Convergence of Optimal Expected Utility
for a Sequence of Binomial Models

Dedicated to the memory of Mark Davis

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We analyze the convergence of expected utility under the approximation of
the Black-Scholes model by binomial models. In a recent paper by D. Kreps
and W. Schachermayer a surprising and somewhat counter-intuitive exam-
ple was given: such a convergence may, in general, fail to hold true. This
counterexample is based on a binomial model where the i.i.d. logarithmic
one-step increments have strictly positive third moments. This is the case,
when the up-tick of the log-price is larger than the down-tick.

In the paper by D. Kreps and W. Schachermayer it was left as an open
question how things behave in the case when the down-tick is larger than the
up-tick and — most importantly — in the case of the symmetric binomial
model where the up-tick equals the down-tick. Is there a general positive
result of convergence of expected utility in this setting?

In the present note we provide a positive answer to this question. It is
based on some rather fine estimates of the convergence arising in the Central
Limit Theorem.

1 Introduction and Main Result

We adopt the setting of the paper [KS20] which in turn is based on David Kreps’ mono-
graph [Kre19]. We assume that the reader is familiar with [KS20] and use the notation
and definitions of this paper.

In [KS20, Section 9] a counterexample is presented which shows that for the positively
skewed, asymmetric binomial model the following fact holds true: the approximation
of the Black-Scholes model by this binomial model does not yield convergence of the
 corresponding portfolio optimization problems if one does not impose extra conditions
 on the utility function such as the condition of reasonable asymptotic elasticity. In the

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present note we show the reassuring fact that for the symmetric binomial model we have a positive result without imposing extra conditions on the utility function $U(\cdot)$.

We consider a general utility function $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying Assumption (3.1) of [KS20], i.e., being strictly increasing, strictly concave, smooth, and satisfying the Inada conditions $\lim_{x \to 0} U'(x) = \infty$ and $\lim_{x \to \infty} U'(x) = 0$. As in [KS20] and [KS99] we denote by $V : \mathbb{R}_+ \to \mathbb{R}$ the conjugate function of $U$, i.e., $V(y) = \sup_{x>0} \{U(x) - xy\}$.

For arbitrary $p \in (0, 1)$ we consider an i.i.d. sequence $(\alpha_n)_{n=1}^\infty$ of Bernoulli variables with

$$P[\alpha_n = 0] = 1 - p, \quad P[\alpha_n = 1] = p,$$

where $p \in (0, 1)$. Denote by $\zeta_n$ the corresponding standardized variables

$$\zeta_n = \frac{\alpha_n - p}{\sqrt{p(1-p)}},$$

so that $E[\zeta_n] = 0$ and $\text{Var}[\zeta_n] = 1$. Denote by $\xi_{n,k}$ the scaled partial sums

$$\xi_{n,k} = \frac{1}{\sqrt{n}} \sum_{j=1}^k \zeta_j$$

and set

$$z_{n,k} = \frac{k - np}{\sqrt{np(1-p)}}, \quad f_{n,k} = \left(\frac{n}{k}\right) p^k (1-p)^{n-k}, \quad k = 0, \ldots, n.$$  

Then $f_{n,k} = P[\xi_{n,n} = z_{n,k}]$.

If we set $S^{(n)}(k/n) = e^{\xi_{n,k}}$ and extend $S^{(n)}$ by interpolation to continuous-time processes as in [KS20], then $(S^{(n)}(t))_{0 \leq t \leq 1}$ approximates the Black-Scholes-Merton (BSM) model. The latter is defined by

$$S(t) = \exp(\omega(t)), \quad 0 \leq t \leq 1,$$

where $(\omega(t))_{0 \leq t \leq 1}$ is a standard Brownian motion.

In fact, for the present purposes, we only need the random variable $\omega(1)$ under the standard normal measure $\mathcal{P}$ as well as under the normalized binomial measure, i.e., the law of $\xi_{n,n}$, denoted by $\mathcal{P}_n$.

