Shadow prices, fractional Brownian motion, and portfolio optimisation under transaction costs

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Abstract

The present paper accomplishes a major step towards a reconciliation of two conflicting approaches in mathematical finance: on the one hand, the mainstream approach based on the notion of *no arbitrage* (Black, Merton & Scholes); and on the other hand, the consideration of non-semimartingale price processes, the archetype of which being *fractional Brownian motion* (Mandelbrot). Imposing (arbitrarily small) proportional transaction costs and considering logarithmic utility optimisers, we are able to show the existence of a semimartingale, frictionless *shadow price process* for an exponential fractional Brownian financial market.

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1 Introduction

The classical framework of mathematical finance is given by *frictionless financial markets*, where at each time t arbitrary amounts of stock can be bought and sold at the *same* price S_t. Here, the mathematical structure of maximisation of expected utility essentially implies that an optimal trading strategy only exists, if the discounted price processes $S = (S_t)_{0\leq t\leq T}$ of the underlying financial instruments are *semimartingales*, that is, stochastic processes which are "good integrators" (see [1, 27, 24]). This rules out non-semimartingale models based on *fractional Brownian motion* proposed by Mandelbrot [28].

While fractional models provide arbitrage opportunities for frictionless trading, Guasoni [16] proved that they are arbitrage-free as soon as *proportional transaction costs* are taken into account. This allows to use these models as price processes for portfolio optimisation under transaction costs, as illustrated by Guasoni [15]. Studying the primal problem, this author shows the existence of an optimal trading strategy.

In this paper, we continue the analysis of the existence of a so-called *shadow price* for portfolio optimisation under transaction costs. That is, a semimartingale price process $\hat{S} = (\hat{S}_t)_{0\leq t\leq T}$ taking values in the bid-ask spread such that frictionless trading for that price process leads to the same optimal trading strategy and utility as in the original problem under transaction costs. See [11, 10] for references and an overview of the literature. For utility functions $U\colon (0,\infty) \to \mathbb{R}$ on the positive half-line (satisfying the condition of reasonable asymptotic elasticity), we show that the condition of "two-way crossing" (TWC) (Definition 2.2) is sufficient for the existence of a shadow price (provided the indirect utility is finite) by using duality results established in [11, 12]. The two-way crossing condition, introduced by Bender [3], has the intuitive interpretation that a process "cannot move a.s. in a fixed direction". This property has been established for the fractional Black-Scholes model by Peyre [29]. Still in the case of the fractional Black-Scholes model, we obtain the finiteness of the indirect utility by establishing an estimate on the tail probabilities of the number of fluctuations of size $\delta > 0$ of fractional Brownian motion. This allows us to show that shadow prices exist for the fractional Black-Scholes model for *all* utility functions on the positive half-line (not even needing the condition of reasonable asymptotic elasticity), which is a fairly complete answer to this question. Though in this article, we only focus on the archetypical fractional Black-Scholes model, for the sake of simplicity, our results can likely also be applied to a much broader class of non-semimartingale price processes.

The shadow price $\hat{S}$ is given by an Itô process

$$\mathrm{d}\hat{S}_t = \hat{S}_t(\hat{\mu}_t \mathrm{d}t + \hat{\sigma}_t \mathrm{d}W_t), \quad 0 \leq t \leq T, \tag{1.1}$$

where $\hat{\mu} = (\hat{\mu}_t)_{0\leq t\leq T}$ and $\hat{\sigma} = (\hat{\sigma}_t)_{0\leq t\leq T}$ are predictable processes such that the solution to (1.1) is well-defined in the sense of Itô integration. For logarithmic utility, this implies that the optimal trading strategy is related to the coefficients of the Itô process (1.1) via

$$\hat{\pi}_t = \frac{\hat{\mu}_t}{\hat{\sigma}_t^2} = \frac{\hat{\varphi}^1_{t-}\hat{S}_t}{\hat{\varphi}^0_{t-} + \hat{\varphi}^1_{t-}\hat{S}_t}, \qquad 0 \leq t \leq T.$$

It is a special feature of logarithmic utility to allow for such a crisp relation between the "dual" variables $\hat{\mu}, \hat{\sigma}$ and the "primal" variables $(\hat{\varphi}^0, \hat{\varphi}^1)$. Indeed, only the logarithmic utility maximiser is myopic. This underlines the central importance of understanding the shadow price in the case of logarithmic utility, a case not covered by [10].

It is well known that the existence of a shadow price is related to the solution of a suitable dual problem; see [20, 9, 11, 12, 10]. Under transaction costs, this duality goes back to the pioneering work [7] of Cvitanić and Karatzas and has been subsequently extended to dynamic duality results [7, 8, 11, 12, 10] for utility functions on the positive half-line as well as static duality results [13, 4, 5, 6, 2] for (possibly) multi-variate utility functions.

To apply this duality in our setup, we need to ensure the existence of so-called λ-*consistent local martingale deflators*. These processes are used as dual variables similarly as equivalent martingale measures [22, 18, 23, 25] and local martingale deflators in the frictionless theory [21]. For this, we provide a local version of the fundamental theorem of asset pricing for continuous processes under small transaction costs of [17] using the condition (NOIA) of "no obvious immediate arbitrage" (see Definition 3.1).

The remainder of the article is organised as follows. We formulate the problem and state our main results in Section 2. Their proofs are given in Section 4. Section 3 recalls duality results and provides the local version of the fundamental theorem of asset pricing. In Appendix A, we establish the estimates of the tail probabilities of the fluctuations of fractional Brownian motion.

2 Main results

We consider a minimal financial market consisting of one riskless bond and one risky stock. The riskless asset is assumed to be normalised to one. Trading the risky asset incurs proportional transaction costs $\lambda \in (0,1)$. This means that one has to pay a (higher) ask price S_t when buying risky shares but only receives a lower bid price $(1-\lambda)S_t$ when selling them. Here, $S = (S_t)_{0\le t\le T}$ denotes a strictly positive, adapted, *continuous* stochastic process defined on some underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ with fixed finite time horizon $T \in (0,\infty)$, satisfying the usual assumptions of right-continuity and completeness. As usual, equalities and inequalities between random variables, resp. between stochastic processes, hold up to $\mathbb{P}$-nullsets, resp. up to $\mathbb{P}$-evanescent sets.

Trading strategies are modelled by $\mathbb{R}^2$-valued, càdlàg and adapted processes $\varphi = (\varphi^0_t, \varphi^1_t)_{0_-\le t\le T}$ of finite variation indexed by $[0_-, T] := \{0_-\} \cup [0,T]$, where φ^0_t and φ^1_t describe the holdings in the riskless and the risky asset, respectively, after rebalancing the portfolio at time t. As explained in [12] in more detail, using $[0_-, T]$ instead of $[0,T]$ as index set allows us to use càdlàg trading strategies. For any process $\psi = (\psi_t)_{0_-\le t\le T}$ of finite variation, we denote by $\psi =: \psi_{0_-} + \psi^\uparrow - \psi^\downarrow$ its Hahn-Jordan decomposition into two non-decreasing processes $\psi^\uparrow$ and $\psi^\downarrow$ starting at zero.

A trading strategy $\varphi = (\varphi^0_t, \varphi^1_t)_{0_-\le t\le T}$ is called *self-financing*, if

$$\int_s^t d\varphi^0_u \le -\int_s^t S_u d\varphi^{1,\uparrow}_u + \int_s^t (1-\lambda) S_u d\varphi^{1,\downarrow}_u \qquad \text{for all } 0_- \le s \le t \le T, \tag{2.1}$$

where the integrals can be defined pathwise as a Riemann-Stieltjes integrals.

A self-financing strategy $\varphi = (\varphi^0, \varphi^1)$ is called *admissible*, if it satisfies

$$V^{\mathrm{liq}}_t(\varphi) := \varphi^0_t + (\varphi^1_t)^+(1-\lambda)S_t - (\varphi^1_t)^- S_t \ge 0 \qquad \text{for all } 0 \le t \le T: \tag{2.2}$$

in this equation, V_t^{liq}, which is called the *liquidation value* at time t, corresponds to the amount that the trader would get if she decided to liquidate instantly her portfolio into cash at that time.

