Asymptotic synthesis of contingent claims in a sequence of discrete-time markets

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Abstract. We prove a fundamental result concerning the connection between discrete-time models of financial markets and the celebrated Black–Scholes–Merton continuous-time model in which "markets are complete." Specifically, we prove that if (a) the probability law of a sequence of discrete-time models converges (in the functional sense) to the probability law of the Black–Scholes–Merton model, and (b) the largest possible one-period step in the discrete-time models converges to zero, then every bounded and continuous contingent claim can be asymptotically synthesized with bounded risk: For any $\epsilon > 0$, a consumer in the discrete-time economy far enough out in the sequence can synthesize a claim that is no more than ϵ different from the target contingent claim x with probability at least $1 - \epsilon$, and which, with probability 1, has norm less or equal to the norm of the target claim. This shows that, in terms of important economic properties, the Black-Scholes-Merton model, with its complete markets, idealizes many more discrete-time models than models based on binomial random walks.

1. Introduction

In the celebrated model of a securities market consisting of a risky asset, the stock, and a riskless bond, originally studied by Black and Scholes (1973) and Merton (1973), "markets are complete," in the (rough) sense that every well-behaved contingent claim based on the history of stock price can be *synthesized* by continuous trading in the stock and a riskless bond (Harrison and Pliska, 1981, 1983). Sharpe (1976) and Cox, Ross, and Rubinstein (1978) show a similar result for discrete-time economies in which the stock price, over each time interval, can move (only) to one of two possible values. But if, in discrete-time models, the stock can move to more than two values over each time interval, markets are "incomplete" and, the arbitrage bounds on the prices of many contingent claims remain

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wide *even when* we look at a sequence of economies where, along the sequence, trading opportunities are increasingly frequent.

These arbitrage bounds are based on the principle that an investor must be capable of synthesizing a claim that lies (weakly) above or below the given contingent claim *with probability one*, sometimes called *super-hedging*. Suppose we ask instead for *asymptotic synthesis*: For a given contingent claim, such as a European call, one wishes (in a discrete-time and possibly incomplete-markets economy) to synthesize a contingent claim that is close to the original contingent claim with probability close to one. This can be done (Duffie and Protter, 1991).

However, if one doesn't control what happens on the set of small probability on which the target contingent claim and the synthesized approximation can be very different, arbitrage opportunities (or free lunches) can be synthesized. A satisfactory theory of (asymptotic) approximate synthesis or replication of contingent claims in this setting requires some reasonable bound on the size of the "replication error." We show, in some generality, that for a sequence of discrete-time markets that asymptotically "resemble" the Black-Scholes-Merton (BSM) model, bounded and continuous contingent claims can be asymptotically synthesized *with bounded risk*: With probability one, the value of the portfolios that approximately synthesize the contingent claim are bounded above and below by, respectively, the upper and lower bounds on the value of the contingent claim being asymptotically synthesized. (This strengthened definition of asymptotic synthesis rules out the asymptotic synthesis of free lunches, of course.)

The economic significance of this result takes us back to Arrow's seminal paper (1963) on the role of securities markets and dynamic trading: In the world of Black, Scholes, and Merton, two securities and dynamic trading give complete markets. Cox, Ross, and Rubinstein (CRR) extend this to discrete-time economies, but for a very special and limited set of discrete-time economies. The results here show that, while more general discrete-time economies than in CRR may not give fully complete markets, if (asymptotically) those economies resemble the BSM economy, then dynamic trading may give *approximately* complete markets.

However, two further issues intrude. First, while each discrete-time economy in a sequence of such economies may be economically viable, the sequence as a whole may allow for the creation of asymptotic free lunches with bounded risk. This issue—known as the problem of *asymptotic arbitrage*—is well known in the literature, so we provide (only) a brief discussion of this issue and how it connects to our main result.

Second, even bounded risk may be too much for consumers with preferences that give no allowance for taking on such risks in the pursuit of desirable contingent claims (although see the concluding remarks). It would be better to show that asymptotic synthesis of contingent claims with *vanishing risk* is feasible. But, alas, this is not possible (outside of the CRR binomial model), as we demonstrate by example. (In an appendix, we discuss how, from a mathematical perspective, asymptotic synthesis of some contingent claims with vanishing risk is possible, but in a framework that poses difficulties in terms of economic interpretation.)

Because the economic framework that underpins our analysis is fairly well known, we proceed with minimal discussion of that framework and the interpretation of our results, concentrating instead on the proof of the main result. A complete discussion of the economics of this analysis is provided in Kreps (2019).

2. General formulation and the main result

We work in the space $\Omega = C_0[0,1]$, the space of all continuous functions ω from $[0,1]$ to R whose value at 0 is 0. We let ω denote a typical element of Ω, with $\omega(t)$ the value of ω at date t. Endow Ω with the sup norm topology, and let $\{F_t; 0 \le t \le 1\}$ be the standard filtration.

Let P be Wiener measure on Ω, so that ω under P is a standard Brownian motion. Expectation with respect to P is denoted by $\mathbf{E}[\cdot]$

The simple Black-Scholes-Merton (BSM) model of one-risky-asset financial market concerns two assets that trade one against the other over the continuous interval $[0,1]$. The bond is the numeraire, whose price (relative to itself) is therefore identically 1. The second security, called the stock, has price $S(t,\omega) = e^{\omega(t)}$ at time t in state ω; that is, under P, the stock price has the law of geometric Brownian motion.

We know that there is a unique probability measure on Ω, denoted P^*, that is equivalent to P and, under which, $S(t)$ is a martingale (Harrison and Kreps, 1979). Expectation with respect to P^* is denoted by $\mathbf{E}^*[\cdot]$.

Contingent claims on ω are functions $x : \Omega \to R$. For our purposes, we restrict attention to contingent claims x that are bounded and continuous (in the sup norm); the space of such claims will be denoted by X.

The well-known "complete markets" result for the BSM model says that, for every $x \in X$, x can be written

$$x = \mathbf{E}^*[x] + \int_0^1 \alpha dS,$$

for a predictable and S-integrable integrand α (Harrison and Pliska, 1981, 1983). The interpretation is that a consumer–investor, living in the BSM economy, can *synthesize* the claim x by a trading strategy α that calls for an initial investment $\mathbf{E}^*[x]$, where $\alpha(t,\omega)$ represents the number of shares of stock held at time t in state ω and where bond holdings are adjusted continuously so that any purchases of stock (after time 0) are financed by the sale of bonds (borrowing) and the proceeds of any sale of stock are used to purchase bonds.

Here, the stochastic integral $\int_0^t \alpha(u)dS(u)$ represents the financial-gains from this strategy up to time t.

Now suppose that for $n = 1, 2, \ldots$, we have different probability measures P^n defined on Ω, with the following structure: For each n, the support of P^n consists of piecewise linear functions that, in particular, are piecewise linear on all intervals of the form $[k/n, (k+1)/n]$, for $k = 0, \ldots, n-1$. The interpretation is that P^n represents a probability distribution on paths of the log of the stock price in an nth discrete-time economy, in which trading between the stock and bond is possible *only* at times $t = k/n$ for $k = 0, \ldots, n-1$. (At time 1, the stock and bond liquidate in state ω at "prices" 1 and $e^{\omega(t)}$.)[1]

Consumer–investors in the nth discrete-time economy can implement (state-dependent) trading strategies $\{\theta(k/n), k = 0, \ldots, n-1\}$, where the interpretation is that $\theta(k/n, \omega)$ is the number of shares of stock held by the consumer–investor *after* she has traded at time k/n, held until time $(k+1)/n$. We require that $\theta(k/n)$ is $F_{k/n}$ measurable; in the nth economy, the consumer–investor only knows at time k/n the evolution of the stock price up to and including that date. And, while θ can involve an initial investment of funds at date 0, it must be self-financing subsequently. This is, in the usual fashion, most easily formulated as follows: If $V_\theta(k/n, \omega)$ is the value of the portfolio formed by the trading strategy θ at time k/n in state ω, then for all $k = 1, \ldots, n$,

$$V_\theta(k/n, \omega) = V_\theta(0) + \sum_{j=1}^{k-1} \theta(j, \omega) \times \Big[S\big((j+1)/n, \omega\big) - S\big(j/n, \omega\big) \Big].$$

Please note that for a given n and trading strategy θ, this defines $V_\theta(k/n, \omega)$ for *all* $\omega \in \Omega$ (and not only for ω in the support of P^n), although we (and our consumer–investor) are interested in this only for those ω that are in the support of P^n.

We maintain throughout the assumption that, for each n, P^n specifies a *viable* model of an economic equilibrium in the usual sense: It is impossible to find in the nth discrete-time trading strategy a self-financing and progressively measurable trading strategy θ with $V_\theta(0) = 0$ and $V_\theta(1) \geq 0$ P^n-a.s. and $V_\theta(1) > 0$ with P^n-positive probability. This is true if and only if there exists a probability measure P^{*n} that is equivalent to P, under which $\{(e^{\omega(k/n)}, F_{k/n}); k = 0, \ldots, n\}$ is a martingale (Dalang, Morton, Willinger, 1990).

Let $X^n := \{x \in X : x(\omega) = V_\theta(1, \omega)$ for some (progressively measurable) trading strategy θ for the nth discrete-time economy$\}$.

[1] The piecewise linearity of ω under the various P^n is a convenient way to have $C_0[0,1]$ be a common state space; in the nth economy, $\omega(t)$ for t not of the form k/n has no economic meaning or consequence. And alternative construction would have $\omega(t)$ piecewise constant over intervals $[k/n, (k+1)/n)$, in which case we would work in the Skorohod space $D[0,1]$.

Definition. *The contingent claim $x \in X$ can be* ***asymptotically synthesized with bounded risk*** *if there exists a sequence $\{x^n\}$, where each $x^n \in X^n$, such that:*

a. For every $\epsilon > 0$, there exists N_ϵ such that, for all $n > N_\epsilon$,

$$P^n\big(\{\omega : |x^n(\omega) - x(\omega)| > \epsilon\}\big) < \epsilon, \quad \textit{and}$$

b. for some finite real number B, $P^n\big(\{\omega : |x^n(\omega)| < B\}\big) = 1$, uniformly in n.

The claim x can be ***asyptotically synthesized with vanishing risk*** *if condition a can be sharpened to: For every $\epsilon > 0$, there exists N_ϵ such that, for all $n > N_\epsilon$,*

$$P^n\big(\{\omega : |x^n(\omega) - x(\omega)| > \epsilon\}\big) = 0$$

(in which case condition b is superfluous).

(Warning: It is tempting to paraphrase *a* of the definition as, $x^n \to x$ in probability. But this not an accurate paraphrase. Convergence in probability is defined relative to a single probability measure; in the definition, there is a different measure P^n for each n.)