To wit, the distribution of the random variable $\omega(1)$ under $\mathcal{P}_n$ equals the distribution of $\xi_{n,n} = \frac{\zeta_1 + \cdots + \zeta_n}{\sqrt{n}}$, where $(\zeta_i)_{i=1}^n$ are independent copies of $\zeta$ and, as in [KS20], we denote by $u(x)$, respectively by $u_n(x)$, the supremal expected utility which an agent can achieve in the BSM economy, respectively in the $n$-th discrete-time economy, if her initial wealth is $x$.

As in [KS20] we define the value functions $u(x)$ and $u_n(x)$ as

$$u(x) = \sup \{ E[\mathcal{P}[U(X) : E[\mathcal{P}[X] \leq x] \}
$$

and

$$u_n(x) = \sup \{ E[\mathcal{P}_n[U(X_n)] : E[\mathcal{P}_n[X_n] \leq x] \},$$

(6)

(7)
where \( P^* \) and \( P^*_n \) are the unique equivalent martingale measures pertaining to the Black-Scholes model and its \( n \)-th approximation, respectively.

When \( p \in (0, 1/2) \) we have \( E[\zeta^3] > 0 \) and things go astray, as demonstrated by the counterexample in Section 9 of [KS20]. Therefore we focus on the other case, when \( p \in [1/2, 1) \).

Here is our main result.

**Theorem 1.** If \( U \) and \( \zeta \) are as above with \( p \in [1/2, 1) \), we have
\[
u(x) = \lim_{n \to \infty} u_n(x), \quad x > 0.
\]
(8)

The theorem will follow from the subsequent more technical version of (8).

**Proposition 1.** Denote by \( v(y) \), respectively \( v_n(y) \), the value functions conjugate to \( u(x) \), respectively \( u_n(x) \), as in [KS20]. We then have
\[
\limsup_{n \to \infty} v_n(y) \leq v(y), \quad y > 0.
\]
(9)

**Proof of Theorem 1 admitting Proposition 1.** We deduce from [KS20, Proposition 2] and standard results on conjugate functions that the reverse inequality to (9) does hold true, i.e.,
\[
\liminf_{n \to \infty} v_n(y) \geq v(y), \quad y > 0.
\]
(10)

Admitting Proposition 1, formulas (9) and (10) imply the equality
\[
\lim_{n \to \infty} v_n(y) = v(y), \quad y > 0.
\]
(11)

Using once again standard results on conjugate functions (compare [KS20]) we obtain (8) from (11). \( \square \)

A key ingredient for the proof of Proposition 1 are estimates for the tails of the standardized binomial distributions in terms of the standard Gaussian tails.

Let \( \xi \) be a standard normal random variable and denote its density by \( \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \).

Set
\[
F_n(x) = P[\xi_{n,n} \leq x], \quad \Phi(x) = P[\xi \leq x]
\]
(12)

and
\[
\bar{F}_n(x) = 1 - F_n(x) = P[\xi_{n,n} > x], \quad \bar{\Phi}(x) = 1 - \Phi(x) = P[\xi > x].
\]
(13)

**Proposition 2.** Suppose \( p \in [1/2, 1) \), then there is \( C > 0 \) such that, for \( n \geq 1 \), we have
\[
f_{n,k} \leq C \cdot \frac{2}{\sqrt{n}} \phi(z_{n,k-1}), \quad 0 \leq k \leq \lceil np \rceil,
\]
(14)

\[
f_{n,k} \leq C \cdot \frac{2}{\sqrt{n}} \phi(z_{n,k+1}), \quad \lceil np \rceil \leq k \leq n.
\]
(15)

Furthermore
\[
F_n(x) \leq C\Phi(x), \quad x \leq 0
\]
(16)

and
\[
\bar{F}_n(x) \leq C\bar{\Phi}(x), \quad x \geq 0.
\]
(17)
Remark 1. The terms $\frac{2}{\sqrt{n}} \phi(z_{n,k+1})$ in (14) and (15) are a lower bound for the area under the density $\phi(x)$ between $z_{n,k-1}$ and $z_{n,k}$ on the left and $z_{n,k}$ and $z_{n,k+1}$ on the right tail, respectively.

