathbb{R}^2)$.

While in the one-dimensional setting considered in [BS 99] there is just one natural order structure of $L^0(\mathbb{R})$, in the two-dimensional setting the situation is more complicated (see [BM 03]). We define a partial order on $L^0(\mathbb{R}^2)$ by letting $\varphi_T = (\varphi_T^0, \varphi_T^1) \succeq \psi_T = (\psi_T^0, \psi_T^1)$ if the difference $\varphi_T - \psi_T$ may be liquidated to the zero-portfolio, i.e. $V_T(\varphi_T - \psi_T) \geq 0$. This partial order is designed in such a way that, for $\varphi_T \in \mathcal{A}_{nf}^M$, we have that $\varphi_T \succeq (-M, -M)$.

Following [KS 09] we say that a sequence $(\varphi_T^n)_{n=1}^\infty$ in $L^0(\mathbb{R}^2)$ *Fatou-converges* to $\varphi_T \in L^0(\mathbb{R}^2)$ if there is $M > 0$ such that each φ_T^n dominates $(-M, -M)$ and $(\varphi_T^n)_{n=1}^\infty$ converges a.s. to φ_T.

By (a version of) Fatou's lemma this convergence implies that, for each $Z_T = (Z_T^0, Z_T^1) \in \mathcal{B}_{nf}$,

$$\liminf_{n\to\infty} \langle \varphi_T^n, Z_T \rangle := \liminf_{n\to\infty} \mathbb{E}[\varphi_T^{0,n} Z_T^0 + \varphi_T^{1,n} Z_T^1] \geq \mathbb{E}[\varphi_T^0 Z_T^0 + \varphi_T^1 Z_T^1] = \langle \varphi_T, Z_T \rangle,$$

as $\varphi_T^{0,n} Z_T^0 + \varphi_T^{1,n} Z_t^1 \geq -M(Z_T^0 + Z_T^1)$ and the latter function is $\mathbb{P}$-integrable.

Denote by $\mathcal{A}_{nf}^b$ the set of bounded elements in $\mathcal{A}_{nf}$, i.e. $\mathcal{A}_{nf}^b = \mathcal{A}_{nf} \cap L^\infty(\mathbb{R}^2)$. It is straightforward to deduce from Theorem 3.6 that under the assumption of Theorem 1.5 the following properties are satisfied.

(*i*) $\mathcal{A}_{nf}$ is Fatou-closed, i.e. contains all limits of its Fatou-convergent sequences.

(*ii*) $\mathcal{A}_{nf}^b$ is Fatou-dense in $\mathcal{A}_{nf}$, i.e. for $\varphi_T \in \mathcal{A}_{nf}$, there is a sequence $(\varphi_T^n)_{n=1}^\infty \in \mathcal{A}_{nf}^b$ which Fatou-converges to φ_T.

(*iii*) $\mathcal{A}_{nf}^b$ contains the negative orthant $-L^\infty(\mathbb{R}_+^2)$.

Define the polar of $\mathcal{A}_{nf}$ by

$$\mathcal{A}_{nf}^0 = \{Z_T = (Z_T^0, Z_T^1) \in L^1(\mathbb{R}^2) : \langle \varphi_T, Z_T \rangle \leq 1\}.$$

As $\mathcal{A}_{nf}$ is a cone we may equivalently write

$$\mathcal{A}_{nf}^0 = \{(Z_T = (Z_T^0, Z_T^1) \in L^1(\mathbb{R}^2) : \langle \varphi_T, Z_T \rangle \leq 0\}.$$

Proposition 4.2 states that $\mathcal{A}_{nf}^0$ equals the cone generated by $\mathcal{B}_{nf}$.