Theorem 1. *Suppose that*

a. $P^n \Rightarrow P$, and

b. For some sequence $\{\delta_n; n = 1, \dots,\}$ of positive numbers tending to zero,

$$P^n\big(\{\omega : \mathit{sup}_{0 \le k \le n} |\omega(k/n) - \omega((k+1)/n)| \le \delta_n\}\big) = 1.$$

Then every (continuous and bounded) $x \in X$ can be asymptotically synthesized with bounded risk, where the bound on the risk, the parameter B in the definition, can be taken to be $\|x\|_\infty$. In fact, if we write $\overline{x} := \mathit{sup}_{\omega \in C_0[0,1]} x(\omega)$ and $\underline{x} := \mathit{inf}_{\omega \in C_0[0,1]} x(\omega)$, then the contingent claim x^n synthesized in the nth discrete-time economy can be synthesized so that $x^n(\omega) \in (\underline{x}, \overline{x})$, P^n-a.s. Moreover, letting θ_n be a trading strategy that synthesizes x^n in the nth economy, this can be done where $V_{\theta_n}(0) = \mathbf{E}^*[x]$ for each n.

The two assumptions in Theorem 1 have the following explanation. That $P^n \Rightarrow P$ is saying that, in a somewhat coarse sense, the discrete-time economies asymptotically resemble the BSM economy or, put the other way around, the BSM economy is, in terms of its viewed-from-afar features, an idealization of the nth discrete-time economy for large n. The second assumption is the key to bounded risk. Because, in the discrete-time economies, a consumer-investor cannot instantaneously intercede in the face of an "unusual" event, it is necessary that the damage done to her portfolio by the time she can react can be contained. In the BSM model, with continuous-time trading, she can intervene instantaneously. In a sense, while assumption *a* says that the nth discrete-time economy for large n is similar to the BSM economy when viewed on a "macroscopic" scale, assumption *b* is the required similarity in terms of "microscopic" features.

Before proving Theorem 1, the next two sections provide a specific example and discussion of asymptotic arbitrage and its connection to Theorem 1.

3. An example

The prototypical example of a sequence $\{P^n\}$ that satisfies the conditions of Theorem 1 is based on the following construction which is the basic model in Kreps (2019). Fix a real-valued random variable ζ with expected value 0, variance 1, and bounded support. Let $\{\zeta_k; k = 1, 2, \dots\}$ be an i.i.d. sequence of random variables, all having the distribution of ζ. For $n = 1, 2, \dots$, let

$$B^n(k/n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} \zeta_j \quad \text{and} \quad S^n(k/n) = e^{B^n(k/n)}, \quad \text{for } k = 0, \dots, n.$$

As above, we imagine that, for each n, we have an economy in which two financial assets, a risky stock and a riskless bond are traded one against the other at times $t = 0, 1/n, 2/n, \dots, (n-1)/n$.

Embed these models (one for each n) into a single state space $\Omega = C[0,1]$, where $\omega = \big(\omega(t)\big)_{t\in[0,1]}$ denotes a typical path in $C[0,1]$. Do this by creating for each n a probability measure P^n on $C[0,1]$, where the support of P^n consists of paths that are piecewise linear over intervals of the form $[k/n, (k+1)/n]$, and the finite-dimensional distribution of paths under P^n at time $0, 1/n, \dots, 1$ match the distributions of the B^n as defined above.

Donsker's Theorem applies and tells us that $P^n \Rightarrow P$, the Wiener measure on $C_0[0,1]$. As for condition *b* in Theorem 1, because the support of ζ is bounded, the condition is clearly met.

4. Asymptotic arbitrage and the rationale for imposing bounded risk

Theorem 1 says that every bounded and continuous contingent claim x can be asymptotically synthesized with bounded risk for an initial investment of $\mathbf{E}^*[x]$, the price of x in the "limiting" BSM economy. However, it *does not* say that x cannot be asymptotically synthesized for a smaller (or, for that matter, larger) initial investment. This points us in the direction of of the issue of *asymptotic arbitrage* (Kabanov and Kramkov,1994; Klein and Schachermayer, 1997).

To introduce this issue, we first back up to discuss a remark made in the introduction, concerning our rationale for imposing condition *b*, bounded risk, in our fundamental definition. Suppose we had defined *asymptotic synthesis* without the bounded-risk part; that is, x can be asymptotically synthesized if there exists a sequence $\{x^n\}$, $x^n \in X^n$, where for every $\epsilon > 0$, there is N_ϵ such that, if $n > N_\epsilon$, $P^n\{\omega : |x(\omega) - x^n(\omega)| \geq \epsilon\}\big) < \epsilon$. Consider a sequence of $\{P^n\}$ generated in the fashion of the first example of Section 3, in which $\zeta = +1$ with probability 1/2 and -1 with probability 1/2. In the usual fashion, a consumer investor in the nth economy can employ the well-known and infamous doubling strategy: She initially buys as many shares of stock as she sells bonds so that, at time $1/n$, she is either ahead by 1 or down by 1. If she is ahead by 1, she liquidates her portfolio (sells her stock and buys bonds) and awaits time 1. And if she is behind by 1, she "doubles down," buying enough stock, financed with the purchase of bonds, so that if the second movement in the stock is an uptick, her (leveraged) portfolio has value 1. And so forth. For any ϵ, there is N_ϵ sufficiently large so that, following this strategy, she ends with a portfolio worth precisely 1 with probability greater than $1 - \epsilon$. Although, of course, with positive probability, she is in a very, very deep hole.

By imposing the bounded-risk part of the definition, we rule this sort of thing out.

However, the ability to create asymptotic arbitrage in this rough fashion is *not* precluded by conditions *a* and *b* in Theorem 1. An example attributed to K. Pötzelberger and Th. Schlumprecht (independently) by Hubalek and Schachermayer (1998) illustrates this. This example is similar to the first sort of example from last section, except that the "scaled" distribution of $\zeta_k/\sqrt{n}$ is different in even and odd periods. In even-numbered periods (for $k = 2, 4, 6, \ldots$), $\zeta_k/\sqrt{n} = 1/\sqrt{n}$ with probability 0.2 and $= -1.5/\sqrt{n}$ with probability 0.8, so for these periods, the expected value of $\zeta_k/\sqrt{n} = -0.3/\sqrt{n}$. In odd-numbered periods, $\zeta_k/\sqrt{n} = 1.5/\sqrt{n}$ with probability 0.8 and $= -1/\sqrt{n}$ with probability 0.2; the expected value is $0.3/\sqrt{n}$. Hence, while the expectations of the ζ_k alternate between $\pm 0.3/\sqrt{n}$, $\zeta_k + \zeta_{k+1}$ has expectation 0 and variance 2. We assert that

i. the resulting $P^n \Rightarrow P$ and

ii. (clearly), as $n \to \infty$, the "diameter" of the support of $\omega((k+1)/n) - \omega(k/n)$ approaches zero, however,

iii. for an initial investment of 0, a consumer–investor can asymptotically end with a portfolio of value 1 and with *vanishing* risk.

She can do this by investing in the stock in odd periods and holding only bonds in the even periods. (The first period, from $t = 0$ to $t = 1/n$, is an odd period.) Specifically, in the first period (at time 0), she purchases $1/\sqrt{n}$ shares of stock, financing this purchase by selling $1/\sqrt{n}$ bonds. She ends this period with a portfolio worth either $e^{1.5/\sqrt{n}}/\sqrt{n} - 1/\sqrt{n}$ or $e^{-1/\sqrt{n}}/\sqrt{n} - 1/\sqrt{n}$, with probabilities 0.8 and 0.2, respectively. At time $1/n$, she converts this portfolio to bonds, and waits until time $2/n$, when she puts all that portion of her portfolio that was previously stock back into stock (continuing to be short $1/\sqrt{n}$ bonds. And so forth. Then her wealth after $n = 2k$ periods has the distribution of

$$\prod_{j=0}^{k-1} \frac{e^{2j+1/\sqrt{n}}}{\sqrt{n}} \; - \; \frac{1}{\sqrt{n}},$$

where $\{\zeta_j\}$ is the (alternating distribution) sequence for this example. Ignoring momentarily the $-1/\sqrt{n}$, the distribution of the log of her wealth (that is, the log of her wealth plus $1/\sqrt{n}$) at time n is

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{k-1} \zeta_{2j+1} \; - \; \ln(1/\sqrt{n}) \;\; = \;\; \sqrt{\frac{2}{n}} \sum_{j=0}^{n/2-1} \zeta_{2j+1} \; - \; \ln(1/\sqrt{n}).$$

By the strong law of large numbers,

$$\lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \zeta_{2j+1} = \lim_{n \to \infty} \frac{2}{n} \sum_{j=0}^{n/2-1} \zeta_{2j+1} = 0.3, \text{ a.s.}$$

so the log of her wealth, hence the her wealth, approaches infinity a.s. at an exponential rate. Of course, for given n, this process stops before k reaches ∞, but as $n \to \infty$, she can with probability arbitrarily close to 1 wind up with a portfolio whose value is as large as she desires, and with vanishing risk.

This is better (for her) than is claimed in *iii*; but by judiciously stopping the process of going back and forth between stock and bond when her wealth first hits 1, if it ever does, and scaling back her holdings of stock when she is within one step of achieving 1 so that she hits 1 on the nose if there is an uptick, she can produce a strategy that, as $n \to \infty$, gets her to 1 precisely with probability that approaches 1, and still (of course) with vanishing risk.

Hence, for this sequence of $\{P^n\}$, the two conditions of Theorem 1 are met, and every $x \in X$ can be asymptotically synthesized with bounded risk for an initial investment of $\mathbf{E}^*[x]$. But every $x \in X$ can also be asymptotically synthesized with bounded risk for *every* level of initial investment, both greater than and less than $\mathbf{E}^*[x]$.

This is hardly a satisfactory state of affairs from the perspective of the economics of the situation. Happily, we can easily plug into Klein and Schachermayer (1997) to add a third condition to the sequence $\{P^n\}$ that eliminates this sort of thing. We adapt their more technical definition to the current context:

Definition. *The sequence* $\{P^n\}$ *admits an* ***asymptotic arbitrage***[2] *if, for some* $B > 0$*, there exists for every* $\epsilon > 0$ *an* n *and a trading strategy* θ^n *for* n *such that*

a. $V_{\theta^n}(0) = 0$,

b. $P^n\big(\{V_{\theta^n}(k/n) \geq -B\}\big) = 1$, *and*

c. $P^n\big(\{V_{\theta^n}(1) \geq 1\}\big) \geq 1 - \epsilon$.