We prove Proposition 1 and Proposition 2 first for the symmetric case $p = 1/2$ in Section 2, as this case allows for several simplifications and the main arguments are more transparent. We then provide the slightly more technical details for the asymmetric case $p \in (1/2, 1)$ in Section 3.

2 The symmetric case

Proof of Proposition 1 (symmetric case) admitting Proposition 2. As in [KS20, Section 3] we write

$$v(y) = \mathbb{E}_P[V(yZ)], \quad y > 0,$$

and

$$v_n(y) = \mathbb{E}_{P^n}[V(yZ_n)], \quad y > 0,$$

where $V(y) = \sup \{U(x) - xy : x > 0\}$ is the conjugate function of $U$.

The random variables $Z$ and $Z_n$ are the densities of the (unique) equivalent martingale measures $P^*$ and $P_n^*$ with respect to $P$ and $P_n$, respectively, i.e., $Z = \frac{dP^*}{dP}$ and $Z_n = \frac{dP_n^*}{dP_n}$. They are of the form

$$Z = \exp\left(-\frac{\omega(1)}{2} - \frac{1}{8}\right)$$

and

$$Z_n = \exp\left(-a_n\omega(1) - b_n\right).$$

In the symmetric case the calculations from [KS20, Section 6] simplify, and we have that

$$a_n = \frac{1}{2}, \quad b_n = n \log \cosh\left(\frac{1}{2\sqrt{n}}\right).$$

It follows that $b_n$ increases to $1/8$ as $n \to \infty$.

Fix $y > 0$ such that $v(y) < \infty$, otherwise (9) is certainly true. Denote by $H : \mathbb{R} \to \mathbb{R}$ the function

$$H(x) = V\left(y \exp\left(-\frac{x}{2} - \frac{1}{8}\right)\right), \quad x \in \mathbb{R},$$

and by $H_n : \mathbb{R} \to \mathbb{R}$ the function

$$H_n(x) = V\left(y \exp\left(-\frac{x}{2} - b_n\right)\right), \quad x \in \mathbb{R}.$$

Clearly, these functions are increasing on $\mathbb{R}$. Note, however, that they are not necessarily concave. We know that

$$v(y) = \mathbb{E}_P[H(\omega(1))] = \int_{\mathbb{R}} H(x)\phi(x)dx < \infty$$

(25)
while
\[
v_n(y) = \mathbb{E}_{P_n}[H_n(\omega(1))] = \sum_{k=0}^{n} H_n(z_{n,k})f_{n,k},
\]
where \(\phi(x)\) is the standard normal density and \(f_{n,k}\) are the binomial probabilities as in Section 1.

As \(H_n(x) \leq H(x)\) for all \(x \in \mathbb{R}\), in order to show (9), it will suffice to show
\[
\limsup_{n \to \infty} \mathbb{E}_{P_n}[H(\omega(1))] \leq \mathbb{E}_{P}[H(\omega(1))].
\]

In order to show (27) the crucial estimate is the uniform integrability of the random variables \(H(\omega(1))\) under \(P_n\). More precisely, we need the following estimates (28) and (29). For \(\varepsilon > 0\) there is \(M > 0\) such that
\[
\mathbb{E}_{P_n}
\left[H(\omega(1))1\{H(\omega(1)) > M\}\right] = \sum_{k=0}^{n} H(z_{n,k})1\{H(z_{n,k}) > M\}f_{n,k} < \varepsilon,
\]
and
\[
\mathbb{E}_{P_n}
\left[|H(\omega(1))|1\{H(\omega(1)) < -M\}\right] = \sum_{k=0}^{n} |H(z_{n,k})|1\{H(z_{n,k}) < -M\}f_{n,k} < \varepsilon,
\]
uniformly in \(n \in \mathbb{N}\). Formulas (28) and (29) correspond to the formulas [KS20, (8.12) and (8.14)].