It is shown in ([KS 09], Th. 5.5.3) that the three properties above imply that, for the set $\mathcal{A}_{nf}$ in $L^0(\mathbb{R}^2)$ which satisfies (*i*), (*ii*) and (*iii*), the bipolar theorem holds true, i.e. an element $X_T = (X_T^0, X_T^1) \in L^0(\mathbb{R}^2)$ such that $X_T \succeq (-M, -M)$, for some $M > 0$, is in $\mathcal{A}_{nf}$ if and only if,

$$\langle X_T, Z_T \rangle := \mathbb{E}[X_T^0 Z_T^0 + X_T^1 Z_T^1] \leq 0, \quad \text{for every} \quad Z_T \in \mathcal{A}_{nf}^0, \tag{36}$$

By normalising, it is equivalent to require the validity of (36) for all $Z_T \in \mathcal{B}_{nf}$.

We thus have assembled all the ingredients for a proof of the numéraire-free version of the super-hedging theorem.

Proof of Theorem 1.5: The above discussion actually yields the following two-dimensional result which is more general than the one-dimensional statement of Theorem

1.5. Under the hypotheses of Theorem 1.5 consider a contingent claim $X_T = (X_T^0, X_T^1)$ which delivers X_T^0 many bonds and X_T^1 many stocks at time T. Then there is a self-financing, admissible (in the numéraire-free sense) strategy φ, starting with $(\varphi_0^0, \varphi_0^1) = (0,0)$ and ending with $(\varphi_T^0, \varphi_T^1) = (X_T^0, X_T^1)$ if and only if

$$\langle X_T, Z_T \rangle = \mathbb{E}_{\mathbb{P}}[X_T^0 Z_T^0 + X_T^1 Z_T^1] = \mathbb{E}_Q[X_T^0 + X_T^1 \widetilde{S}_t] \leq 0, \tag{37}$$

for every $Z_T \in \mathcal{B}_{nf}$. This is just statement (36), where $(\widetilde{S}, Q)$ is given by (34), i.e. Q is a probability measure, absolutely continuous w.r. to $\mathbb{P}$, and $\widetilde{S}$ is a (true) Q-martingale taking values in $[(1-\lambda)S, S]$.

We still need two observations. In (37) we may equivalently assume that the probability measure Q is actually *equivalent* to $\mathbb{P}$, i.e. the corresponding martingale Z satisfies $Z_T^0 > 0$ almost surely. Indeed, fix $Z_T \in \mathcal{B}_{nf}$ as in (37). By assumption (CPS^λ) (in the non-local sense) there is *some* $\overline{Z}_T \in \mathcal{B}_{nf}$ verifying $\overline{Z}_T^0 > 0$ almost surely. Note that $\langle X_T, \overline{Z}_T \rangle$ takes a finite value. For $0 < \mu < 1$ the convex combination $\mu \overline{Z}_T + (1-\mu) Z_T$ is in $\mathcal{B}_{nf}$ and still satisfies the strict positivity condition. Sending μ to zero we see that in (37) we may assume w.l.g. that Z_T^0 is a.s. strictly positive.

A second remark pertains to the initial endowment $(\varphi_0^0, \varphi_0^1)$ which in (37) we have normalised to $(0,0)$. If we replace $(0,0)$ by an arbitrary pair $(X_0^0, X_0^1) \in \mathbb{R}^2$ then (37) trivially translates to the equivalence of the following two statements for a contingent claim (X_T^0, X_T^1) verifying

$$V_T(X_T^0, X_T^1) \geq -M(1 + S_T), \quad \text{for some} \quad M > 0.$$

(*i*) There is a self-financing, admissible (in the numéraire-free sense) trading strategy $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ such that

$$\varphi_0 = (X_0^0, X_0^1) \quad \text{and} \quad \varphi_T = (X_T^0, X_T^1)$$

(*ii*) For every consistent price system, i.e. each probability measure Q, equivalent to $\mathbb{P}$ such that there is a martingale $\widetilde{S}$ under Q, taking its values in the bid-ask spread $[(1-\lambda)S, S]$, we have

$$\mathbb{E}_Q[(X_T^0 - X_0^0) + (X_T^1 - X_0^1)\widetilde{S}_T] \leq 0.$$

Specialising to the case where X_T^0 and X_T^1 is equal to zero we obtain the assertion of Theorem 1.5. ∎

5 The proof of Theorem 1.4

We now deduce the numéraire-based super-replication theorem from its numéraire-free counterpart.