Theorem 2.

a. If the sequence $\{P^n\}$ *does not admit asymptotic arbitrage, and if* $x \in X$ *is asymptotically synthesized with bounded risk by a sequence of trading strategies* $\{\theta^n\}$*, then* $\lim_{n\to\infty} V_{\theta^n}(0) = \mathbf{E}^*[x]$.

b. Any sequence $\{P^n\}$ *created in the fashion of Section 3 (based on scaled random walks constructed from a random variable* ζ *with expectation zero, variance 1, and bounded support) does not admit asymptotic arbitrage.*

Proof. The proof of part *a* is straightforward (and is found in Kreps, 2019, Chapter 7). Kreps (2019) also provides an indirect proof of part *b*, but here is a sketch of direct proof:

Fix the (unscaled) increment distribution ζ. We will show that there are equivalent martingale measures P^{*n} for the P^n such that the sequences $\{P^n; n = 1, 2, \ldots\}$ and $\{P^{*n}; n = 1, 2, \ldots\}$ are continguous, hence by Klein and Schachermayer (1997), there can be no asymptotic arbitrage.

We do this by applying a discrete version of Girsanov's Theorem. For each $n = 1, 2, \ldots$, there are unique constants c_n and d_n such that the "Esscher transform"

$$\begin{aligned} Z^n(\omega) &:= e^{-c_n\omega(1)-d_n} \\ &= \exp\left\{\sum_{k=1}^{n}\left[-c_n\left(\omega\left(\frac{k+1}{n}\right)-\left(\frac{k}{n}\right)\right)-d_n\left(\frac{k+1}{n}\right)-\left(\frac{k}{n}\right)\right]\right\} \end{aligned} \tag{4.1}$$

defines the density of a martingale measure P^{*n} for $\{S(t,\omega) = e^{\omega(t)}; t \in [0,1]\}$ and that is equivalent to P^n; that is $dP^{*n}/dP^n = Z^n$. It is straightforward to check that the assumptions on the unscaled increment ζ—namely that $\mathbf{E}[\zeta] = 0$, $\mathbf{Var}[\zeta] = 1$, and ζ has bounded support—imply that $c_n \to 1/2$ and $d_n \to 1/8$, where $Z := e^{-\omega(1)/2-1/8} = \exp\{-(1/2)\int_0^1 d(\omega(t)) - (1/8)\int_0^1 dt\}$ is the Radon-Nikodym derivative dP^*/dP of P^* to P (Wiener measure) by Girsanov's formula.[3] Because $\|Z^n\|_{L^2(P)}$ and $\|(Z^n)^{-1}\|_{L^2(P^*)}$ are both uniformly bounded in n, mutual continguity of the sequences $\{P^n; n = 1, 2, \ldots\}$

[2] In Klein and Schachermayer (1997), following Kabanov and Kramkov (1994), this is called an asymptotic arbitrage of the second kind.

[3] The required calculations are provided in Kreps (2019, Chapter 5).

and $\{P^{*n}; n = 1, 2, \ldots\}$ follows, which implies that the sequence $\{P^n\}$ does not admit asymptotic arbitrage (Klein and Schachermayer, 1997). ∎

5. Proof of Theorem 1.

5.1. Preliminaries

Throughout, $\int_0^t \alpha\, dS(u)$ will mean the stochastic integral of α with respect to the process S over the interval from 0 to t, under the usual conditions (α is predictable and S-integrable). Note in particular that if α is constant on intervals of the form $[k/n, (k+1)/n)$, then $\int_0^t \alpha dS(u)$ is just the forward Itô sum

$$\int_0^t \alpha dS(u) = \sum_{j=0}^{k-1} \alpha\left(\frac{j}{n}\right)\left[S\left(\frac{j+1}{n}\right) - S\left(\frac{j}{n}\right)\right] + \alpha\left(\frac{k}{n}\right)\left[S(t) - S\left(\frac{k}{n}\right)\right], \tag{5.1}$$

where k is such that $k/n \leq t \leq (k+1)/n$.

In the standard (Strasbourg) way of doing stochastic integration, simple integrands are meant to be continuous from the left and having right limits (or càglàd), so α would be constant on $(k/n, (k+1)/n]$. Done this way, the interpretation of $\alpha(t)$ in this context would be that it is the portfolio holding at time t prior to any trading. For integrands α that are a.s. continuous, it does not matter, but our interpretation is that, for a trading strategy θ that is piecewise constant, $\theta(t, \omega)$ is the portfolio holding after time t trading is done, and so for such trading strategies, the formula (5.1) for the forward Itô sum is correct.

Theorem 1 is stated for contingent claims x that are bounded and continuous. It is without loss of generality – and saves on notation – to assume as well that $\mathbf{E}^*[x] = 0$: Suppose x is a bounded and continuous contingent claim. Then so is $x' := x - \mathbf{E}^*[x]$. Of course $\mathbf{E}^*[x'] = 0$. And if we can asymptotically replicate x' with bounded risk (in the sense of Theorem 1), then it is clear that we can do so for x as well: In addition to whatever sequence of trading strategies are employed to asymptotically replicate x' with bounded risk, add the purchase of a side portfolio of $\mathbf{E}^*[x]$ bonds, a side portfolio the composition of which never changes.

It is perhaps worth adding here that, as we assume that $\mathbf{E}^*[x] = 0$, our construction of trading strategies that asymptotically synthesize x to follow works entirely with zero-net initial endowment strategies. Hence that part of Theorem 1 that states that, for a given x, we are asymptotically synthesizing x with strategies with an initial net investment of $\mathbf{E}^*[x]$ follows immediately from the argument just provided.

As a final preliminary, we have the following:

Lemma 1. *Let x be a bounded and continuous function on $\big(C_0[0,1], \|\cdot\|_\infty\big)$. Then, for each t, there is a bounded and continuous version of $x(t) = \mathbf{E}^*[x|F_t]$ (defined in the proof to follow). This version of $\mathbf{E}^*[x|F_t]$ is uniquely determined for all continuous trajectories $\omega \in C_0[0,1]$. And if x is Lipschitz continuous with Lipschitz constant Λ, then (for each t) $x(t)$ is Lipschitz continuous with the same Lipschitz constant.*[4]

Proof. Let Ψ be a second copy of $C_0[0,1]$ with generic element ψ. Let $P^* \otimes Q^*$ be the product measure on $\Omega \times \Psi$, such that $(\omega, \psi) \in \Omega \times \Psi$ is two-dimensional Brownian motion with drift $-1/2$ in each coordinate and such that $P^* \otimes Q^*\big(\{(\omega,\psi) : \omega(0) = \psi(0) = 0\}\big) = 1$. That is, $\{\psi(t); 0 \le t \le 1\}$ under Q^* is a Brownian motion independent of and identically distributed as $\{\omega(t); 0 \le t \le 1\}$ under P^*. For the balance of this proof, write $\mathbf{E}^*$ as $\mathbf{E}^{P^*}$ to distinguish from $\mathbf{E}^{Q^*}$.

Define the concatenation at $t \in [0,1]$ of two paths ω and ψ, denoted $\omega \oplus_t \hat{\psi}$, as follows:

$$(\omega \oplus_t \psi)(u) := \omega(u)\, \mathbf{1}_{[0,t)}(u) + \big(\omega(t) + \psi(u) - \psi(t)\big)\, \mathbf{1}_{[t,1]}(u).$$

It is clear from the independence properties of Brownian motion that, fixing a path ω up to time t, the law that governs ω over $[t,1]$ is the same as the law that governs $\omega \oplus_t \psi$. Hence $\mathbf{E}^{P^*}[x|F_t](\omega) = \mathbf{E}^{Q^*}[x(\omega \oplus_t \psi)]$. That is, if we *define* $x(t,\omega)$ pointwise by

$$x(t,\omega) := \mathbf{E}^{Q^*}[x(\omega \oplus_t \psi)],$$

then $x(t,\cdot)$ is a version of $\mathbf{E}^{P^*}[x|F_t]$. Fix this specific version of $\mathbf{E}^{P^*}[x|F_t]$.

Suppose x is continuous and that $\{\omega_n\}$ is a sequence in $C_0[0,1]$ with limit ω. Then

$$\begin{aligned}\lim_n x(t,\omega_n) &= \lim_n \mathbf{E}^{Q^*}\big[x(\omega_n \oplus_t \psi)\big] \\ &= \mathbf{E}^{Q^*}\big[\lim_n x(\omega_n \oplus_t \psi)\big] = \mathbf{E}^{Q^*}\big[x(\omega \oplus_t \psi)\big] = x(\omega, t),\end{aligned}$$

where the key step is taking the limit instead the integral, a simple application of bounded convergence and the continuity of x.

To show that this version is the unique continuous version: Suppose $x'(t,\omega)$ is another continuous version of $\mathbf{E}^{P^*}[x|F_t]$. For each ω and $\ell = 1, 2, \ldots$, because P^* has full support on $C_0[0,1]$, there must be within the $1/\ell$ neighborhood of ω a path ω_ℓ such that $x'(t,\omega_\ell) = x(t,\omega_\ell)$. But then $x'(t,\omega) = \lim_{\ell\to\infty} x'(t,\omega_\ell) = \lim_{\ell\to\infty} x(t,\omega_\ell) = x(t,\omega)$, where the two outside equalities follow from the continuity of $x(t,\cdot)$ and the supposed continuity of $x'(t,\omega)$.

[4] It is also true that this version $x_t(\omega)$ is continuous in t for each ω, but we do not need this. We are very grateful to Rama Cont, who showed us how to prove this.

To complete the proof of the lemma, we must show that if x is Liptschitz continuous with Lipschitz constant Λ, then so is $x(t)$. Write

$$\begin{aligned}
|x(t,\omega)-x(t,\omega')| &= \left|\mathbf{E}^{Q^*}\left[x(\omega\oplus_t\psi)\right]-\mathbf{E}^{Q^*}\left[x(\omega'\oplus_t\psi)|\right]\right| \\
&\le \mathbf{E}^{Q^*}\left[\left|x(\omega\oplus_t\psi)-x(\omega'\oplus_t\psi)\right|\right] \\
&\le \mathbf{E}^{Q^*}\Big[\Lambda\Big\|\Big((\omega(u)\mathbf{1}_{[0,t)}(u)+\big(\omega(t)+\psi(u)-\psi(t)\big)\mathbf{1}_{[t,1]}\Big) \\
&\quad -\Big((\omega'(u)\mathbf{1}_{[0,t)}(u)+(\omega'(t)+\psi(u)-\psi(t))\mathbf{1}_{[t,1]}(u)\Big)\Big\|_\infty\Big] \\
&\text{(by the presumed Lipschitz continuity of } x) \\
&= \mathbf{E}^{Q^*}\left[\Lambda\left\|(\omega(u)-\omega'(u))\mathbf{1}_{[0,t)}(u)\right\|_\infty\right] \\
&\text{(because, path by path, the continuation portion } \psi \text{ cancels out)} \\
&= \Lambda\left\|(\omega-\omega')\mathbf{1}_{[0,t)}\right\|_\infty \\
&\text{(the integrand is constant with respect to } Q^*) \\
&\le \Lambda\|\omega-\omega'\|_\infty.
\end{aligned}$$

■

Although it is probably obvious, observe that $x(1,\omega)=x(\omega)$.