First we consider \(M > H(1)_+\). If \(H(z_{n,k}) > M\), then \(n/2 < k \leq n\) and we are in a position to invoke formula (15) of Theorem 2, which gives an estimate on the right tail of the binomial distribution as compared to the normal one. More precisely, there is a universal constant \(C > 0\) such that, for every \(n \in \mathbb{N}\) and \(0 \leq k < n/2\) and all \(x \in (z_{n,k}, z_{n,k+1})\)
\[
f_{n,k} \leq C \cdot \frac{2}{\sqrt{n}} \phi(x), \quad 0 \leq H(z_{n,k}) \leq H(x).
\]
Thus
\[
\sum_{k=0}^{n} H(z_{n,k})1\{H(z_{n,k}) > M\}f_{n,k} \leq C \int_{-\infty}^{+\infty} H(x)1\{H(x) > M\}\phi(x)dx.
\]
It follows from (25) that the right-hand side of (31) can be made smaller than \(\varepsilon\) for sufficiently large \(M\).

A similar estimate applies for the left tail. We consider now \(M > H(-1)_-\). If \(H(z_{n,k}) < -M\) then \(0 \leq k < n/2\) and we invoke formula (14) of Theorem 2. We get now for every \(n \in \mathbb{N}\) and \(0 \leq k < n/2\) and all \(x \in (z_{n,k-1}, z_{n,k})\)
\[
f_{n,k} \leq C \cdot \frac{2}{\sqrt{n}} \phi(x), \quad H(x) \leq H(z_{n,k}) \leq 0.
\]
Thus
\[
\sum_{k=0}^{n} |H(z_{n,k})|1\{H(z_{n,k}) < -M\}f_{n,k} \leq C \int_{-\infty}^{+\infty} |H(x)|1\{H(x) < -M\}\phi(x)dx.
\]
Again, it follows from (25) that the right-hand side of (33) can be made smaller than \( \varepsilon \) for sufficiently large \( M \).

Using the well-known weak convergence of \( P_n \) to \( P \) and the uniform integrability conditions we can deduce (26), see [vdV98, Thm 2.20, p.17].

Finally we consider the case \( y = y_0 \), where \( y_0 = \inf \{ y > 0 : v(y) < \infty \} \), for the case \( y_0 > 0 \). Either \( v(y_0) = \infty \) in which case (9) holds trivially. Or \( v(y_0) < \infty \), in which case \( v \) is right continuous at \( y_0 \), see [KS99]; we therefore may repeat the above argument with \( y = y_0 \).

This finishes the proof of Proposition 1.

\[ \Box \]

**Proof of Proposition 2 (symmetric case).** Let us start with (15). It is enough to prove that there is \( n_0 > 0 \) such that (15) holds for \( n \geq n_0 \), since \( \varphi \) is strictly positive and there are only finitely many remaining cases that can be incorporated in the value of the constant \( C \).

Let us consider first the extreme case \( k = n \). In this case (15) follows since \( p_{n,n} = 2^{-n} \), which decays faster than \( n^{-1/2} \varphi(z_{n,n+1}) \approx e^{-n^2/2} \) for \( n \to \infty \), as \( \log 2 > 1/2 \). Here we used the convention \( z_{n,n+1} = \sqrt{n+2}/\sqrt{n} \), although \( z_{n,n+1} \) is not a possible value for \( Y_n \).

Passing to the other extreme case, we deduce from the central limit theorem, that for \( k = \lfloor n/2 \rfloor \) we have \( \sqrt{n} f_{n,k}/\varphi(z_{n,k+1}) \to 1 \) as \( n \to \infty \).