For $x > 0$, we denote by $\mathcal{A}(x)$ the set of all self-financing and admissible trading strategies under transaction costs λ starting from initial endowment $(\varphi^0_{0_-}, \varphi^1_{0_-}) = (x, 0)$. We consider an economic agent whose goal is to maximise her expected utility from terminal wealth

$$\mathbb{E}\big[U(V_T^{\mathrm{liq}}(\varphi))\big] \to \max!, \quad \varphi \in \mathcal{A}(x). \tag{2.3}$$

Here, $U\colon (0,\infty) \to \mathbb{R}$ denotes an increasing, strictly concave, continuously differentiable utility function, satisfying the Inada conditions

$$U'(0) := \lim_{x \searrow 0} U'(x) = \infty \quad \text{and} \quad U'(\infty) := \lim_{x \nearrow \infty} U'(x) = 0. \tag{2.4}$$

(If $\lim_{x\searrow 0} U(x) > -\infty$, we will implicitly extend U to $[0,\infty)$ by continuity).

In this paper, we continue the analysis of problem (2.3) by using the concept of a shadow price.

Definition 2.1. A semimartingale price process $\hat{S} = (\hat{S}_t)_{0\le t\le T}$ is called a *shadow price process*, if all the following conditions hold:

1. $\hat{S}$ is valued in the bid-ask spread $[(1-\lambda)S, S]$;
2. A solution $\hat{\vartheta} = (\hat{\vartheta}_t)_{0\le t\le T}$ to the frictionless utility maximisation problem

$$\mathbb{E}\left[U(x + \vartheta \cdot \hat{S}_T)\right] \to \max!, \quad \vartheta \in \mathcal{A}(x; \hat{S}) \tag{2.5}$$

   exists (in the sense of [25]), where $\mathcal{A}(x;\hat{S})$ denotes the set of all self-financing and admissible trading strategies $\vartheta = (\vartheta_t)_{0\le t\le T}$ for $\hat{S}$ without transaction costs. That is, $\hat{S}$-integrable (in the sense of Itô), predictable processes $\vartheta = (\vartheta_t)_{0\le t\le T}$ such that $X_t = x + \vartheta\cdot\hat{S}_t \ge 0$ for all $0 \le t \le T$.
3. An optimal trading strategy $\hat{\vartheta} = (\hat{\vartheta}_t)_{0\le t\le T}$ to the frictionless problem (2.5) coincides with (the left limit of) the holdings in stock $\hat{\varphi}^1_- = (\hat{\varphi}^1_{t-})_{0\le t\le T}$ of an optimal trading strategy to the utility maximisation problem (2.3) under transaction costs so that $x + \hat{\vartheta}\cdot\hat{S}_T = V_T^{\mathrm{liq}}(\hat{\varphi})$.

In Theorem 3.2 of [12], the existence of a shadow price for a continuous price process $S = (S_t)_{0\le t\le T}$ satisfying the condition (NUPBR) of "no unbounded profit with bounded risk" (without transaction costs) is established. The assumption of (NUPBR) implies that S has to be a semimartingale. Therefore, the result did not yet apply to price processes driven by fractional Brownian motion $B^H = (B^H_t)_{0\le t\le T}$ such as the *fractional Black-Scholes model*

$$S_t = \exp(\mu t + \sigma B^H_t), \qquad 0 \le t \le T, \tag{2.6}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$n, and $H \in (0,1)\setminus\{\frac{1}{2}\}$ denotes the Hurst parameter of the fractional Brownian motion B^H. In the present article, we combine a recent result of Peyre [29] with a strengthening of the existence result in Theorem 3.2 of [12] to fill this gap.

For this, we need a weaker no-arbitrage-type condition than (NUPBR) that is nevertheless in some sense stronger than stickiness. It turns out that the condition (TWC) of "two-way crossing" is the suitable one to work with.

Definition 2.2. Let $X = (X_t)_{0 \le t \le T}$ be a real-valued continuous stochastic process. For σ a finite stopping time, set

$$\sigma_+ := \inf\{t > \sigma \mid X_t - X_\sigma > 0\},$$
$$\sigma_- := \inf\{t > \sigma \mid X_t - X_\sigma < 0\}.$$

Then, we say that X satisfies the condition (TWC) of "two-way crossing", if $\sigma_+ = \sigma_-$ $\mathbb{P}$-a.s. for *all* finite stopping time σ.

The two-way crossing condition was introduced by Bender in [3] for the analysis of the condition of "no simple arbitrage" (without transaction costs), that is, no arbitrage by (finite) linear combinations of buy and hold strategies. Using it in the context of portfolio optimisation under transaction costs allows us to establish the following results. For better readability, their proofs are deferred to Section 4.

Theorem 2.3. *Fix a strictly positive continuous process $S = (S_t)_{0 \le t \le T}$ satisfying* (TWC) *and transaction costs $\lambda \in (0, 1)$. Let $U: (0, \infty) \to \mathbb{R}$ be an increasing, strictly concave, continuously differentiable utility function, satisfying the Inada conditions* (2.4)*, and having reasonable asymptotic elasticity, that is* $\limsup_{x \to \infty}(xU'(x) \,/\, U(x)) < 1$*, and suppose that*

$$u(x) := \sup_{\varphi \in \mathcal{A}(x)} \mathbb{E}\big[U(V_T^{\mathrm{liq}}(\varphi))\big] < \infty \tag{2.7}$$

for some $x > 0$.

Then, there exists an optimal trading strategy $\hat{\varphi} = (\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{0- \le t \le T}$ for (2.3)*, for which there exists a shadow price $\hat{S} = (\hat{S}_t)_{0 \le t \le T}$.*

The significance of the condition (TWC) in the above result is that it does not require S to be a semimartingale, as it holds for the fractional Black-Scholes model (2.6). This allows us to conclude the existence of a shadow price process for the fractional Black-Scholes model and utility functions that are bounded from above, like power utility $U(x) = \alpha^{-1}x^\alpha$ with risk aversion parameter $\alpha < 0$. For utility functions $U: (0, \infty) \to \mathbb{R}$ that are not bounded from above like logarithmic utility $U(x) = \log(x)$ or power utility $U(x) = \alpha^{-1}x^\alpha$ with risk aversion parameter $\alpha \in (0, 1)$, it remains to show that the indirect utility (2.7) is finite in order to apply Theorem 2.3. We do this below by controlling the number of fluctuations of size $\delta > 0$ of fractional Brownian motion, which allows us to obtain the following complete answer to the question whether or not there exists a shadow price for the fractional Black-Scholes model.

Theorem 2.4. *Fix the fractional Black-Scholes model* (2.6) *and transaction costs $\lambda \in (0, 1)$. Let $U: (0, \infty) \to \mathbb{R}$ be an increasing, strictly concave, continuously differentiable utility function, satisfying the Inada conditions* (2.4)*.*

Then,

$$u(x) := \sup_{\varphi \in \mathcal{A}(x)} \mathbb{E}\big[U(V_T^{\mathrm{liq}}(\varphi))\big] < \infty \tag{2.8}$$

for all $x > 0$, and there exists an optimal trading strategy $\hat{\varphi} = (\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{0- \le t \le T}$ for (2.3)*, for which there exists a shadow price $\hat{S} = (\hat{S}_t)_{0 \le t \le T}$.*

It is worth pointing out that the above theorem does not need a condition on the asymptotic elasticity. The finiteness of the indirect utility function and the existence of an optimal trading strategy even holds for linear utility functions. The existence of a shadow price, however, needs that the Inada conditions are satisfied.

3 Local duality theory

In this section, we establish a local version of the fundamental theorem of asset pricing for continuous processes under small transaction costs; compare [17]. To that end, we recall the following notions.