5.2. A sketch of the proof of Theorem 1

Because the reader may get lost in the details of the proof, here is an overview:

We begin by assuming that the claim x to be asymptotically synthesized is not only continuous and bounded, but also Lipschitz continuous as a function on $\big(C_0[0,1],\|\cdot\|_\infty\big)$. And—in many steps—we prove the following more technical and precise result:

Proposition 1. *Suppose that x is bounded and Lipschitz continuous. Denote by $x(t)$ the Lipschitz-continuous version of $\mathbf{E}^*[x|F_t]$ provided by Lemma 1. (Whenever we write $\mathbf{E}^*[x|F_t]$, we mean this version.)*

Then for every $\epsilon>0$, there exists N such that for all $n>N$, there is a predictable integrand α^n that is constant on the intervals $\big[k/n,(k+1)/n\big)$ and a stopping time τ^n, taking values in $\big\{k/n;k=1,\dots,n\big\}\cup\big\{\infty\big\}$, such that

$$P^n\big(\{\omega:\tau^n(\omega)=\infty\}\big)>1-\epsilon,\quad \textit{and} \tag{5.2a}$$

$$P^n\left(\left\{\omega:\left|\int_0^{\tau^n\wedge 1}\alpha^n dS(u)-x(\tau^n\wedge 1)\right|<\epsilon\right\}\right)=1, \tag{5.2b}$$

where $\tau^n\wedge 1$ means min$\{\tau^n,1\}$.

We know from the theory of the BSM model that, for the fixed x, there is a predictable integrand α such that $\int_0^1\alpha dS=x$ holds true P-almost surely. Moreover, if we write

$x(t,\omega) = \mathbf{E}^*[x|F_t]$ for the specific version of $\mathbf{E}^*[x|F_t]$ provided in Step 2, then $x(t)$ is a version of $\left[\int_0^t \alpha(u)dS(u)\right](\omega)$.

Our first step in proving the proposition is then to find an integer M and a predictable continuous-time process $\left(\alpha^M(t); 0 \leq t \leq 1\right)$ that is constant on each interval $\left[j/M, (j+1)/M\right)$ and such that

$$\int_0^1 \alpha^M dS \approx x = \int_0^1 \alpha dS = x(1), \text{ under the probability } P$$

where the symbol $\approx$ has to be made precise. Note in this regard that, because α^M is constant on intervals $\left[j/M, (j+1)/M\right)$, the value of $\left[\int_0^1 \alpha^M dS\right](\omega)$ can be defined path by path for all $\omega \in C[0,1]$ as the Itô sum,

$$\left[\int_0^1 \alpha^M dS\right](\omega) := \sum_{j=0}^{M-1} \alpha^M\left(\frac{j}{M}, \omega\right) \times \left[S\left(\frac{j+1}{M}, \omega\right) - S\left(\frac{j}{M}, \omega\right)\right],$$

where $S(j/M, \omega) = e^{\omega(j/M)}$. *And*, we can replace $\int_0^1 \alpha dS$ with $x = x(1)$. Using Doob's Maximal Inequality, we can extend this to show that

$$\int_0^t \alpha^M dS(u) \approx x(t) = \int_0^t \alpha dS(u) = x(t), \text{ uniformly in } t, \text{ under the probability } P.$$

We then pass to a finer mesh $\{k/n; k = 0, \ldots, n\}$ which splits each of the intervals from j/M to $(j+1)/M$ into ℓ pieces; that is, $n = \ell M$. If ℓ is large enough ($\ell > L$, for L to be determined), then the estimates in the first part of the proof show that

$$\int_0^{k/n} \alpha^M dS(u) \approx x(k/n), \quad \text{uniformly in } k = 1, \ldots, n,$$

under the probability P^n.

Finally, we stop the process α^M at the first time k/n where either the integral or the stock price is not behaving in a suitably desirable fashion. Because the support of ζ is bounded, for large enough L, stopping allows us to control the damage that can occur over the just-before-stopping interval, $(k - 1/n, k/n]$, which gives us (5.2). And we show that, for sufficiently large n, the probability (under P^n) that we must intercede in this fashion goes to zero, which is (5.1).

This will finish the proof of Proposition 1. It should be evident (but we'll give some details) that this proves Theorem 1 for Lipschitz-continuous and bounded contingent claims

x. To complete the proof of the theorem, we show that if Theorem 1 holds for Lipschitz-continuous and bounded contingent claims x, it holds for continuous and bounded claims.

5.3. Proof of Proposition 1

Throughout, Λ denotes the Lipschitz constant for the contingent claim x.

Step 1. *For $\epsilon > 0$, there is an integer M and a predictable integrand $\alpha^M = \big(\alpha^M(t); 0 \le t \le 1\big)$ that is uniformly bounded and constant on all intervals of the form $\big[j/M, (j+1)/M\big)$ and, for each $t \in \big\{j/M; j = 0, \ldots, M\big\}$, Lipschitz in the variable $\omega \in C_0[0,1]$, with the following property:*

$$P\bigg(\bigg\{\omega : \sup_{0\le t\le 1}\bigg|\int_0^t \alpha^M dS(u) - x(t)\bigg| < \epsilon\bigg\}\bigg) > 1 - \epsilon/2, \tag{5.3}$$

where $x(t)$ is the continuous version of $\mathbf{E}^*[x|F_t]$ given by Lemma 1.

Proof of Step 1. It is convenient to work under the equivalent martingale measure P^* of P. We therefore have that $dS(t) = S(t)dW^*(t)$, where $W^*(t) = W(t) + t/2$ is a P^* Brownian motion. Noting that $S(t)$ has quadratic variation $d\langle S\rangle(t) = S(t)^2dt$, we obtain the following version of Itô's isometry. Denote by R^* the measure on $[0,1]\times C_0[0,1]$ with density

$$\frac{dR^*}{d\big(\lambda\otimes P^*\big)}(t,\omega) = S(t,\omega)^2,$$

where $\lambda\otimes P^*$ denotes the product of Lebesgue measure λ on $[0,1]$ and P^*. We then have the Itô isometry

$$\big\|\beta(t,\omega)\big\|_{L^2([0,1]\times C_0[0,1],R^*)} = \bigg\|\int_0^1 \beta dS(t)\bigg\|_{L^2(P^*)}, \tag{5.4}$$

for every predictable process β for which the left-hand side is finite.

Let $\mathcal{P}$ denote the predictable sigma-algebra on $[0,1]\times C[0,1]$, generated by the filtration $\big\{F_t; 0 \le t \le 1\big\}$. That is, $\mathcal{P}$ is the sigma-algebra generated by the stochastic intervals $(\tau, 1]$, where τ runs through the stopping times pertaining to the filtration $\big\{F_t\big\}$. Note that, in the present case of the filtration of a Brownian motion, the predictable sigma-algebra coincides with the optional sigma-algebra which, by definition, is generated by the stochastic intervals of the form $[\tau, 1]$, where τ runs through the stopping times with respect to the filtration $\{F_t\}$. This is so because, in this case, every stopping time τ is predictable; i.e., $\tau = \lim_{\ell\to\infty}\tau_\ell$, where $\{\tau_\ell; \ell = 1, 2, \ldots\}$ is a sequence of strictly increasing stopping times. Let $\mathcal{G}^M$ denote the sigma-algebra generated by stochastic intervals of the form $[\tau^M, 1]$ for stopping times τ^M taking values in $\{j/M; j = 0, \dots, M\}$; we have that $\bigcup_M \mathcal{G}^M$ generates $\mathcal{P}$.

Define by $\overline{\alpha}^M$ the conditional expectation of α with respect to $\mathcal{G}^M$; that is

$$\overline{\alpha}^M = \mathbf{E}^{R^*}[\alpha|^M]. \tag{5.5}$$

In fact, a little care is needed here as R^* is not normalized to have mass 1. Hence (5.5) must be interpreted as the conditional expectation with respect to the re-normalized probability measure

$$\frac{R^*}{R^*\big([0,1] \times C_0[0,1]\big).}$$

In any case, as $\alpha \in L^2(R^*)$, the sequence $\big(\overline{\alpha}^M; M = 1, 2, \dots\big)$ converges to α in the norm of $L^2(R^*)$. Indeed, the sigma-algebras $\big(\mathcal{G}^M; M = 1, 2, \dots\big)$ generate the sigma-algebra $\mathcal{P}$ and $\|\alpha\|_{L^2(R^*)}$ is finite as $x = \int_0^1 \alpha dS(t)$ is bounded. Hence, by Itô's isometry (5.5), the sequence of random variables $x^M = \int_0^1 \overline{\alpha}^M dS(t)$ converges in the norm of $L^2(P^*)$ to $x = \int_0^1 \alpha dS(t)$.

We still have to pass from $\overline{\alpha}^M$ to an F^M-adapted process $\alpha^M = \alpha^M(t, \omega)$ that is uniformly bounded and Lipschitz continuous in ω. To do so, it suffices to approximate each of the finitely many $F^M(j/M)$-measurable random variables $\overline{\alpha}^M(j/M) \in L^2(P^*)$ by an $F^M(j/M)$-measurable and bounded Lipschitz function $\alpha^M(j/M)$ on $\big(C_0[0,1], \|\cdot\|_\infty\big)$ with respect to the norm of $L^2(P^*)$. The process that results by keeping these values during the respective intervals $\big(j/M, (j+1)/M\big], denoted by\, \alpha^M$, does what we want. Indeed, we can make the error $x - x^M$ arbitrarily small with respect to the norm of $L^2(P^*)$. Finally, we apply the L^2-version of Doob's maximal inequality[5] to not only make $\big\| \int_0^1 (\alpha - \alpha^M) dS \big\|_{L^2(P^*)}$ small, but also $\big\| \sup_{0 \le t \le 1} \int_0^t (\alpha - \alpha^M) dS(u) \big\|_{L^2(P^*)}$. Using the fact that P and P^* are equivalent, we obtain inequality (6.3).

Step 2. *Because the integrand α^M is Lipschitz in $\omega \in C_0[0,1]$ and changes value only finitely many times, we know that the function x^M defined pathwise by*

$$\begin{aligned} x^M(t)(\omega) := \left[\int_0^t \alpha^M dS(u)\right](\omega) &= \alpha^M\left(\frac{j}{m}, \omega\right) \cdot \left[S(t,\omega) - S\left(\frac{j}{m}, \omega\right)\right] \\ &+ \sum_{i=0}^{j} \alpha^M\left(\frac{i}{m}, \omega\right) \cdot \left[S\left(\frac{i+1}{m}, \omega\right) - S\left(\frac{i}{m}, \omega\right)\right] \end{aligned},$$

[5] Doob (1990, Theorem 3.4); see also Acciaio et al. (2013).

where j is such that $j/M \le t < (j+1)/M$, is also Lipschitz on bounded subsets of $C[0,1]$ under the sup norm, uniformly in $t \in [0,1]$.