For the remaining cases, i.e., \( \lfloor n/2 \rfloor < k \leq n-1 \), we take logarithms and show that

\[
\log \left( \frac{\sqrt{n} f_{n,k}}{2 \varphi(z_{n,k+1})} \right)
\]

is bounded from above. To estimate the numerator in (34) we use a fine version of Stirling’s Formula as given in [AS92, 6.1.38, p.257], namely

\[
x! = \sqrt{2\pi} x^{x+\frac{1}{2}} \exp \left( -x + \frac{\theta(x)}{12x} \right), \quad x > 0,
\]

with \( 0 < \theta(x) < 1 \) for all \( x > 0 \). We also note that \( \lim_{x \to \infty} \theta(x) = 0 \). We obtain the estimates

\[
\log(n!) \leq \log \sqrt{2\pi} + \left( n + \frac{1}{2} \right) \log n - n + \frac{1}{12n}
\]

(36)

\[
\log(k!) \geq \log \sqrt{2\pi} + \left( k + \frac{1}{2} \right) \log k - k
\]

(37)

\[
\log((n-k)!) \geq \log \sqrt{2\pi} + \left( n - k + \frac{1}{2} \right) \log(n-k) - (n-k)
\]

(38)

This yields an upper bound for the numerator of (34), as

\[
\log f_{n,k} = \log(n!) - \log(k!) - \log((n-k)!) - n \log 2.
\]

(39)

As regards the denominator of (34), we have

\[
\log \varphi(z_{n,k+1}) = 2 + 2k - \frac{2}{n} - 4k - 2k^2/n - \frac{n}{2} - \log \sqrt{2\pi}.
\]

(40)
Writing \( \log k = \log n + \log(k/n) \) and \( \log(n - k) = \log n + \log(1 - k/n) \) and combining (39), (36), (37), (38), and (40) yields
\[
\log \left( \frac{\sqrt{n} f_{n,k}}{2 \varphi(z_{n,k+1})} \right) \leq g_n \left( \frac{k}{n} \right), \quad (41)
\]
with \( g_n(w) = \alpha(n)w + \beta_n(w) \), where \( w \in \left[ \frac{1}{2}, 1 \right] \), and
\[
\alpha(w) = -w \log w - (1 - w) \log(1 - w) - 2w(1 - w) + \frac{1}{2} - \log 2 \quad (42)
\]
and
\[
\beta_n(w) = -\frac{1}{2} \log w - \frac{1}{2} \log(1 - w) + 4w - 2 + \frac{25}{12n}. \quad (43)
\]
It remains to show that \( g_n(w) \) is bounded from above uniformly in \( n \in \mathbb{N} \) and \( w \in [1/2, 1 - 1/n] \).

We have \( \alpha(\frac{1}{2}) = \alpha'(\frac{1}{2}) = \alpha''(\frac{1}{2}) = \alpha'''(\frac{1}{2}) = 0 \) and for the forth derivative we have \( \alpha^{iv}(w) = -2/(1 - w)^3 - 2/w^3 < 0 \) for \( w \in (\frac{1}{2}, 1) \), and thus each of the functions \( \alpha''(w), \alpha''(w), \) and \( \alpha(w) \) is strictly negative and decreasing for \( w \in (\frac{1}{2}, 1) \).

We have \( \beta_n(1/2) = 25/(12n) \) and \( \beta_n'(w) = 4 - 1/(2w) + 1/(2(1 - w)) > 0 \) for \( w \in (\frac{1}{2}, 1) \), thus \( \beta_n(w) \) is strictly positive and strictly increasing for \( w \in (\frac{1}{2}, 1) \).

For \( w \in [1/2, 3/4] \) we have \( g_n(w) \leq b_n(3/4) \). As \( \lim_{n \to \infty} \beta_n(3/4) = 1 + \log(2/\sqrt{3}) < \infty \) it follows that \( g_n(w) \) is bounded from above for the interval under consideration.