A λ-*consistent price system* is a pair of stochastic processes $Z = (Z_t^0, Z_t^1)_{0 \le t \le T}$ such that $Z^0 = (Z_t^0)_t$ is the density process of an equivalent measure $\mathbb{Q} \sim \mathbb{P}$ under which $(\tilde{S}_t)_t := (Z_t^1 \,/\, Z_t^0)_t$ is a local martingale, and such that $\tilde{S}$ evolves in the bid-ask spread $[(1-\lambda)S, S]$. Requiring that $\tilde{S}$ is a local martingale under $\mathbb{Q}$ is tantamount to requiring the product $Z^1 = Z^0 \tilde{S}$ to be a local martingale under $\mathbb{P}$. Under transaction costs, λ-consistent price systems ensure "absence of arbitrage" in the sense of "no free lunch with vanishing risk" (NFLVR) similarly as equivalent local martingale measures in the frictionless case; see, for example, [19] and the references therein. A λ-*consistent local martingale deflator* is a pair of strictly positive local martingales $Z = (Z_t^0, Z_t^1)_{0 \le t \le T}$ such that $\tilde{S} := Z^1 \,/\, Z^0$ is evolving within the bid-ask spread $[(1-\lambda)S, S]$ and $\mathbb{E}[Z_0^0] = 1$. We denote the set of all λ-consistent local martingale deflators by $\mathcal{Z}$. Note that, if $(\tau_n)_{n=1}^\infty$ is a localising sequence of stopping times such that the stopped process $(Z^0)^{\tau_n} = (Z^0_{\tau_n \wedge t})_{0 \le t \le T}$ is a true martingale, then $Z^{\tau_n} = (Z^0_{\tau_n \wedge t}, Z^1_{\tau_n \wedge t})_{0 \le t \le T}$ is a λ-consistent price system for the stopped process $S^{\tau_n} = (S_{\tau_n \wedge t})_{0 \le t \le T}$. In this sense, the condition that S admits a λ-consistent local martingale deflator is indeed the local version of the condition that S admits a λ-consistent price system.

Moreover, we use the subsequent no arbitrage concepts.

Definition 3.1. Let $S = (S_t)_{0 \le t \le T}$ be a strictly positive, continuous process. We say that S allows for an "obvious arbitrage", if there are $\alpha > 0$ and $[0,T] \cup \{\infty\}$-valued stopping times $\sigma \le \tau$ with $\mathbb{P}[\sigma < \infty] = \mathbb{P}[\tau < \infty] > 0$ such that either

$$S_\tau \ge (1+\alpha) S_\sigma, \quad \text{a.s. on } \{\sigma < \infty\}, \tag{a}$$

or

$$S_\tau \le (1+\alpha)^{-1} S_\sigma, \quad \text{a.s. on } \{\sigma < \infty\}. \tag{b}$$

In the case of (b), we also assume that $(S_t)_{\sigma \le t \le \tau}$ is uniformly bounded.

We say that S allows for an "obvious immediate arbitrage", if, in addition, we have

$$S_t \ge S_\sigma, \quad \text{for all } t \in [\![\sigma, \tau]\!], \text{ a.s. on } \{\sigma < \infty\}, \tag{a'}$$

or

$$S_t \le S_\sigma, \quad \text{for all } t \in [\![\sigma, \tau]\!], \text{ a.s. on } \{\sigma < \infty\}. \tag{b'}$$

We say that S satisfies the condition (NOA) (respectively, (NOIA)) of "no obvious arbitrage" (respectively, "no obvious immediate arbitrage"), if no such opportunity exists. The name "obvious arbitrage" comes from the fact that it is indeed obvious how to make an arbitrage if (NOA) fails, provided the transaction costs λ are smaller than α.

Using the above, we obtain the following slight strengthening of Theorem 1 of [17].

Theorem 3.2. *Let $S = (S_t)_{0 \le t \le T}$ be a strictly positive, continuous process. Then, the following assertions are equivalent.*

(i) *Locally, S has no obvious immediate arbitrage, i.e. satisfies* (NOIA).

(ii) *Locally, S has no obvious arbitrage, i.e. satisfies* (NOA).

(iii) *Locally, S admits a μ-consistent price system for all $\mu \in (0,1)$.*[1]

(iv) *For each $\mu \in (0,1)$, there exists a μ-consistent local martingale deflator for S.*

Proof. Obviously, we have (ii) $\Rightarrow$ (i). The equivalent (ii) $\Leftrightarrow$ (iii) follows directly from Theorem 1 of [17]. As explained above, (iv) implies (iii).

The converse (iii) $\Rightarrow$ (iv) follows by exploiting that (iii) asserts locally the existence of a μ-consistent price system for *each* $0 < \mu < 1$. Indeed, fix $0 < \mu < 1$ and a localising sequence $(\tau_n)_{n=1}^{\infty}$ of stopping times. Let $\bar{Z} = (\bar{Z}^0_{\tau_n \wedge t}, \bar{Z}^1_{\tau_n \wedge t})_{0 \le t \le T}$ be a $\bar{\mu}$-consistent price system for $S^{\tau_n} := (S_{\tau_n \wedge t})_{0 \le t \le T}$ with $0 < \bar{\mu} < \mu$. Then, we can extend $\bar{Z}$ to a $\tilde{\mu}$-consistent price system $\tilde{Z} = (\tilde{Z}^0_{\tau_{n+1} \wedge t}, \tilde{Z}^1_{\tau_{n+1} \wedge t})_{0 \le t \le T}$ for $S^{\tau_{n+1}} = (S_{\tau_{n+1} \wedge t})_{0 \le t \le T}$ with $0 < \bar{\mu} < \tilde{\mu} < \mu$ by setting

$$\tilde{Z}^0_t = \begin{cases} \bar{Z}^0_t & \text{for } 0 \le t < \tau_n, \\ \frac{\bar{Z}^0_{\tau_n}}{\check{Z}^0_{\tau_n}} \check{Z}^0_{\tau_{n+1} \wedge t} & \text{for } \tau_n \le t \le T, \end{cases}$$
$$\tilde{Z}^1_t = \begin{cases} (1 - \check{\mu}) \bar{Z}^1_t & \text{for } 0 \le t < \tau_n, \\ (1 - \check{\mu}) \frac{\bar{Z}^1_{\tau_n}}{\check{Z}^1_{\tau_n}} \check{Z}^1_{\tau_{n+1} \wedge t} & \text{for } \tau_n \le t \le T, \end{cases}$$

where $\check{Z} = (\check{Z}^0_{\tau_{n+1} \wedge t}, \check{Z}^1_{\tau_{n+1} \wedge t})_{0 \le t \le T}$ is a $\check{\mu}$-consistent price system for $S^{\tau_{n+1}} = (S_{\tau_{n+1} \wedge t})_{0 \le t \le T}$ with $0 < \check{\mu} < \frac{1}{2}(\tilde{\mu} - \bar{\mu})$. Repeating this extension allows us to establish the existence of a μ-consistent local martingale deflator.

(i) $\Rightarrow$ (iii): As (iii) is a local property, we may assume that S satisfies (NOIA).

To prove (iii), we do a similar construction as in the proof of Proposition 1 in [17]. We suppose in the sequel that the reader is familiar with the aforementioned proof.

Define the stopping time $\bar{\varrho}_1$ by

$$\bar{\varrho}_1 := \inf \left\{ t > 0 \;\middle|\; \frac{S_t}{S_0} \ge 1 + \mu \text{ or } \frac{S_t}{S_0} \le (1+\mu)^{-1} \right\}.$$

Define the sets $\bar{A}^+_1$, $\bar{A}^-_1$ and $\bar{A}^0_1$ as

$$\begin{aligned} \bar{A}^+_1 &:= \{\bar{\varrho}_1 < \infty \text{ and } S_{\bar{\varrho}_1} = S_0 \times (1+\mu)\}, \\ \bar{A}^-_1 &:= \{\bar{\varrho}_1 < \infty \text{ and } S_{\bar{\varrho}_1} = S_0 \,/\, (1+\mu)\}, \\ \bar{A}^0_1 &:= \{\bar{\varrho}_1 = \infty\}. \end{aligned}$$

It was observed in [17] that the assumption (NOA) rules out the case $\mathbb{P}[\bar{A}^+_1] = 1$ and $\mathbb{P}[\bar{A}^-_1] = 1$. But under the present weaker assumption (NOIA) we cannot a priori exclude the above possibilities. To refine the argument from [17] in order to apply to the present setting, we distinguish two cases. Either we have $\mathbb{P}[\bar{A}^+_1] < 1$ and $\mathbb{P}[\bar{A}^-_1] < 1$, or one of the probabilities $\mathbb{P}[\bar{A}^+_1]$ or $\mathbb{P}[\bar{A}^-_1]$ equals one.

[1] Here we mean that there exists *one* localised version of S, not depending on μ, which admits a μ-consistent price system for *all* $\mu \in (0,1)$.

In the first case, we let $\varrho_1 := \bar{\varrho}_1$ and proceed exactly as in the proof of Proposition 1 in [17] to complete the first inductive step.

For the second case, we assume without loss of generality that $\mathbb{P}[\bar{A}_1^+] = 1$, the other case can be treated in an analogous way.