Proof of Step 2. Consider a bounded set $\mathcal{B}$ in $C[0,1]$, so that there is a constant $C \ge 0$ such that $S(t)(\omega) = e^{\omega(t)} \le C$ for every $\omega \in \mathcal{B}$. We know that there is a uniform Lipschitz constant $\mathcal{L}$ for the functions $\{\alpha^M((j-1)/M); j = 1, \dots, M\}$ on $(C_0[0,1], \|\cdot\|_\infty)$ as well as a uniform bound on $\{\|\alpha^M((j-1)/M)\|_\infty; j = 1, \dots, M\}$, which we may assume is the same $C > 0$. We must show that there is a constant $\mathcal{C} > 0$ such that

$$\left|x^M(t)(\omega) - x^M(t)(\omega')\right| \le \mathcal{C}\, \|\omega - \omega'\|, \tag{5.6}$$

for all $\omega, \omega' \in B$ and $t \in [0,1]$. Clearly, it will suffice to show that there is a constant, denoted by $\mathcal{K}$, such that

$$\left|\int_{(j-1)/M}^{t} \alpha^M dS(u)(\omega) - \int_{(j-1)/M}^{t} \alpha^M dS(u)(\omega')\right| \le \mathcal{K}\, \|\omega - \omega'\|, \tag{5.7}$$

for every $j = 1, \dots, M$, $t \in \big[(j-1)/M, j/M\big]$, and for all $\omega, \omega', j = 1, \dots M$, and $t \in [(j-1)/M, j/M]$. For then (5.7) will imply (5.6), where we take $\mathcal{C} = \mathcal{K}M$.

And to show (5.7), note that, for $d = \|\omega - \omega'\|_\infty$, we have the estimates

$$\left|\alpha^M\left(\frac{j-1}{M}\right)(\omega) - \alpha^M\left(\frac{j-1}{M}\right)(\omega')\right| \le \mathcal{L}d,$$

$$\left|S(t)(\omega) - S(t)(\omega')\right| \le 2Cd, \text{ and } \left|S\left(\frac{j-1}{M}\right)(\omega) - S\left(\frac{j-1}{M}\right)(\omega')\right| \le 2Cd.$$

Putting these together, we have

$$\begin{aligned}
&\left|\int_{(j-1)/M}^{t} \alpha^M dS(u)(\omega) - \int_{(j-1)/M}^{t} \alpha^M dS(u)(\omega')\right| \\
&= \left|\alpha^M\left(\frac{j-1}{M}\right)(\omega) \times \left[S(t)(\omega) - S\left(\frac{j-1}{M}\right)(\omega)\right]\right. \\
&\quad \left. - \alpha^M\left(\frac{j-1}{M}\right)(\omega) \times \left[S(t)(\omega) - S\left(\frac{j-1}{M}\right)(\omega)\right]\right| \le 2C(\mathcal{L}+C)d,
\end{aligned}$$

where we have used the inequality $|ab - a'b'| \le 2C(\mathcal{L}+C)d$, provided that $|a-a'| \le \mathcal{L}d$, $|b-b'| \le Cd$, and $\max\big\{|a|, |a'|, |b|, |b'|\big\} \le 2C$.

This finishes the proof of Step 2.

Step 3. Because $P^n \Rightarrow P$, the family of distributions $\{P^n; n = 1, \ldots\} \cup \{P\}$ is tight, and so for given $\epsilon > 0$ there is a bound B such that the event

$$D_B := \{\omega : |\omega(t)| < B \text{ for all } t \in [0,1]\}$$

has $P^n(D_B) \ge 1 - \epsilon/2$ for all n and $P(D_B) \ge 1 - \epsilon/2$.

Step 4. We know from (5.3) that the Borel set

$$D^{M,\epsilon} := \left\{\omega : \sup_{0 \le t \le 1} \big|x(t)(\omega) - x^M(t)(\omega)\big| < \epsilon\right\}$$

has $P\big(D^{M,\epsilon}\big) > 1 - \epsilon/2$. Recall the uniform Lipschitz continuity of the functions $x(t)$ established in Lemma 1 and the uniform Liptschitz continuity of the function $x^M(t)$ shown to hold true on bounded subsets of $C[0,1]$ shown in Step 3. We therefore obtain that the set $D^{M,\epsilon} \cap D_B$ is *open* in $\big(C_0[0,1], \|\cdot\|_\infty\big)$. Because $P(D^{M,\epsilon} \cap D_B) > 1 - \epsilon$, by applying the Portmanteau Theorem for functional weak convergence (Billingsley, 1999, Theorem 2.1), we conclude that, for large enough n,

$$P^n\big(D^{M,\epsilon} \cap D_B\big) > 1 - \epsilon. \tag{5.8}$$

Therefore, for large enough L and all $n = \ell M$ for $\ell > L$, (5.8) is true.

Step 5. Define stopping times τ_1^n, τ_2^n, and τ^n for each $\omega \in C[0,1]$ by

$$\begin{aligned}
&\tau_1^n(\omega) := \inf\left\{t = k/n : \big|x(k/n)(\omega) - x^M(k/n)(\omega)\big| \ge \epsilon\right\}, \\
&\tau_2^n(\omega) := \inf\left\{t = k/n : \ln\big(S(t,\omega)\big) \notin \big[-B, B\big]\right\}, \quad \text{and} \\
&\tau^n(\omega) = \min\big\{\tau_1^n(\omega), \tau_2^n(\omega)\big\},
\end{aligned}$$

where, by convention, $\tau_1^n(\omega) = \infty$ if $\big|x(k/n)(\omega) - x^M(k/n)(\omega)\big| < \epsilon$ for $k = 0, \ldots, n$, and $\tau_2^n(\omega) = \infty$ if $\ln\big(S(k/n,\omega)\big) \in \big[-B, B\big]$ for $k = 0, \ldots, n$. These stopping times are all well defined for all $\omega \in C[0,1]$.

In particular, an investor using the portfolio strategy α^M in the nth discrete-time economy can stop according to these stopping rules: Clearly, she will know the current stock price, so implementing τ_2 is trivial. As for implementing τ_1, $x^M(k/n)(\omega)$ is just the value of her portfolio at time k/n, for the path ω she has observed (where we fill in between times k/n and $(k+1)/n$ with linear interpolation), while $x(k/n)(\omega)$ is calculated from the path ω up to time k/n as in Lemma 1.

Step 6. Note that for any path $\omega \in D_B \cap D^{M,\epsilon}$, $|\omega(t)| < B$ for all t, hence for all t of the form k/n. And for $\omega \in D_B \cap D^{M,\epsilon}$, $\left|x(t)(\omega) - x^M(t)(\omega)\right| < \epsilon$ for all t, hence for all t of the form k/n. This implies that for all $\omega \in D_B \cap D^{M,\epsilon}$, $\tau_1^n = \tau_2^n = \tau^n = \infty$. Hence, for large enough L (which gives large enough n),

$$P^n\big(\{\omega : \tau^n(\omega) = \infty\}\big) \geq P^n\big(D_B \cap D^{M,\epsilon}\big) > 1 - \epsilon.$$

That is, we have $(5.2a)$.

Step 7. It remains to show that $(5.2b)$ holds. Please recall that, by assumption, there is a sequence $\{\delta_n\}$ of positive numbers such that $\delta_n \to 0$ and

$$P^n\big(\{\omega : |\omega(k/n) - \omega((k-1)/n)| > \delta_n\}\big) = 0.$$

If either $\tau_1^n(\omega) = \infty$ or if $\tau_2^n(\omega) < \tau_1^n(\omega) < \infty$, then at time $\tau^n \wedge 1$, we know that $\left|x(\tau^n \wedge 1)(\omega) - x^M(\tau^n \wedge 1)(\omega)\right| < \epsilon$. Hence for all such ω, $(5.2b)$ holds.

This leaves the case of paths ω such that $\tau_1^n(\omega) \leq \tau_2^n(\omega)$ and $\tau_1^n(\omega) \leq 1$. For such an ω, let $\tau_1^n(\omega) = k/n$, and consider the state of affairs at time $(k-1)/n$. Since neither stopping time τ_1^n nor τ_2^n has triggered, we know that

$$S((k-1)/n) \leq e^B \quad \text{and} \quad \left|x((k-1)/n) - x^M((k-1)/n)\right| < \epsilon.$$

We must show that

$$\left|x(k/n) - x^M(k/n)\right| < 2\epsilon,$$

which we do by showing that, for large enough L,

$$\left|x(k/n) - x((k-1)/n)\right| \leq \epsilon/2 \text{ and } \left|x^M(k/n) - x^M((k-1)/n)\right| \leq \epsilon/2.$$

The first of these follows easily from the Lipschitz continuity of x. Fixing the path of ω up to time $(k-1)/n$, consider two possible continuations, ω and ω'. That is, ω and ω' are partial paths up to time k/n that coincide up to time $(k-1)/n$. Then

$$\sup\left\{\left|\omega(i/n)-\omega'(i/n)\right|; i=0,\ldots,k\right\} = \left|\omega(k/n)-\omega'(k/n)\right|,$$

and, since ω and ω' coincide up to time $(k-1)/n$, $\left|\omega(k/n)-\omega'(k/n)\right|$ can be no larger than $2\delta_n$, P^n-a.s. By choosing L to be large enough, this can be made as small as needed so that, taking into account the Lipschitz constant for x, we get the desired bound on $\left|x(k/n)-x((k-1)/n)\right|$.

And, finally, to bound $\left|x^M(k/n)-x^M((k-1)/n)\right|$, write

$$x^M\left(\frac{k}{n}\right)-x^M\left(\frac{k-1}{n}\right) = \alpha^M\left(\frac{k-1}{n}\right)\times\left[S\left(\frac{k}{n}\right)-S\left(\frac{k-1}{n}\right)\right]$$

$$= \alpha^M\left(\frac{k-1}{n}\right)\times S\left(\frac{k-1}{n}\right)\times\left[\frac{S(k/n)-S((k-1)/n)}{S((k-1)/n)}\right]$$

$$= \alpha^M\left(\frac{k-1}{n}\right)\times S\left(\frac{k-1}{n}\right)\times\left[\frac{e^{\omega(k/n)}}{e^{\omega((k-1)/n)}}-1\right]$$

$$= \alpha^M\left(\frac{k-1}{n}\right)\times S\left(\frac{k-1}{n}\right)\times\left[e^{\omega(k/n)-\omega((k-1)/n)}-1\right].$$

We are looking at paths ω such that $\tau_1^n \le \tau_2^n$, so we know that $S((k-1)/n) \le e^B$. And we know that α^M is uniformly bounded. And the final term in the product can be made a small as necessary to make the product less than $\epsilon/2$, because $|\omega(k/n)-\omega((k-1)/n)| < \delta_n$, P^n-a.s. This completes Step 7.