(*i*) $\Rightarrow$ (*ii*) This is the easy implication. Suppose that X_T and $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ are given as in (*i*) of Theorem 1.4. Let $(\widetilde{S}, Q)$ be a consistent local price system. By Proposition 1.6 in [S 13], the process $\widetilde{V}_t = (\varphi_t^0 + \varphi_t^1 \widetilde{S}_t)_{0 \leq t \leq T}$ is an optional strong super-martingale under Q which implies (6).

$(ii) \Rightarrow (i)$ Conversely, let $X_T \geq -M$ be as in the statement of Theorem 1.4 and suppose that (ii) holds true. Define the $[0,T] \cup \{\infty\}$-valued stopping time τ_n by

$$\tau_n = \inf\{t : S_t \geq n\}.$$

Also define

$$X_T^n = \begin{cases} X_T, & \text{on} \quad \{\tau_n = \infty\}, \\ -M, & \text{on} \quad \{\tau_n \leq T\}, \end{cases}$$

so that $(X_T^n)_{n=1}^\infty$ is $\mathcal{F}_{\tau_n}$-measurable and increases a.s. to X_T.

Let $0 < \lambda_n < \lambda$ be a sequence of reals increasing to λ.

For fixed $n \in \mathbb{N}$ we may apply Theorem 1.5 to the stopped process S^{τ_n}, the random variable X_T^n and transaction costs λ_n. To verify that the conditions of Theorem 1.5 are indeed satisfied note that under the hypotheses of Theorem 1.4, for every $0 < \lambda' < 1$, condition $(CPS^{\lambda'})$ is satisfied for S in a local sense and therefore – by stopping – also for S^{τ_n}. By Proposition 6.1 below we conclude that $(CPS^{\lambda'})$ is, in fact, satisfied in a non-local sense for the process S^{τ_n} as required by Theorem 1.5.

Next we show that condition (ii) of Theorem 1.5 is satisfied for the process S^{τ_n} and transaction costs λ_n. Indeed, fix n and let $Q \sim \mathbb{P}$ be such that there is a Q-martingale $\widetilde{S} = (\widetilde{S}_t)_{0 \leq t \leq T}$ taking its values in $[(1-\lambda_n)S^{\tau_n}, S^{\tau_n}]$ and associate the martingales Z^0, Z^1 to $(Q, \widetilde{S})$.

We may concatenate this λ_n-consistent price systems for S^{τ_n} to a λ-consistent local price system $\overline{Z} = (\overline{Z}_t^0, \overline{Z}_t^1)_{0 \leq t \leq T}$ for the process S.

Here are the details. Fix $0 < \lambda' < \frac{\lambda - \lambda_n}{2}$. By the assumption of Theorem 1.4 there is a λ'-consistent local price system $\check{Z} = (\check{Z}_t^0, \check{Z}_t^1)_{0 \leq t \leq T}$ for S. Define $\overline{Z}$ by

$$\overline{Z}_t^0 = \begin{cases} Z_t^0, & 0 \leq t \leq \tau_n \\ \check{Z}_t^0 \frac{Z_\tau^0}{\check{Z}_\tau^0}, & \tau_n \leq \tau \leq T, \end{cases}$$

$$\overline{Z}_t^1 = \begin{cases} (1-\lambda')Z_t^1, & 0 \leq t \leq \tau_n \\ (1-\lambda')\check{Z}_t^1 \frac{Z_\tau^1}{\check{Z}_\tau^1}, & \tau_n \leq \tau \leq T. \end{cases}$$