Define the real number $\beta \leq 1$ as the essential infimum of the random variable $\min_{0\leq t\leq \bar{\varrho}_1}(S_t \,/\, S_0)$. We must have $\beta < 1$, otherwise the pair $(0, \bar{\varrho}_1)$ would define an immediate obvious arbitrage. We also have the obvious inequality $\beta \geq (1+\mu)^{-1}$.

We define for $1 > \gamma \geq \beta$ the stopping time

$$\bar{\varrho}_1^\gamma := \inf\left\{ t > 0 \,\middle|\, \frac{S_t}{S_0} \geq 1+\mu \text{ or } \frac{S_t}{S_0} \leq \gamma \right\}.$$

Defining $\bar{A}_1^{\gamma,+} := \{S_{\bar{\varrho}_1^\gamma} = (1+\mu)S_0\}$ and $\bar{A}_1^{\gamma,-} := \{S_{\bar{\varrho}_1^\gamma} = \gamma S_0\}$, we find an almost surely partition of $\bar{A}_1^+$ into the sets $\bar{A}_1^{\gamma,+}$ and $\bar{A}_1^{\gamma,-}$. Clearly $\mathbb{P}[\bar{A}_1^{\gamma,-}] > 0$, for $1 > \gamma > \beta$. We claim that

$$\lim_{\gamma \searrow \beta} \mathbb{P}\big[\bar{A}_1^{\gamma,-}\big] = 0.$$

Indeed, supposing that this limit were positive, we again could find an obvious immediate arbitrage, as in this case we have that $\mathbb{P}[\bar{A}_1^{\beta,-}] > 0$. Hence, the pair of stopping times

$$\begin{aligned}
\sigma &:= \bar{\varrho}_1^\beta \mathbb{1}_{\{S_{\bar{\varrho}_1^\beta} = \beta S_0\}} + \infty \mathbb{1}_{\{S_{\bar{\varrho}_1^\beta} = (1+\mu)S_0\}} \\
\tau &:= \bar{\varrho}_1 \mathbb{1}_{\{S_{\bar{\varrho}_1^\beta} = \beta S_0\}} + \infty \mathbb{1}_{\{S_{\bar{\varrho}_1^\beta} = (1+\mu)S_0\}}
\end{aligned}$$

would define an obvious immediate arbitrage, which is contrary to our assumption.

We thus may find $1 > \gamma > \beta$ such that

$$0 < \mathbb{P}\big[\bar{A}_1^{\gamma,-}\big] < \frac{1}{2}. \tag{3.1}$$

After having found this value of γ we can define the stopping time ϱ_1 in its final form as

$$\varrho_1 := \bar{\varrho}_1^\gamma.$$

Next, Ω can be decomposed into the partition $\bar{A}_1^{\gamma,+} \cup \bar{A}_1^{\gamma,-}$, which is made of two sets of positive measure. We will denote $\bar{A}_1^{\gamma,+} =: A_1^+$, resp. $\bar{A}_1^{\gamma,-} =: A_1^-$. As in the proof of Proposition 1 in [17], we define a probability measure $\mathbb{Q}_1$ on $\mathcal{F}_{\varrho_1}$ by letting $d\mathbb{Q}_1/d\mathbb{P}$ be constant on these two sets, where the constants are chosen such that

$$\mathbb{Q}_1[A_1^+] = \frac{1-\gamma}{1+\mu-\gamma} \quad \text{and} \quad \mathbb{Q}_1[A_1^-] = \frac{\mu}{1+\mu-\gamma}.$$

We then may define the $\mathbb{Q}_1$-martingale $(\tilde{S}_t)_{0\leq t\leq \varrho_1}$ by

$$\tilde{S}_t := E_{\mathbb{Q}_1}[S_{\varrho_1} | \mathcal{F}_t], \qquad 0 \leq t \leq \varrho_1,$$

to obtain a process remaining in the interval $[\gamma S_0, (1+\mu)S_0]$. The above weights for $\mathbb{Q}_1$ were chosen in such a way to obtain $\tilde{S}_0 = E_{\mathbb{Q}_1}[S_{\varrho_1}] = S_0$. This completes the first inductive step similarly as in the proof of Proposition 1 of [17].

Summing up, we obtained ϱ_1, $\mathbb{Q}_1$ and $(\tilde{S}_t)_{0\le t\le\varrho_1}$ precisely as in the proof of Proposition 1 in [17] with the following additional possibility: it may happen that ϱ_1 does not stop when S_t first hits $(1+\mu)S_0$ or $(1+\mu)^{-1}S_0$, but rather when S_t first hits $(1+\mu)S_0$ or γS_0, for some $\gamma \in \big((1+\mu)^{-1}, 1\big)$. In the case we have $\mathbb{P}[A_1^0] = 0$, we made sure that $\mathbb{P}[A_1^-] < \frac{1}{2}$, i.e., we have a control on the probability of $\{S_{\varrho_1} = \gamma S_0\}$.

We now proceed as in the proof of Proposition 1 in [17] with the inductive construction of $\varrho_n, \mathbb{Q}_n$ and $(\tilde{S}_t)_{0\le t\le\varrho_n}$. The new ingredient is that again we have to take care (conditionally on $\mathcal{F}_{\varrho_{n-1}}$) of the additional possibility $\mathbb{P}[A_n^+] = 1$ or $\mathbb{P}[A_n^-] = 1$. Supposing again without loss of generality that we have the first case, we deal with this possibility precisely as for $n = 1$ above, but now we make sure that $\mathbb{P}[A_n^-] < 2^{-n}$ instead of $\mathbb{P}[A_1^-] = \mathbb{P}[\bar{A}_1^{\gamma,-}] < \frac{1}{2}$ above.

This completes the inductive step and we obtain, for each $n \in \mathbb{N}$, an equivalent probability measure $\mathbb{Q}_n$ on $\mathcal{F}_{\varrho_n}$ and a $\mathbb{Q}_n$-martingale $(\tilde{S}_t)_{0\le t\le\varrho_n}$ taking values in the bid-ask spread $\big([(1+\mu)^{-1}S_t, (1+\mu)S_t]\big)_{0\le t\le\varrho_n}$. We note in passing that there is no loss of generality in having chosen this normalization of the bid-ask spread instead of the usual normalization $[(1-\mu')S', S']$ by passing from S to $S' = (1+\mu)S$ and from μ to $\mu' = 1-(1+\mu)^{-2}$.

There is one more thing to check to complete the proof of *(iii)* : we have to show that the stopping times $(\varrho_n)_{n=1}^\infty$ increase almost surely to infinity. This is verified in the following way: suppose that $(\varrho_n)_{n=1}^\infty$ remains bounded on a set of positive probability. On this set we must have that $S_{\varrho_{n+1}} / S_{\varrho_n}$ equals $(1+\mu)$ or $(1+\mu)^{-1}$, except for possibly finitely many n's. Indeed, the above requirement $\mathbb{P}[A_n^-] < 2^{-n}$ makes sure that a.s. the novel possibility of moving by a value different from $(1+\mu)$ or $(1+\mu)^{-1}$ can only happen finitely many times. Therefore we may, as before Proposition 1 in [17], conclude from the uniform continuity and strict positivity of the trajectories of S on $[0,T]$ that ϱ_n increases a.s. to infinity which completes the proof of (iii). □

4 Proofs of the main results

Proof of Theorem 2.3. Like in the proof of Theorem 3.2 of [12], we show the existence of a shadow price by duality. To that end, we observe that, for continuous price processes $S = (S_t)_{0\le t\le T}$, the condition (TWC) of "two-way crossing" implies the no obvious immediate arbitrage condition (NOIA) locally. It follows by part (iv) of Theorem 3.2 that there exists a μ-consistent local martingale deflator for S for each $\mu \in (0,1)$. Therefore, the assumptions of Theorem 2.10 of [12] are satisfied; and thus there exists an optimal trading strategy $\hat{\varphi} = (\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{0-\le t\le T}$ that attains the supremum in (2.7), as well as a so-called "λ-consistent supermartingale deflator"[2] $\hat{Y} = (\hat{Y}_t^0, \hat{Y}_t^1)_{0\le t\le T}$ given by an optimiser to a suitable dual problem, such that the process

$$\hat{Y}^0(\hat{y})\hat{\varphi}^0(x) + \hat{Y}^1(\hat{y})\hat{\varphi}^1(x) = (\hat{Y}_t^0(\hat{y})\hat{\varphi}_t^0(x) + \hat{Y}_t^1(\hat{y})\hat{\varphi}_t^1(x))_{0\le t\le T}$$