Step 8. *We therefore have the result for all* n *that are of the form* ℓM*, for any* $\ell > L$*. To finish the proof of Proposition 1, we must show that, enlarging* L *still further as necessary, the result is true for all* $n > (L+1)M$*.*

This is accomplished as follows. We have fixed M and L. For every $n > (L+1)M$, and for $j = 0,\ldots,M$, let $k_{j,n}$ be the least integer such that $k_{j,n}/n \ge j/m$. Of course, $k_{j,n} - j/m < 1/n$. Modify the construction given above: writing $\alpha_j^M(\omega)$ for the value of α^M previously applied on the interval $[j/M,(j+1)/M)$ (which is based on the path of stock prices up to time j/M), delay slightly the shift from α_{j-1}^M to α_j^M by (instead) holding $\alpha_j^M(\omega)$ shares of stock over the interval $[k_{j,n}/n, k_{j+1,n}/n)$. The stopping rule τ^n is defined just as before.

This (slight) shift in when the portfolio's composition changes is properly adapted to the information received. And, as $n \to \infty$ for fixed M, the change it causes in the value of the portfolio uniformly tends to zero: The stopping rule puts a bound on the price of the stock and the number of shares of stock held is uniformly bounded, so the "error" introduced by this slight delay vanishes as the amount by which the stock price can move over any single interval of length $1/n$ vanishes (in n, again relying on the bound in stock prices before stopping). And, since M is fixed, there is a fixed number of such "errors" that are introduced. Enlarging L as necessary (holding M fixed), the sum of these M errors become (uniformly) arbitrarily small, completing the proof of Proposition 1.

5.4. Completing the proof of Theorem 1

We have proved Proposition 1 for Lipschitz continuous x. This almost immediately gives Theorem 1 for such x: Given ϵ, find N sufficiently large so that for all $n > N$, there exist α^n and τ^n that satisfy (5.1) and (5.2). Re-interpret α^n and τ^n as a trading strategy $\hat{\alpha}^n$ for the nth discrete-time economy where

$$\hat{\alpha}^n(t,\omega) = \begin{cases} \alpha^n(t,\omega), & \text{if } t < \tau^n(\omega), \mathit{and} \\ 0, & \text{if } t \geq \tau^n(\omega). \end{cases}$$

In words, "stopping" according to τ^n is interpreted as converting the value of the portfolio held at that time entirely into bonds and doing no further trading until time 1. On the event $\{\omega : \tau^n(\omega) = \infty\}$, which has probability greater than $1-\epsilon$, the integral $\big[\int_0^1 \hat{\alpha}^n dS(u)\big](\omega) = \big[\int_0^1 \alpha^n dS(u)\big](\omega) = x(1,\omega)$ will be within ϵ of $x(1,\omega) = x(\omega)$. And on the event $\{\omega : \tau^n(\omega) < \infty\}$, $\big[\int_0^1 \hat{\alpha}^n dS(u)\big](\omega) = \big[\int_0^{\tau^n} \hat{\alpha}^n dS(u)\big](\omega)$, which is within ϵ of $x(\tau^n)$, which of course has expected value less or equal to $\|x\|_\infty$.

This almost gives us Theorem 1, except that, for fixed $\epsilon > 0$, the probability-one bound on the synthesized claims are that their values lie in the interval $(\underline{x} - \epsilon, \overline{x} + \epsilon)$. We want the synthesized claims to have values that lie in the interval $(\underline{x}, \overline{x})$.[6]

[6] Theorem 1 as given here is reported in Kreps (2019) as Proposition 4.1*b*. Readers who compare these two will see that in Kreps (2019), Proposition 4.1*b* is stated with slightly wider probability-one bounds on the x^n. The statement of Theorem 1 given here is "cleaner" than its counterpart in Kreps (2019), although, the careful reader will note, this has no discernible effect on the uses to which this result can be put.

Recall that we assumed, without loss of generality, $\mathbf{E}^*[x] = 0$. Except for the trivial case where $x \equiv 0$, this implies that $\underline{x} < 0$ and $\overline{x} > 0$. For the fixed ϵ, let α be close enough to 1 so that $(1-\alpha)\overline{x} < \epsilon/2$ and $(\alpha - 1)\underline{x} < \epsilon/2$. Let $\delta = \min\{(\alpha-1)\underline{x}, (1-\alpha)\overline{x}\}$. Of course, $\delta < \epsilon/2$.

Consider the claim $x' = \alpha x$: (a) if x is Lipschitz continuous, then so is x'; (b) $\mathbf{E}^*[x] = 0$ implies $\mathbf{E}^*[x'] = 0$; and (c) $\underline{x}' = \alpha \underline{x}$ and $\overline{x}' = \alpha\overline{x}$. Employ Proposition 1 for the claim x' and for δ in place of ϵ. This guarantees that there exists N sufficiently large so that, for all $n > N$, a claim x^n can be synthesized in the nth discrete time economy such that, with P^n-probability $1-\delta$ or greater, $|x^n(\omega) - x'(\omega)| < \delta$, and $\underline{x}' - \delta < x^n(\omega) < \overline{x}' + \delta$ with P^n-probability 1,. Since $|x'(\omega) - x(\omega)| = |\alpha x(\omega) - x(\omega)| = |(1-\alpha)x(\omega)| < \epsilon/2$ by the choice of α, we know that $|x^n(\omega) - x(\omega)| \leq |x^n(\omega) - x'(\omega)| + |x'(\omega) - x(\omega)| < \delta + \epsilon/2 < \epsilon$ with P^n-probability $1-\delta$ or more, which is certainly $1-\epsilon$ or more under P^n. And if $x^n(\omega) \in (\underline{x}' - \delta, \overline{x}' + \delta)$, since $\underline{x}' - \delta = \alpha\underline{x} - \delta > \underline{x}$, and $\overline{x}' + \delta = \alpha\overline{x} + \delta < \overline{x}$, we know that, with P^n-probability one, $x^n \in (\underline{x}, \overline{x})$. Hence we have the tighter probability-one bounds on the synthesized claims.

As a final step, we need to extend Theorem 1 from Liptschitz-continuous and bounded contingent claims x to continuous and bounded contingent claims. We employ the following Lemma:

Lemma 2. *Fix a bounded and continuous function x on $\big(C_0[0,1], \|\cdot\|_\infty\big)$. For $\Lambda > 0$, define $x^\Lambda(\omega) := \mathit{inf}\big\{x(\omega') + \Lambda\|\omega - \omega'\|; \omega' \in C_0[0,1]\big\}$ for each $\Lambda > 0$. Then:*

a. *If $\Lambda < \Lambda'$, then $x^\Lambda(\omega) \leq x^{\Lambda'}(\omega) \leq x(\omega)$. And if $\underline{x} = \mathit{inf}_\omega\, x(\omega)$, then $x^\Lambda(\omega) \geq \underline{x}$. Hence for all Λ, $\|x^\Lambda\| \leq \|x\|$.*

b. *For all ω, $\mathit{lim}_{\Lambda\to\infty} x^\Lambda(\omega) = x(\omega)$. Hence, by monotone convergence, $\mathit{lim}_{L\to\infty} \mathbf{E}^*[x^\Lambda] = \mathbf{E}^*[x]$. And for any compact set $\mathcal{K}$ in $\big(C_0[0,1], \|\cdot\|_\infty\big)$ and for any $\epsilon > 0$, there is sufficiently large Λ (depending on $\mathcal{K}$ and ϵ) such that $x(\omega) - x^\Lambda(\omega) \;\leq \epsilon$ for all $\omega \in \mathcal{K}$.*

c. *x^Λ is Lipschitz continuous with Lipschitz constant Λ.*

This is a standard construction, so we omit the proof.

Now fix a bounded and continuous claim x such that $\mathbf{E}^*[x] = 0$ and some $\epsilon > 0$. We have the following (asymptotic) estimates.

i. Since $\mathbf{E}^*[x^\Lambda] \nearrow \mathbf{E}^*[x] = 0$, for all large enough v, $|\mathbf{E}^*[x^\Lambda]| \leq \epsilon/3$.

ii. Since $P^n \Rightarrow P$, the tightness of probability measures $\{P^n\}$ allows us to produce a compact subset $\mathcal{K}$ of $C[0,1]$ such that $P^n(\mathcal{K}) > 1 - \epsilon/3$ for all n.

iii. Apply Lemma 2: For this compact set $\mathcal{K}$ and for all large enough Λ, $|x^\Lambda(\omega) - x(\omega)| \leq \epsilon/3$ for all $\omega \in K$.

iv. Since x^Λ is Lipschitz, so is $\hat{x}^\Lambda := x^\Lambda - \mathbf{E}^*[x^\Lambda]$. And $\|\hat{x}^\Lambda\| \leq \|x^\Lambda\| + |\mathbf{E}^*[x^\Lambda]| \leq \|x\| + \epsilon/3$

v. Theorem 1 for Lipschitz continuous and bounded functions ensures that for all sufficiently large n, we can produce in the nth discrete time economy a contingent claim x^n such that

$$P^n\big(\{|x^n(\omega)| \ \leq \|\hat{x}^\Lambda\| + \epsilon/3\}\big) = 1 \ \text{ and } \ P^n\big(\{|x^n(\omega) - \hat{x}^\Lambda(\omega)| \leq \epsilon/3\}\big) \geq 1 - \epsilon/3.$$

Combining these estimates, we have that, for all sufficiently large n:

vi. $P^n\big(\{|x^n(\omega)| \ \leq \|x\| + 2\epsilon/3\}\big) = 1$, and

vii. If we denote by $\mathcal{J}^n$ the set $\{|x^n(\omega) - \hat{x}^\Lambda(\omega)| \leq \epsilon/3\}$, then on the set $\mathcal{J}^n \cap \mathcal{K}$, $\big|x^n(\omega) - x(\omega)\big| \leq \big|x^n(\omega) - \hat{x}^\Lambda(\omega)\big| + \big|\hat{x}^\Lambda(\omega) - x^\Lambda(\omega)\big| + \big|x^\Lambda(\omega) - x(\omega)\big| \leq \epsilon$.

viii. And $P^n(\mathcal{J}^n \cap \mathcal{K}) \geq 1 - 2\epsilon/3$.

This completes the proof of Theorem 1. ∎

6. Vanishing risk?

Theorem 1 establishes the ability to asymptotically synthesize bounded and continuous contingent claims with bounded risk. It would be better (in terms of saying, for instance, that markets are asymptotically complete) if we could replace "bounded risk" with "vanishing risk" as defined earlier. But, this is not possible, even for examples in which the P^n are created in the fashion of Section 3, for ζ with bounded support.