Clearly, $\overline{Z}^0$ (resp.$\overline{Z}^1$) is an $\mathbb{R}_+$-valued martingale (resp. local martingale) under $\mathbb{P}$ and $\frac{d\overline{Q}}{d\mathbb{P}} = \overline{Z}_T^0$ defines a probability measure on $\mathcal{F}$ equivalent to $\mathbb{P}$. To show that $\frac{\overline{Z}^1}{\overline{Z}^0}$ takes its values in $[(1-\lambda)S, S]$ note that, for $0 \leq t \leq \tau_n$, the quotient $\frac{\overline{Z}_t^1}{\overline{Z}_t^0}$ lies in $[(1-\lambda_n)(1-\lambda')S_t, (1-\lambda')S_t]$. For $\tau_n \leq t \leq T$ we still obtain that $\frac{\overline{Z}_t^1}{\overline{Z}_t^0}$ lies in $[(1-\lambda_n)(1-\lambda')^2 S_t, \frac{1-\lambda'}{1-\lambda'} S_t]$ which is contained in $[(1-\lambda)S_t, S_t]$ as $\lambda' < \frac{\lambda - \lambda_n}{2}$. By assumption (ii) of Theorem 1.4 we conclude that

$$\begin{aligned} \mathbb{E}_Q[X_T^n] &= \mathbb{E}_{\overline{Q}}[X_T^n] \\ &\leq \mathbb{E}_{\overline{Q}}[X_T] \leq X_0. \end{aligned}$$

Hence we may apply Theorem 1.5 to conclude that there is a λ_n-self-financing trading strategy $\varphi^n = (\varphi_t^{0,n}, \varphi_t^{1,n})_{0 \leq t \leq T}$ for S such that $\varphi_0^n = (X_0, 0)$ and $\varphi_T^n = \varphi_{\tau_n}^n = (X_T^n, 0)$ and which is M-admissible in the sense of (7). Applying Theorem 2.5 in [S 13] to the case

$x = M$ and $y = 0$ we may conclude that each φ^n is, in fact, M-admissible in the sense of (5). Finally, we apply Theorem 3.4 and the subsequent Remark 3.5, which yields the desired self-financing trading strategy φ as a limit of $(\varphi^n)_{n=1}^{\infty}$. This strategy φ has the properties stated in Theorem 1.4 (i). ■

6 Appendix

The following proposition seems to be a well-known folklore type result. As we are unable to give a reference we provide a proof.

Proposition 6.1. *Let $(X_t)_{0\le t\le T}$ be an $\mathbb{R}_+$-valued local martingale, τ a stopping time, and $C > 0$ a constant such that $X_t \le C$, for $0 \le t < \tau$. Then the stopped process X^τ is a martingale.*

Proof. It follows from Fatou's lemma and the boundedness from below that X is a supermartingale. Hence it will suffice to show that

$$\mathbb{E}[X_\tau] = X_0. \tag{38}$$

By hypothesis there is a sequence $(\sigma_k)_{k=1}^{\infty}$ of $[0, T]\cup\{\infty\}$-valued stopping times, increasing to ∞, such that

$$\mathbb{E}[X_{\sigma_k\wedge\tau}] = X_0, \quad \text{for} \quad k \ge 1.$$

As $\lim_{k\to\infty} \mathbb{P}[\sigma_k < \tau] = 0$ and X_{σ_k} is bounded by C on $\{\sigma_k < \tau\}$ we obtain from the monotone convergence theorem:

$$\begin{aligned} X_0 &= \lim_{k\to\infty} \mathbb{E}[X_\tau \mathbb{1}_{\{\sigma_k\ge\tau\}} + X_{\sigma_k}\mathbb{1}_{\{\sigma_k<\tau\}}] \\ &= \mathbb{E}[X_\tau]. \end{aligned}$$

This gives (38). □

Acknowledgement. *We thank Irene Klein for her insistency on the topic as well as fruitful discussions on the proof of Theorem 1.4, and Christoph Czichowsky for his advise and careful reading of the paper. We also thank an anonymous referee for a careful reading of the paper and helpful remarks.*

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