[2]The set $\mathcal{B}(y)$ of all λ-*consistent supermartingale deflators* consists of all pairs of non-negative càdlàg supermartingales $Y = (Y_t^0, Y_t^1)_{0\le t\le T}$ such that $\mathbb{E}[Y_0^0] = y$, $Y^1 = Y^0\tilde{S}$ for some $[(1-\lambda)S, S]$-valued process $\tilde{S} = (\tilde{S}_t)_{0\le t\le T}$, and $Y^0(\varphi^0 + \varphi^1\tilde{S}) = Y^0\varphi^0 + Y^1\varphi^1$ is a non-negative càdlàg supermartingale for all $\varphi \in \mathcal{A}(1)$. Note that $y\mathcal{Z} \subseteq \mathcal{B}(y)$ for $y > 0$ by Proposition 2.6 of [12].

is a martingale. By Theorem 2.10 of [12], this martingale property implies that, for $\hat{S} := \hat{Y}^1 / \hat{Y}^0$, we have

$$\{\mathrm{d}\hat{\varphi}^1 > 0\} \subseteq \{\hat{S} = S\}, \quad \{\mathrm{d}\hat{\varphi}^1 < 0\} \subseteq \{\hat{S} = (1-\lambda)S\}. \tag{4.1}$$

To obtain that $\hat{S} = (\hat{S}_t)_{0 \le t \le T}$ is a shadow price for problem (2.7) (in the sense of Definition 2.1), it is by Proposition 3.7 of [11] sufficient to show that the dual optimiser $\hat{Y} = (\hat{Y}_t^0, \hat{Y}_t^1)_{0 \le t \le T}$ is a local martingale. By Proposition 3.3 of [12], this follows as soon as we have that the liquidation value

$$V_t^{\mathrm{liq}}(\hat{\varphi}) := \hat{\varphi}_t^0 + (\hat{\varphi}_t^1)^+(1-\lambda)S_t - (\hat{\varphi}_t^1)^- S_t$$

is strictly positive almost surely for all $t \in [0, T]$, i.e.,

$$\inf_{0 \le t \le T} V_t^{\mathrm{liq}}(\hat{\varphi}) > 0, \quad \text{a.s.}. \tag{4.2}$$

To show (4.2), we argue by contradiction. Define

$$\sigma_\varepsilon := \inf\{t \in [0, T] \mid V_t^{\mathrm{liq}}(\hat{\varphi}) \le \varepsilon\}, \tag{4.3}$$

and let $\sigma := \lim_{\varepsilon \searrow 0} \sigma_\varepsilon$. We have to show that $\sigma = \infty$, almost surely. Suppose that $\mathbb{P}[\sigma < \infty] > 0$ and let us work towards a contradiction.

First observe that $V_\sigma^{\mathrm{liq}}(\hat{\varphi}) = 0$ on $\{\sigma < \infty\}$. Indeed, as $(V_t^{\mathrm{liq}}(\hat{\varphi}))_{0 \le t \le T}$ is càdlàg, we have that $0 \le V_\sigma^{\mathrm{liq}}(\hat{\varphi}) \le \lim_{\varepsilon \searrow 0} V_{\sigma_\varepsilon}^{\mathrm{liq}}(\hat{\varphi}) \le 0$ on the set $\{\sigma < \infty\}$.

So suppose that $V_\sigma^{\mathrm{liq}}(\hat{\varphi}) = 0$ on the set $\{\sigma < \infty\}$ with $\mathbb{P}[\sigma < \infty] > 0$. We may and do assume that S "moves immediately after σ", i.e., $\sigma = \inf\{t > \sigma \mid S_t \ne S_\sigma\}$: indeed, we may replace σ on $\{\sigma < \infty\}$ by the stopping time $\sigma_+ = \sigma_-$ (the equality "$\sigma_+ = \sigma_-$" coming from the (TWC) assumption). As $V_T^{\mathrm{liq}}(\hat{\varphi}) > 0$ a.s., we have $\sigma_+ < T$ on $\{\sigma < \infty\}$.

We shall repeatedly use the fact established in Theorem 2.10 in [12] that the process

$$\hat{V} := (\hat{\varphi}_t^0 \hat{Y}_t^0 + \hat{\varphi}_t^1 \hat{Y}_t^1)_{0 \le t \le T}$$

is a uniformly integrable $\mathbb{P}$-martingale satisfying $\hat{V}_T > 0$ almost surely. This implies that $\hat{\varphi}_\sigma^1 \ne 0$ a.s. on $\{\sigma < \infty\}$. Indeed, otherwise $\hat{V}_\sigma = \hat{Y}_\sigma^0 V_\sigma^{\mathrm{liq}}(\hat{\varphi}) = 0$ on $\{\sigma < \infty\}$. As $\hat{V}$ is a uniformly integrable martingale with strictly positive terminal value $\hat{V}_T > 0$, we arrive at the desired contradiction.

We consider here only the case that $\hat{\varphi}_\sigma^1 > 0$ on $\{\sigma < \infty\}$ almost surely: the case $\hat{\varphi}_\sigma^1 < 0$ with strictly positive probability on $\{\sigma < \infty\}$ can be dealt with in an analogous way. We show that we cannot have $\hat{S}_\sigma = (1-\lambda)S_\sigma$ with strictly positive probability on $\{\sigma < \infty\}$. Indeed, this again would imply that $\hat{V}_\sigma = \hat{Y}_\sigma^0 V_\sigma^{\mathrm{liq}}(\hat{\varphi}) = 0$ on this set which yields a contradiction as in the previous paragraph.

Hence, we assume that $\hat{S}_\sigma > (1-\lambda)S_\sigma$ on $\{\sigma < \infty\}$. This implies by (4.1), see also Theorem 3.5 in [10] and formula (249) in [30], that the utility-optimising agent defined by $\hat{\varphi}$ cannot sell stock at time σ as well as for some time after σ, as S is continuous and $\hat{S}$ càdlàg. Note, however, that the optimising agent may very well buy stock. But, we shall see that this is not to her advantage.

Define the stopping time ϱ_n as the first time after σ when one of the following events happens:

(i) $\hat{S}_t - (1-\lambda)S_t < \frac{1}{2}\big(\hat{S}_\sigma - (1-\lambda)S_\sigma\big)$, or

(ii) $S_t < S_\sigma - n^{-1}$.

By the hypothesis of (TWC) of "two-way crossing", we conclude that, a.s. on $\{\sigma < \infty\}$, we have that ϱ_n decreases to σ and that we have $S_{\varrho_n} = S_\sigma - n^{-1}$, for n large enough. Choose n large enough such that $S_{\varrho_n} = S_\sigma - n^{-1}$ on a subset of $\{\sigma < \infty\}$ of positive measure. Then $V^{\mathrm{liq}}_{\varrho_n}(\hat{\varphi})$ is strictly negative on this set which will give the desired contradiction. Indeed, the assumption $\hat{\varphi}^1_\sigma > 0$ implies that the agent suffers a strict loss from this position as $S_{\varrho_n} < S_\sigma$. The condition (i) makes sure that the agent cannot have sold stock between σ and ϱ_n. The agent may have bought additional stock during the interval $[\![\sigma, \varrho_n]\!]$. However, this cannot result in a positive effect either as the subsequent calculation, which holds true on $\{S_{\varrho_n} = S_\sigma - n^{-1}\}$, reveals:

$$\begin{aligned}
V^{\mathrm{liq}}_{\varrho_n}(\hat{\varphi}) &= \hat{\varphi}^0_{\varrho_n} + (1-\lambda)\hat{\varphi}^1_{\varrho_n} S_{\varrho_n} \\
&\leq \hat{\varphi}^0_\sigma - \int_\sigma^{\varrho_n} S_u \mathrm{d}\hat{\varphi}^{1,\uparrow}_u + (1-\lambda)\Big(\hat{\varphi}^1_\sigma + \int_\sigma^{\varrho_n} \mathrm{d}\hat{\varphi}^{1,\uparrow}_u\Big) S_{\varrho_n} \\
&= V^{\mathrm{liq}}_\sigma(\hat{\varphi}) + \hat{\varphi}^1_\sigma(1-\lambda)\underbrace{(S_{\varrho_n} - S_\sigma)}_{=-n^{-1}} - \int_\sigma^{\varrho_n} \underbrace{\big(S_u - (1-\lambda)S_{\varrho_n}\big)}_{\geq S_u - S_{\varrho_n} \geq 0} \mathrm{d}\hat{\varphi}^{1,\uparrow}_u < 0.
\end{aligned}$$

This contradiction finishes the proof of the theorem. □

To apply Theorem 2.3 to the fractional Black-Scholes model (2.6), it remains to show that condition (2.7), requiring that the indirect utility is finite, is satisfied. This is established in the following lemma by using the estimate on the fluctuations of fractional Brownian motion from Proposition A.1.