Consider, for instance, ζ with the following distribution:

$$\zeta = \begin{cases} 1.5, & \text{with probability } 2/9, \\ 0, & \text{with probability } 5/9, \text{ and} \\ -1.5, & \text{with probability } 2/9. \end{cases}$$

Imagine trying to synthesize a European put option with strike price 1 on the stock, $x(\omega) = \big(1 - S(1,\omega)\big)^+$. Asymptotically synthesizing x with vanishing risk implies doing so with bounded risk, so by Theorem 2*a*, the initial investment for doing in the nth model must converge to $\mathbf{E}^*[x] \approx 0.38239 > 0$. But, for any n, in the nth discrete-time economy, there is positive probability of the path ω^o for which $S(t)$ never moves from 1. Along this path, the stock produces neither capital gains nor capital losses, and so every portfolio strategy θ gives $V_\theta(t, \omega^o) = V_\theta(0)$. Synthesis with vanishing risk would require that, for every path ω with positive probability, and in particular, for ω^o, $V_\theta(1, \omega)$ is close to $x(\omega)$. But $x(\omega^o) = (1 - S(1, \omega^o))^+ = 0$. Since $V_\theta(1, \omega^o) = V_\theta(0)$, we can't have both $V_\theta(1, \omega^o)$ close to 0 and $V_\theta(0)$ close to 0.38239.

7. Concluding remarks

Theorem 1 is easily extended to claims that are unbounded on one side. Suppose x is a continuous claim with $x \geq 0$ (for all ω) and $\mathbf{E}^*[x] < \infty$. Then a simple corollary to Theorem 1 is that, for any $\epsilon > 0$, there exists N such that for all $n > N$, a claim x^n can be synthesized in the nth discrete-time economy for an initial investment of $\mathbf{E}^*[x]$ and such that $P^n\big(\{\omega : x^n(\omega) - x(\omega) > \epsilon\}\big) < \epsilon$ and $P^n\big(\{\omega : x^n(\omega) > -\epsilon\}\big) = 1$. (Proof: Define x^B by $x^B(\omega) := \min\{x(\omega), B\}$. Choose B large enough so that $P^n\big(\{\omega : x(\omega) > B\}\big) \leq \epsilon/3$ uniformly in n (recall that $P^n \Rightarrow P$ for the uniformity), and such that $\mathbf{E}^*[x] - \mathbf{E}^*[x^B] < \epsilon/3$. And then apply Theorem 1 to x^B for $\epsilon/3$ in place of ϵ.[7]

And, Theorem 1 has the following application to the consumer's problem of choosing an optimal consumption bundle. The context is where all consumption is done at time 1; the consumer is endowed with initial wealth in the form of an endowment of stock and bond, whose value at time 0 is denoted W, and she trades (in self-financing fashion) from time 0 to time 1, where she consumes the "dividend" paid by her portfolio at time 1. Suppose the consumer is an expected utility maximizer, with continuous and nondecreasing utility function, endowment wealth W, and subjective beliefs given by P. We are agnostic as to whether she is constrained to nonnegative consumption or not. In the continuous-time limit model, she has access to complete markets, with price of claims given by $\mathbf{E}^*[\cdot]$. Let

$$U^* := \sup\big\{\mathbf{E}[u(x)] : x \text{ such that } \mathbf{E}^*[x] \leq W \text{ and, if the nonnegativity constraint is imposed, } x \geq 0\big\}.$$

Under mild conditions on u (for instance, concavity), she has the same supremal expected-utility level if she restricts attention to bounded and continuous claims x. More-

[7] One must be careful in general how to interpret this. Kreps (2019, Chapter 4) provides an example of a nonnegative and continuous (but not Lipschitz continuous) x and where ζ is binomial—hence for each n there is a unique emm P^{*n} — such that $\mathbf{E}^*[x]$ is finite but $\mathbf{E}^{*n}[x] \to \infty$.

over, if $x \geq 0$ is imposed, she can achieve the same supremal level of expected utility with claims that are bounded and continuous and uniformly (in ω) bounded away from zero and that satisfy her budget constraint. Fix ϵ and let $\overline{x}$ be such a claim that comes within $\epsilon/2$ of U^* in terms of expected utility. (If $U^* = \infty$, instead fix K arbitrarily large and have $\hat{x}$ be a claim whose expected utility is at least $K + 1$.) Then, as corollary to Theorem 1, there is N large enough so that, for $n \geq N$, she can in the nth discrete-time economy synthesize a claim whose expected utility is within $\epsilon/2$ of the expected utility of $\hat{x}$. That is, she can do at least as well asymptotically in the discrete-time economies, despite any incomplete markets, as she can in the complete-markets limit economy. For full details, see Kreps (2019, Proposition 5.2).

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Appendix. Vanishing risk—a positive result

We can provide a positive result concerning asymptotic synthesis of contingent claims with vanishing risk, if the probability measure, subsequently denoted by $P^{\mathcal{L}}$, that governs (via scaled copies of a fixed ζ) the evolution of stock prices supports paths (and markets) that can, at stopping times, move from paths on a coarser grid to paths on a finer grid. A discussion of the economic meaning of this construction will follow definitions and analysis.

In this construction, we restrict attention to models built in the fashion of the example of Section 3: Fix a random variable ζ with expectation zero, unit variance, and bounded support.

Consider an ensemble $\mathcal{L}$ consisting of a sequence of stopping times $\{\tau_n; n = 0, 1, \ldots\}$

and a sequence of random variables $\{\nu_n; n = 0, 1, \ldots\}$, together with a probability measure $P^{\mathcal{L}}$ on $C_0[0,1]$, with the following properties:

1. ν_0 is a deterministic number of the form $\nu_0 = 2^{\eta_0}$. For each $n = 0, 1, \ldots$, $\nu_{n+1} > \nu_n$. Each ν_n takes values of the form 2^{η_n} for positive integer η_n.
2. $\tau_0 = 0$, and for each $n = 1, 2, \ldots$, τ_n is either of the form $\tau_{n-1} + \kappa_{n-1}/\nu_{n-1}$ or ∞, where if $\tau_n \neq \infty$, then $\tau_n < 1$. Put differently, given $\tau_{n-1} < 1$, we have either $\tau_n = \infty$ or κ_{n-1} is integer valued with $\kappa_{n-1} < \big(1 - \tau_{n-1}\big)/\nu_{n-1}$, in which case $\tau_n = \tau_{n-1} + \kappa_{n-1}/\nu_{n-1}$.
3. If $\tau_n < 1$, then for all $k = 1, 2, \ldots, \kappa_n$, the distribution of $\omega(\tau_n + k/\nu_n) - \omega(\tau_n + (k-1)/\nu_n)$ under $P^{\mathcal{L}}$ is the distribution of $\zeta/\sqrt{\nu_n}$, independent of any prior history of the path ω. For $t \in [\tau_n + (k-1)/\nu_n, \tau_n + k/\nu_n]$, paths of ω are filled in by linear interpolation. (This condition defines $P^{\mathcal{L}}$, as explained below.)
4. The random variables τ_n and ν_n are stoppping times with respect to the filtration $\{\mathcal{F}(t); 0 \leq t \leq 1\}$ and the ν_n are $\mathcal{F}(\tau_n)$-measurable.
5. $\lim_{n\to\infty} \tau_n = \infty$, $P^{\mathcal{L}}$-almost surely.

This description of $\mathcal{L}$ and the implicit definition of the corresponding probability measure $P^{\mathcal{L}}$ on $C_0[0,1]$ is fairly opaque, hiding a relatively simple (iterative) construction. At time 0, a value $\nu_0 = 2^{\eta_0}$ is fixed. Paths ω are built for the interval $[0, 1/\nu_0]$ to be linear, starting at $\omega(0) = 0$ and ending at $\omega(1/\nu_0)$, which has the distribution of $\zeta/\sqrt{\nu_0}$. If, given the value of $\omega(1/\nu_0)$, the stopping time τ_1 is not $1/\nu_0$, then the path continues over the interval $[1/\nu_0, 2/\nu_0]$ in (piecewise) linear fashion, with $\omega(2/\nu_0) - \omega(1/\nu_0)$ having the distribution of $\zeta/\sqrt{\nu_0}$, independent of what happened previously.

The step-by-step construction of paths (and the measure $P^{\mathcal{L}}$) thus continues. Either $\tau_1 = \infty$ (for the path that is constructed step-by-step), in which case the construction continues for ν_0 steps, when time 1 is reached. But if $\tau_1 < \infty$, in which case $\tau_1 = \kappa_0/\tau_0 < 1$, the construction of paths from time τ_1 until τ_2, if $\tau_2 < 1$, or until time 1, if $\tau_2 = \infty$, continues, but with a finer "grid size" $1/\nu_1$, for $\nu_1 = 2^{\eta_1} > \nu_0$. Paths over this (random) interval are piecewise linear, with the distribution of $\omega\big(\tau_0 + (k+1)/\nu_1\big) - \omega\big(\tau_0 + k/\nu_1\big)$ having the law of $\zeta/\sqrt{\nu_1}$, independent of everything previous. If $\tau_2 = \infty$, this proceeds until time 1; if $\tau_2 = \kappa_0/\nu_0 + \kappa_1/\nu_1 < 1$, then starting from τ_2, the grid size shrinks again to $1/\nu_2$, and so forth. The final requirement, Condition 5, is that for $P^{\mathcal{L}}$-almost every path constructed in this fashion, time 1 is reached after a finite number of "switches."

For each such ensemble $\mathcal{L}$, let $\mathcal{T}(\omega)$ be the set of $t \in [0,1]$ of the form $\tau_n(\omega) + k/\nu_n(\omega)$, for $k = 0, \ldots, \kappa_n(\omega)$. We imagine that $\mathcal{T}(\omega)$ is the set of "available trading times" for consumer-investors along the path ω, times when they can rearrange their portfolios of stock and bond (in self-financing fashion). There is a one-to-one correspondence between ensembles $\mathcal{L}$ and the set $\mathcal{T}$ of trading times consistent with $\mathcal{L}$; we could have had $\mathcal{T}$ as the primitive. But it is harder to write down the "rules" in terms of structure and measurability of a "legitimate" set $\mathcal{T}$.

That said, note that $\mathcal{L}$ is a directed set, with $\mathcal{L} = \{\tau_n, \nu_n\} \succ \mathcal{L}' = \{\tau'_n, \nu'_n\}$ if the times satisfy $\tau_n \leq \tau'_n$ and the fineness scales satisfy $\nu_n \geq \nu'_n$. Expressing this in terms of the $\mathcal{T}$ is simple: If $\mathcal{T}$ are the trading times for $\mathcal{L}$ and $\mathcal{T}'$ are the trading times for $\mathcal{L}'$, then $\mathcal{L} \succ \mathcal{L}'$ if and only if $\mathcal{T} \supset \mathcal{T}'$.

Fixing $\mathcal{L}$ or, equivalently, $\mathcal{T}$ or, equivalently (for fixed ζ) the measure $P^{\mathcal{L}}$, we imagine a consumer–investor who takes an initial position in the stock and bond at time 0 and trades at available trading times in a self-financing and non-anticipatory manner. Letting $\theta(t, \omega)$ denote the number of shares of the "stock" in her portfolio at time t after she trades (if allowed to do so) at time t, we have that $\theta(t, \omega)$ is piecewise constant (in our manner of doing things, right continuous with left limits), with $\theta(t, \omega)$ changing values *only* at available trading times $t \in \mathcal{T}(\omega)$. The value of her portfolio, denoted $V_\theta(t, \omega)$, is computed as the value of its initial value $V_\theta(0)$ plus the sum of capital gains (or losses) accrued; because of Condition 5, this value is a well-defined finite sum with $P^{\mathcal{L}}$ probability 1.