Fix $H \in (0,1)$, $\mu \in \mathbb{R}$, $\sigma > 0$, the fractional Black-Scholes model (2.6), that is,

$$S_t = \exp\big(\mu t + \sigma B^H_t\big), \qquad 0 \leq t \leq T.$$

Let $X_t := \log(S_t) = \mu t + \sigma B^H_t$. For $\delta > 0$, define the *δ-fluctuation times $(\tau_j)_{j\geq 0}$ of X* inductively by $\tau_0 \equiv 0$ and

$$\tau_{j+1}(\omega) := \inf\big\{t \geq \tau_j(\omega) \;\big|\; |X_t(\omega) - X_{\tau_j(\omega)}(\omega)| \geq \delta\big\}. \tag{4.4}$$

The *number of δ-fluctuations of X up to time T* is then given by the random variable

$$\bar{F}^{(\delta)}_T(\omega) := \sup\{j \geq 0 \mid \tau_j(\omega) \leq T\}. \tag{4.5}$$

Lemma 4.1. *Fix $H \in (0,1)$, $\mu \in \mathbb{R}$, $\sigma > 0$, the fractional Black-Scholes model* (2.6)*, that is,*

$$S_t = \exp\big(\mu t + \sigma B^H_t\big), \qquad 0 \leq t \leq T,$$

as well as $\lambda > 0$, and $\delta > 0$ such that $(1-\lambda)e^{2\delta} < 1$.

Then, there exists a constant $K > 0$, depending only on δ and λ, such that, for each $\varphi \in \mathcal{A}(x)$, we have

$$V^{\mathrm{liq}}_T(\varphi) \leq xK^n \quad \text{on} \quad \{\bar{F}^{(\delta)}_T = n\}. \tag{4.6}$$

In particular, for any concave function $U\colon (0,\infty) \mapsto \mathbb{R}$, the set

$$\left\{ \left(U(V_T^{\mathrm{liq}}(\varphi))\right)^+ \;\middle|\; \varphi \in \mathcal{A}(x) \right\}$$

is dominated by an integrable random variable.

Proof. We first observe that the mapping $t \mapsto \mu t$ can at most have $2\mu\delta^{-1}T$ fluctuations of size $\frac{1}{2}\delta$ up to time T. Since $\delta = |X_{\tau_j} - X_{\tau_{j+1}}| \leq |\mu(\tau_j - \tau_{j+1})| + \sigma|B^H_{\tau_j} - B^H_{\tau_{j+1}}|$, we therefore have that

$$\bar{F}_T^{(\delta)} \leq 2\mu T\delta^{-1} + F_T^{(\delta/2\sigma)},$$

where $F_T^{(\delta/2\sigma)}$ denotes the number of $(\delta/2\sigma)$-fluctuations of B^H up to time T as defined in Appendix A. Combining the previous estimate with Corollary A.2 gives that the random variable $\bar{F}_T^{(\delta)}$ has exponential moments of all orders, that is, $\mathbb{E}[\exp(a\bar{F}_T^{(\delta)})] < \infty$ for all $a \in \mathbb{R}$. As regards the final sentence of the lemma, it follows from (4.6) and (A.2) that

$$0 \leq V_T^{\mathrm{liq}}(\varphi) \leq xK^{\bar{F}_T^{(\delta)}} = x\exp(\log(K)\bar{F}_T^{(\delta)}) \in L^1(P) \tag{4.7}$$

and hence $\{V_T^{\mathrm{liq}}(\varphi) \mid \varphi \in \mathcal{A}(x)\} \subseteq L^1_+(P)$ is dominated by an integrable random variable. This implies the final assertion as any concave function U is dominated by an affine function.

It remains to show (4.6). Fix an admissible trading strategy φ starting at $\varphi_{0_-} = (x, 0)$ and ending at $\varphi_T = (\varphi_T^0, 0)$. Define the “optimistic value” process $(V^{\mathrm{opt}}(\varphi_t))_{0\leq t\leq T}$ by

$$V^{\mathrm{opt}}(\varphi_t) := \varphi_t^0 + (\varphi_t^1)^+ S_t - (\varphi_t^1)^-(1-\lambda)S_t :$$

the difference to the liquidation value V^{liq} as defined in (2.2) is that we interchanged the roles of S and $(1-\lambda)S$. Clearly $V^{\mathrm{opt}} \geq V^{\mathrm{liq}}$.

Fix a trajectory $(X_t(\omega))_{0\leq t\leq T}$ of X as well as $j \in \mathbb{N}$ such that $\tau_j(\omega) < T$. We claim that there is a constant $K = K(\lambda, \delta)$ such that, for every $\tau_j(\omega) \leq t \leq \tau_{j+1}(\omega) \wedge T$,

$$V^{\mathrm{opt}}(\varphi_t(\omega)) \leq KV^{\mathrm{opt}}(\varphi_{\tau_j}(\omega)). \tag{4.8}$$

To prove this claim we have to do some rough estimates. Fix t as above. Note that $S_t(\omega)$ is in the interval $[e^{-\delta}S_{\tau_j}(\omega), e^{\delta}S_{\tau_j}(\omega)]$ as $\tau_j(\omega) \leq t \leq \tau_{j+1}(\omega) \wedge T$. To fix ideas suppose that $S_t(\omega) = e^{\delta}S_{\tau_j}(\omega)$. We try to determine the trajectory $(\varphi_u)_{\tau_j(\omega)\leq u\leq t}$ which maximises the value on the left-hand side of (4.8) for given $V := V^{\mathrm{opt}}(\varphi_{\tau_j}(\omega))$ on the right-hand side. As we are only interested in an upper bound we may suppose that the agent is clairvoyant and knows the entire trajectory $(S_u(\omega))_{0\leq u\leq T}$.

In the present case where $S_t(\omega)$ is assumed to be at the upper end of the interval $[e^{-\delta}S_{\tau_j}(\omega), e^{\delta}S_{\tau_j}(\omega)]$ the agent who is trying to maximise $V^{\mathrm{opt}}(\varphi_t(\omega))$ wants to exploit this up-movement by investing into the stock S as much as possible. But she cannot make $\varphi_u^1 \in \mathbb{R}_+$ arbitrarily large as she is restricted by the admissibility condition $V_u^{\mathrm{liq}} \geq 0$ which implies that $\varphi_u^0 + \varphi_u^1(1-\lambda)S_u(\omega) \geq 0$, for all $\tau_j(\omega) \leq u \leq t$. As for these u we have $S_u(\omega) \leq e^{\delta}S_{\tau_j}(\omega)$ this implies the inequality

$$\varphi_u^0 + \varphi_u^1(1-\lambda)e^{\delta}S_{\tau_j}(\omega) \geq 0, \quad \tau_j(\omega) \leq u \leq t. \tag{4.9}$$

As regards the starting condition $V^{\text{opt}}(\varphi_{\tau_j}(\omega))$ we may assume without loss of generality that $\varphi_{\tau_j}(\omega) = (V, 0)$ for some number $V > 0$. Indeed, any other value of $\varphi_{\tau_j}(\omega) = (\varphi^0_{\tau_j}(\omega), \varphi^1_{\tau_j}(\omega))$ with $V^{\text{opt}}(\varphi_{\tau_j}(\omega)) = V$ may be reached from $(V, 0)$ by either buying stock at time $\tau_j(\omega)$ at price $S_{\tau_j}(\omega)$ or selling it at price $(1-\lambda)S_{\tau_j}(\omega)$. Hence we face the elementary deterministic optimization problem of finding the trajectory $(\varphi^0_u, \varphi^1_u)_{\tau_j(\omega)\le u\le t}$, starting at $\varphi_{\tau_j}(\omega) = (V, 0)$ and respecting the self-financing condition (2.1) as well as inequality (4.9), which maximizes $V^{\text{opt}}(\varphi_t)$. Keeping in mind that $(1-\lambda) < e^{-2\delta}$, a moment's reflection reveals that the best (clairvoyant) strategy is to wait until the moment $\tau_j(\omega) \le \bar{t} \le t$ when $S_{\bar{t}}(\omega)$ is minimal in the interval $[\tau_j(\omega), t]$, then to buy at time $\bar{t}$ as much stock as is allowed by the inequality (4.9), and then keeping the positions in bond and stock constant until time t. Assuming the most favourable (limiting) case $S_{\bar{t}}(\omega) = e^{-\delta}S_{\tau_j}(\omega)$, simple algebra gives $\varphi_u = (V, 0)$ for $\tau_j(\omega) \le u < \bar{t}$ and

$$\varphi_u = \left(V - V_{\text{bought}}, \frac{V_{\text{bought}}e^{\delta}}{S_{\tau_j}(\omega)}\right) \quad \text{for } \bar{t} \le u \le t,$$

where

$$V_{\text{bought}} := \frac{V}{1-(1-\lambda)e^{2\delta}}.$$

Using $S_t(\omega) = e^{\delta}S_{\tau_j}(\omega)$ we therefore may estimate in (4.8):

$$V^{\text{opt}}(\varphi_t(\omega)) \le V\left(1 - \frac{1}{1-(1-\lambda)e^{2\delta}} + \frac{e^{2\delta}}{1-(1-\lambda)e^{2\delta}}\right). \tag{4.10}$$

Due to the hypothesis $(1-\lambda)e^{2\delta} < 1$ the term in the bracket is a finite constant K, depending only on λ and δ. We have assumed a maximal up-movement $S_t(\omega) = e^{\delta}S_{\tau_j}(\omega)$. The case of a maximal down-movement $S_t(\omega) = e^{-\delta}S_{\tau_j}(\omega)$ as well as any intermediate case follow by the same token yielding again the estimate (4.8) with the same constant given by (4.10). Observing that $V^{\text{opt}} \ge V^{\text{liq}}$ we obtain inductively (4.6) thus finishing the proof. □

Proof of Theorem 2.4. The fractional Black-Scholes model satisfies (TWC) by Corollary 6.1 of [29]. The finiteness of the indirect utility function $u(x) < \infty$ follows from Lemma 4.1. Using Theorem 3.2, Proposition 2.9 of [12] and Theorem 3.1 of [25], we obtain that the primal value function u and the dual value function v are conjugate.

In order to show the duality results, by Theorem 4 of [26], it is sufficient to show that $v(y) < \infty$ for all $y > 0$. To that end, let $y > 0$ be arbitrary. Since U is strictly concave, strictly increasing and satisfies the Inada conditions, we may find $C > 0$ and $0 < k < y(\mathbb{E}[K^{\bar{F}^{(\delta)}_T}])^{-1}$, depending on y, such that $U(x) \le C + kx$, and therefore, by (4.7), that $u(x) \le C + k\,\mathbb{E}[K^{\bar{F}^{(\delta)}_T}]x$, which implies

$$v(y) = \sup_{x>0}\{u(x) - xy\} \le \sup_{x>0}\big\{C + \big(k\,\mathbb{E}\big[K^{\bar{F}^{(\delta)}_T}\big] - y\big)x\big\} \le C < \infty.$$

By Theorem 4 of [26], we obtain the existence of primal optimiser $(\hat{\varphi}^0, \hat{\varphi}^1)$ and dual optimiser $(\hat{Y}^0, \hat{Y}^1)$ such that $\hat{Y}^0\hat{\varphi}^0 + \hat{Y}^1\hat{\varphi}^1$ is a uniformly integrable martingale. The existence of a shadow price $\hat{S} = (\hat{S}_t)_{0\le t\le T}$ in the sense of Definition 2.1 then follows by the same argument as in the proof of Theorem 2.3. □

A Appendix: Fluctuations of fractional Brownian motion

In this appendix we establish a control on the number of fluctuations of fractional Brownian motion (Proposition A.1). It allows us to show the finiteness of the indirect utility for the proof of Theorem 2.4, and may also be interesting in its own right. Though the results stated here only deal with the case of fractional Brownian motion, our techniques can actually also be applied to a broader class of Gaussian processes.

Let $B^H = (B^H_t)_{t\geq 0}$ be a standard fractional Brownian motion with Hurst parameter $H \in (0,1]$. Fix $\delta > 0$ and define the δ-fluctuation times of B^H, denoted by $(\tau_j(\omega))_{j\in\mathbb{N}}$, resp. the number of δ-fluctuations of B^H up to time $T \in (0,\infty)$, denoted by $F^{(\delta)}_T(\omega)$, exactly as we defined these concepts for X in (4.4) and (4.5).

The main result of this appendix is the following proposition:

Proposition A.1. *With the notation above, there exist finite positive constants $C = C(H), C' = C'(H)$ such that*

$$\mathbb{P}\big[F^{(\delta)}_T \geq n\big] \leq \exp\big(-C^{-1}\delta^2 T^{-2H} n(n^{2H\wedge 1} - C' \ln n)\big) \quad \text{for all } n \geq 2. \tag{A.1}$$

The interest of Proposition A.1 for this article lies in the following corollary:

Corollary A.2. *With the above notation, for any $\delta > 0$, the random variable $F^{(\delta)}_T$ does have exponential moments of all orders, that is,*

$$\mathbb{E}\big[\exp\big(aF^{(\delta)}_T\big)\big] < \infty \quad \text{for all } a \in \mathbb{R}. \tag{A.2}$$

Moreover, if $H \geq 1/2$, this random variable even has a Gaussian moment, that is, there exists $a > 0$ such that

$$\mathbb{E}\big[\exp(a(F^{(\delta)}_T)^2)\big] < \infty.^3 \tag{A.3}$$

Proof of Corollary A.2. For $f(x) = \exp(ax)$ and $f(x) = \exp(ax^2)$, we have

$$\mathbb{E}\big[f(F^{(\delta)}_T)\big] = f(0) + \int_0^\infty f'(x)\,\mathbb{P}\big[F^{(\delta)}_T \geq x\big]\mathrm{d}x$$

by Fubini's Theorem. Combining this with the estimate (A.1) gives (A.2) and (A.3). □

Proof of Proposition A.1. Throughout the proof, we denote by $C, C' > 0$ constants only depending on H, but whose precise value may vary from appearance to appearance.

Let $n, m \in \mathbb{N}$ be such that $m > n \geq 2$. We divide $[0,T]$ into m subintervals $I_k := [\frac{k}{m}T, \frac{k+1}{m}T]$ for $k = 0,\dots,m-1$ and denote their midpoints by $t_k := \frac{k+1/2}{m}T$. Then, we can estimate the probability of the set

$$A_1 := \bigcup_{k=0}^{m-1} \{\exists t \in I_k \quad |B^H_t - B^H_{t_k}| > \tfrac{1}{4}\delta\}$$

[3] Equation (A.3) is actually not used in this article, but this result seemed worth being written to us.

by

$$\begin{aligned}\mathbb{P}\big[A_1\big] &\le m\,\mathbb{P}\big[\exists t\in I_k \quad |B^H_t - B^H_{t_k}| > \tfrac14\delta\big] \\ &= m\,\mathbb{P}\big[\sup_{|t|\le 1}|B^H_t| > (T\,/\,2m)^{-H}\tfrac14\delta\big] \le C' m \exp\big(-C^{-1}\big((m\,/\,T)^H\delta\big)^2\big), \quad \text{(A.4)}\end{aligned}$$

where the penultimate equality comes from translation and scale invariance of fractional Brownian motion,[4] and the last inequality from Fernique's theorem [14, Lemma 2.2.5].

On the complement $A_1^\complement$ of A_1, we then have that

$$\sup_{t\in I_k}|B^H_t - B^H_{t_k}| \le \tfrac14\delta \quad \text{for all } k = 0,\ldots,m-1. \tag{A.5}$$

Suppose now that $F^{(\delta)}_T(\omega) \ge n$. Then there have to be at least $(n+1)$ "random indices" $0 = K_0(\omega) < K_1(\omega) < \cdots < K_n(\omega) < m$ such that $\tau_j(\omega) \in I_{K_j(\omega)}$ for $j = 0,\ldots,n$. Because of (A.5) and $|B^H_{\tau_j} - B^H_{\tau_{j+1}}| = \delta$, we then must have $|B^H_{t_{K_j}} - B^H_{t_{K_{j+1}}}| \ge \frac12\delta$ for $j = 0,\ldots,n-1$ on $\{F^{(\delta)}_T \ge n\}\cap A_1^\complement$.