Theorem 3. *Suppose random variable ζ has expectation 0, variance 1, and bounded support. For every bounded and continuous claim $x : C_0[0, 1] \to dR$, and for every $\epsilon > 0$, there is an ensemble $\mathcal{L}^\epsilon$ as above such that for all ensembles $\mathcal{L}$ such that $\mathcal{L} \succ \mathcal{L}^\epsilon$, there is a trading strategy θ for $\mathcal{L}$ such that*

$$P^{\mathcal{L}}\left(\left\{\omega : \left|V_\theta(1, \omega) - x(\omega)\right| < \epsilon\right\}\right) = 1. \tag{A.1}$$

Moreover, if $\epsilon^i \searrow 0$, and we find $\mathcal{L}^i$ and θ^i such that $(A.1)$ is true for $\mathcal{L}^i$, θ^i, and ϵ^i, then $lim_i\, V^i_\theta(0)$ exists and equals $\mathbf{E}^*[x]$.

In words, consumer-investor can asymptotically synthesize x with vanishing risk for a sequence of these ensembles $\mathcal{L}^i$, and the initial investment required must converge to the price of x in the BSM economy.

We will provide a formal proof of this theorem in another paper, which will deal with further interesting mathematical issues that arise concerning the asymptotic properties of measures $P^{\mathcal{L}}$ as one moves out along the directed set of ensembles $\mathcal{L}$. But we can offer here a sketch of how the proof goes. As in the proof of Theorem 1, we work with Lipschitz-continuous claims x, and then, in a final stage, approximate a given bounded and continuous x by Lipschitz-continuous claims. And, for a Lipschitz-continuous claim x:

Fix $\epsilon > 0$. Invoke the construction in the proof of Theorem 1 for x, but with $\epsilon/2$ instead of ϵ. The n that Theorem 1 produces is the ν_0 here, where (in this setting) we insist on $\nu_0 = 2^{\eta_0}$ for some integer η_0. Of course, there is positive probability – which approaches 1 as ν_0 approaches ∞ –that there is never the need to stop the process, in which case $\tau_1 = \infty$. But if, in the Theorem 1 construction, the process is stopped, this becomes the value of τ_1. Note that, at time $\tau_1 \wedge 1$, the portfolio being constructed for each ω in the support of $P^{\mathcal{L}}$ is at most $\epsilon/2$ away from $x(\tau_1 \wedge 1)$. (Indeed, it is true that, at all $t \leq \tau_1 \wedge 1$) that the value of the portfolio is less than $\epsilon/2$ away from $x(t)$.) And, instead of really stopping the process, as in the proof of Theorem 1, at this point you invoke Theorem 1 from whatever is your current situation, but for $\epsilon/4$ in place of ϵ. This will take, in general, a larger—perhaps significantly larger – value of n, but whatever value is required, this is $\nu_1 = 2^{\eta_1}$. Again, there is positive probability – which approaches 1 as ν_1 approaches ∞ – of making it to time 1 with the fineness grid set at ν_1. But if the Theorem 1 construction calls for a stop—at which point you are at most $3\epsilon/4$ away from $x(\tau_2 \wedge 1)$ – this is τ_2, and you invoke Theorem 1 once more, but for $\epsilon/8$. And so forth. As long as, at each time you stop, you choose a fine-enough grid size – that is, a large enough ν_n – so that your potential error in the sequel is no larger than $\epsilon/2^n$ – hence your accumulated error is no larger than $\epsilon(1-(1/2)^n)$ – and the probability that you reach time 1 before stopping is uniformly (in n) bounded away from zero, then, almost surely, you will reach time 1, with a portfolio within ϵ of x, after a finite (but, in general, unbounded in ω) number of "switches."

Please note: This sketch says how we show that x can be synthesized with vanishing risk. The last piece of Theorem 3, pertaining to the asymptotic uniqueness of the cost of the initiating portfolio—in essence, that there is no asymptotic arbitrage in this context—also requires proof.

One point should be stressed. When, at time 1, the portfolio is within ϵ of the claim x, what is really being said is that the value of the portfolio is within ϵ of $x(\omega)$ *for the path* ω

that has been traversed. And that path is the concatenation, at the different stopping times, of a path that starts off with grid size ν_0, then (perhaps) has increments of grid size ν_1, and so forth.

To understand what is going on here, consider the example of Section 6 in the paper: The consumer-investor attempts to synthesize the European put option $x(\omega) = \big(1 - S(1,\omega)\big)^+$ with vanishing risk, where ζ has the trinomial distribution specified in Section 6. To do so, she must begin with a portfolio whose value (asymptotically) is the Black-Scholes price, or $E^*[x] \approx 0.38239 > 0$. To simplify the exposition, suppose she starts with a portfolio whose value is precisely $\mathbf{E}^*[x]$. Suppose she sets her target ϵ at $\epsilon = 0.08$. And suppose that the first several increments she observes are all $\zeta = 0$; that is, the trajectory $\omega(t)$ does not move from 0 (and, therefore, the stock price trajectory $S(t,\omega)$ does not move from 1).

Because the stock price doesn't change along this path, she accrues neither capital gains nor losses, no matter which strategy she uses: her portfolio does not change value but remains worth the initial $\mathbf{E}^*[x] \approx 0.38239$. But, as time elapses, the Black-Scholes value of the put, $x(t) = \mathbf{E}^*[x|\omega(k/\nu_0) = 0]$, falls as k grows. At some specific time (along this path), long before time 1, $x(t)$ along this path reaches a level such that there is positive probability that, in one more step, it will fall below $0.38239 - \epsilon/2 = 0.34239$. It is at this point that τ_1 fires. The investor increases the grid-size parameter to some ν_1. Suppose she continues to see only $\zeta = 0$; that is, $\omega \equiv 0$ and $S(t,\omega) \equiv 1$. As $\mathbf{E}^*[x|\omega(t) = 0]$ continues to fall, reaching (well before time 1) a point where there is positive probability that, in the next step, it may fall below 0.32239; that is, more than $3\epsilon/4$ away from the value of the investor's portfolio. At the time (of the form $\tau_1 + k/\nu_1$ for some k) when this prospect looms as possible, τ_2 fires, and a new and smaller grid size comes into effect.

Suppose it continues to happen that $\omega(t) \equiv 0$. For as long as this happens, the value of the investor's portfolio keeps its original value, 0.38239. And, as time passes, $\mathbf{E}^*[x|\omega(t) = 0]$ continues to fall. At some time t^* *which is less than* 1, we have $\mathbf{E}^*[x|\omega(t^*) = 0] \leq 0.30239$, which is the (constant) value of the investor's portfolio (along this extraordinary path) less the value of $\epsilon = 0.08$ she set as the error she is willing to tolerate. So, before t^*, along this path, $\tau_n < t^*$ for all $n = 1, 2, \ldots$, which implies that she will have "observed" countably many independent "increments" in the path ω, each having probability 5/9. Of course, this has probability zero; the investor can be sure that, $P^{\mathcal{L}}$-almost surely, the path she observes will deviate from $\omega(t) \equiv 0$ and, in fact, it will do so almost surely prior to time $t^* < 1$, because, if not, there will be countably many "steps" crowded in between time 0 and t^*.

What is the economic story behind this model? There are a variety of (not entirely satisfactory) stories one can tell. Perhaps the story that comes closest to making sense is that there is a countable number of risky securities being marketed, one that is priced every $1/2^k$ units of time, for $k = 1, \ldots$ The consumer-investor can trade in any of these risky securities, but for reasons that don't have an entirely satisfactory economic rationale, she is constrained to hold no more than one risky asset at a time. Think, if you will, of a sequence of "trading rooms," something like a sequence of casinos; she can gamble (invest) in any one of these rooms, but to invest in any one you must be within that room, and if you leave a room, you must "cash out " first. Then the stopping times τ_n become times at which the consumer-investor moves at her own volition (and with tremendous speed) from one room to the next, and the ν_n give the "name" of the room to which she moves. In this interpretation, $\mathcal{L}$ is not handed down exogenously; rather, it is part of the overall investing strategy of the investor.[8]

There is one nice feature of this interpretation. Because the P^n—the probability laws that govern what goes on in the various rooms—converge weakly to Wiener measure P, the Skorohod Representation Theorem tells us that we can embed all this into a single probability space $(\Psi, \mathcal{P})$, with a sequence of random variables $\{\omega^n\}$, each taking values in $C_0[0,1]$, where the "marginal distribution" of ω^n, the measure $\mathcal{P} \circ (\omega^n)^{-1}$, is P^n, and $\omega(n) \to \omega$ $\mathcal{P}$-a.s., where ω is a standard Brownian motion.

This is nice because, while it is a bit hard to attach economic meaning to x evaluated at the concatenated path "created" by the consumer-inventor, with this interpretation, by choosing ν_0 large, Theorem 3 can be extended to say: the consumer-investor can synthesize a claim whose value at time 1 is within ϵ of x evaluated along the "fundamental" path ω.

However, this interpretation is not without problems. To mention three:

1. If the theory is meant to support Arrow's original notion that a few securities, traded frequently, can "span" a large state space, moving to a model with a countable number of securities is hardly in the spirit of Arrow's idea. Put differently, if Arrow's story is, ultimately meant to be about economizing on the activity of market making, there is no economizing here.

2. Why is the consumer–investor restricted to portfolios with at most one risky asset at a

[8] And, in this story, the text of the paper can be thought of as describing a world with all these trading rooms, but where an investor must choose one and only one in which to do her trading.

time? And, if she isn't, is this ensemble of countably many markets viable, in the sense of Cuchiero, Klein, and Teichmann?

3. In the story in the text, the idea is that, as long as there is a single market that meets relatively frequently, markets are relatively close to complete, where the criterion is getting close to the contingent claim with probability close to one, while not winding up very far from the desired claim with small probability. Of course, as in any asymptotic analysis, the linkage between "relatively frequently" and "relatively close" is not shown; without a much more precise rate-of-convergence analysis, we aren't quantifying the linkage.

   However, if the criterion is getting arbitrarily close with probability 1, which is what vanishing risk entails, then having a "very large but finite" number of these trading rooms is insufficient. To get vanishing risk, you need (in general) the full complement of countably many rooms (as illustrated by our example of the put option), even though, with probability one, a consumer–investor will only visit finitely many of them.

Which explains why Theorem 3 is relegated to the Appendix. It is an appealing story mathematically (with further interesting mathematics that we will explore in a subsequent paper), but as economics, it somewhat misses the mark.