In order to estimate $\mathbb{P}[\{F^{(\delta)}_T \ge n\}\cap A_1^\complement]$, it therefore only remains to bound the probability of the event

$$A_2 \coloneqq \bigcap_{j=0}^{n-1}\big\{|B^H_{t_{K_j}} - B^H_{t_{K_{j+1}}}| \ge \tfrac12\delta\big\}.$$

But the event A_2 depends on the realisation of the "random indices" $(K_j(\omega))_{0\le j\le n}$. To get rid of this dependence, we simply estimate the probability of the event

$$A_3 \coloneqq \bigcap_{j=0}^{n-1}\{|B^H_{t_{k_j}} - B^H_{t_{k_{j+1}}}| \ge \tfrac12\delta\}$$

for all $\binom{m-1}{n}$ possible realisations $0 = k_0 < k_1 < \cdots < k_n < m$ of our "random indices".

For this, fix an arbitrary realisation of indices $(k_j)_{0\le j\le n}$ and set

$$\Delta_j \coloneqq B^H_{t_{k_{j+1}}} - B^H_{t_{k_j}} \quad \text{for } j = 0,\ldots,m-1.$$

Then

$$A_3 = \bigcap_{j=0}^{n-1}\{|\Delta_j| \ge \tfrac12\delta\} = \bigcap_{j=0}^{n-1}\{\operatorname{sgn}(\Delta_j)\Delta_j \ge \tfrac12\delta\} \subseteq \Big\{\sum_{j=0}^{n-1}\operatorname{sgn}(\Delta_j)\Delta_j \ge \tfrac12 n\delta\Big\},$$

so that

$$A_3 \subseteq \bigcup\Big\{\Big\{\sum_{j=0}^{n-1}\varepsilon_j\Delta_j \ge \tfrac12 n\delta\Big\}\ \Big|\ (\varepsilon_j)_{0\le j<n} \in \{-1,+1\}^n\Big\}.$$

[4] *Exact* translation and scale invariance of fractional Brownian motion is actually not needed here: more precisely, exact invariance shortens the proof by allowing the use of Fernique's theorem, but a slight refinement of that theorem would make the result work as soon as one has a bound of the type $\operatorname{Var}(B^H_t - B^H_s) \le C|t-s|^{2H}$: see e.g. Lemma 4.2 of [29].

For fixed $(\varepsilon_j)_{0\le j<n} \in \{-1,+1\}^n$, we have that $\sum_{j=0}^{n-1} \varepsilon_j \Delta_j$ is a centred normally distributed random variable with variance

$$\operatorname{Var}\Bigl(\sum_{j=0}^{n-1} \varepsilon_j \Delta_j\Bigr) = \sum_{j,j'=0}^{n-1} \varepsilon_j \varepsilon_{j'} \operatorname{Cov}(\Delta_j, \Delta_{j'}) \le \sum_{j,j'=0}^{n-1} |\operatorname{Cov}(\Delta_j, \Delta_{j'})|. \tag{A.6}$$

To estimate (A.6), we distinguish the cases $H \ge 1/2$ from the case $H < 1/2$. If $H \ge 1/2$, the covariance $\operatorname{Cov}(\Delta_j, \Delta_{j'})$ is always non-negative, so that

$$\sum_{j,j'=0}^{n-1} |\operatorname{Cov}(\Delta_j, \Delta_{j'})| = \operatorname{Var}\Bigl(\sum_{j=0}^{n-1} \Delta_i\Bigr) = \operatorname{Var}(B^H_{t_{k_n}} - B^H_{t_{k_0}}) = |t_{k_0} - t_{k_n}|^{2H} \le T^{2H}.$$

If $H < 1/2$, the covariance $\operatorname{Cov}(\Delta_j, \Delta_{j'})$ is non-positive as soon as $j \ne j'$, so that

$$\begin{aligned}\sum_{j'=0}^{n-1} |\operatorname{Cov}(\Delta_j, \Delta_{j'})| &= \operatorname{Var}(\Delta_j) - \operatorname{Cov}\Bigl(\Delta_j, \sum_{j'<j} \Delta_{j'}\Bigr) - \operatorname{Cov}\Bigl(\Delta_j, \sum_{j'>j} \Delta_{j'}\Bigr) \\ &= \operatorname{Var}(\Delta_j) - \operatorname{Cov}(\Delta_j, B^H_{t_{k_0}} - B^H_{t_{k_j}}) - \operatorname{Cov}(\Delta_j, B^H_{t_{k_{j+1}}} - B^H_{t_{k_n}}).\end{aligned}$$

But for $0 \le t \le u \le v \le T$, it follows from the definition of fractional Brownian motion that

$$-\operatorname{Cov}(B^H_u - B^H_t, B^H_v - B^H_u) = \tfrac{1}{2}(|t-u|^{2H} + |u-v|^{2H} - |t-v|^{2H}) \le \tfrac{1}{2}|t-u|^{2H}.$$

Therefore, we have that

$$\sum_{j'=0}^{n-1} |\operatorname{Cov}(\Delta_j, \Delta_{j'})| \le |t_{k_j} - t_{k_{j+1}}|^{2H} + \tfrac{1}{2}|t_{k_j} - t_{k_{j+1}}|^{2H} + \tfrac{1}{2}|t_{k_j} - t_{k_{j+1}}|^{2H} = 2|t_{k_j} - t_{k_{j+1}}|^{2H}$$

and thus

$$\operatorname{Var}\Bigl(\sum_{j=0}^{n-1} \varepsilon_j \Delta_j\Bigr) \le 2 \sum_{j=0}^{n-1} |t_{k_j} - t_{k_{j+1}}|^{2H}. \tag{A.7}$$

But, since $H < 1/2$, the function $x \mapsto x^{2H}$ is concave, so that the right-hand side of (A.7) is bounded above by

$$2n\Biggl(\frac{\sum_{j=0}^{n-1} |t_{k_j} - t_{k_{j+1}}|}{n}\Biggr)^{2H} = 2n(|t_{k_0} - t_{k_n}| \,/\, n)^{2H} \le 2n(T \,/\, n)^{2H} = 2n^{1-2H} T^{2H}.$$

Hence in both cases we can estimate:

$$\operatorname{Var}\Bigl(\sum_{j=0}^{n-1} \varepsilon_j \Delta_j\Bigr) \le 2T^{2H} n^{(1-2H)_+}.$$

So, using the classical bound that $\mathbb{P}[Z \ge x] \le e^{-x^2/2}$ for any standard normal distributed random variable $Z \sim \mathcal{N}(0,1)$, we have that

$$\mathbb{P}\Bigl[\sum_{j=0}^{n-1} \varepsilon_j \Delta_j \ge \tfrac{1}{2} n\delta\Bigr] \le \exp(-\tfrac{1}{16} T^{-2H} \delta^2 n^{1+(2H\wedge 1)})$$

for all 2^n possible choices of $(\varepsilon_j)_{0\le j<n} \in \{-1,+1\}^n$, and therefore

$$\mathbb{P}\big[A_3\big] \le 2^n \exp(-\tfrac{1}{16}T^{-2H}\delta^2 n^{1+(2H\wedge 1)}).$$

Combining that estimate with (A.4), and using that $\binom{m-1}{n} \le m^n$, we finally get that

$$\begin{aligned}\mathbb{P}\big[F_T^{(\delta)} \ge n\big] &\le \mathbb{P}\big[A_1\big] + \mathbb{P}\big[\big\{F_T^{(\delta)} \ge n\big\} \cap A_1^c\big] \\ &\le C'm\exp\big(-C^{-1}T^{-2H}\delta^2 m^{2H}\big) + 2^n m^n \exp\big(-\tfrac{1}{16}T^{-2H}\delta^2 n^{1+(2H\wedge 1)}\big).\end{aligned}$$

Now, it only remains to choose m to be $\lceil n^{1/H}\rceil$ to obtain

$$\mathbb{P}\big[F_T^{(\delta)} \ge n\big] \le \exp\big(-C^{-1}\delta^2 T^{-2H} n(n^{2H\wedge 1} - C'\ln n)\big),$$

which completes the proof. $\square$

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