For \( w \in [3/4, 1 - 1/n] \) we have \( g_n(w) \leq \alpha(3/4)n + \beta_n(1 - 1/n) \). Now \( \alpha(3/4) < 0 \) and \( \beta_n(1 - 1/n) \sim \frac{1}{2} \log n \) as \( n \to \infty \). Here the second term on the right hand side of (43) is the leading term. Finally we use the fact that \( \alpha(3/4)n \) grows quicker than \( \frac{1}{2} \log n \) to conclude that \( g_n(w) \) is negative for \( w \in [3/4, 1 - 1/n] \) and sufficiently large \( n \).

The proof of (14) is completely symmetric with \( g_n(w) \) for \( w \in [1/2, 1 - 1/n] \) replaced by \( g_n(1 - w) \) for \( w \in [1/n, 1/2] \).

Having proved (14) and (15) we mentioned already in Remark 1 how these two inequalities imply (16) and (17).

For the above proof of Proposition 1 the estimates (14) and (15) involving an unspecified constant \( C > 0 \) is sufficiently strong. But we can do better than that. We may adapt the above argument to yield a constant \( C = 1 + \varepsilon \) for \( n \) sufficiently large. Indeed, analyzing the above proof of Proposition 2, we see that the above argument also works when we split the interval \( (\frac{1}{2}, 1) \) not at \( w = 3/4 \), but at a point \( \theta \in (1/2, 1) \), which is close to \( 1/2 \) to obtain a better constant \( C \), for large enough \( n \). The detailed argument is given in the proof of the following proposition, which sharpens Proposition 2.

**Proposition 3.** For any \( C > 1 \) there is \( n_0(C) > 0 \) such that equations (14)–(15) and (16)–(17) hold for \( n \geq n_0(C) \).

**Proof.** We consider \( \theta \in (\frac{1}{2}, 1) \) and proceed as in the proof of the Proposition 2 above. We distinguish two cases, \( w \in [\frac{3}{2}, \theta] \) and \( w \in [\theta, 1 - 1/n] \). In the first case, when \( w \in [1/2, \theta] \), we have \( g_n(w) \leq \beta_n(w) \) and
\[
\lim_{n \to \infty} \beta_n(\theta) = -\frac{1}{2} \log \theta - \frac{1}{2} \log (1 - \theta) + 4\theta - 2 - \log 2. \quad (44)
\]
The right hand side is increasing in $\vartheta$ and equals zero when $\vartheta = 1/2$. In the second case, for $w \in [\vartheta, 1 - 1/n]$ we have $g_n(w) \leq \alpha(\vartheta)n + \beta_n(1 - 1/n)$. Again $\alpha(\vartheta) < 0$ and $\beta_n(1 - 1/n) \sim \frac{1}{2} \log n$ as $n \to \infty$, so that $g_n(w)$ is negative for $w \in [\vartheta, 1 - 1/n]$ and sufficiently large $n$.

3 The asymmetric case

Proof of Proposition 1 (asymmetric case) admitting Proposition 1. We fix $p \in (\frac{1}{2}, 1)$ and follow the steps from the symmetric case, but now we get instead of (22) the following coefficients in (21):

$$a_n = \frac{\sqrt{n}}{z_{1,1} - z_{1,0}} \log \left( \frac{p - e^{z_{1,1}/\sqrt{n}}}{1 - p - e^{z_{1,0}/\sqrt{n}}} \right),$$

(45)

and

$$b_n = n \log \left( (1 - p)e^{-z_{1,0}a_n/\sqrt{n}} + pe^{-z_{1,1}a_n/\sqrt{n}} \right).$$

(46)

with $z_{1,0} = -\sqrt{p/(1 - p)}$ and $z_{1,1} = \sqrt{(1 - p)/p}$. Straightforward asymptotic expansions for $n \to \infty$ yield

$$a_n = \frac{1}{2} - \frac{2p - 1}{24\sqrt{p(1 - p)}} n^{-\frac{1}{2}} + O(n^{-1}),$$

(47)

which slightly extends the result that is given in [KS20, Sec.6], and

$$b_n = \frac{1}{8} - \frac{1 - p + p^2}{576p(1 - p)} n^{-1} + O(n^{-2}).$$

(48)

Now we fix an arbitrary $\delta > 0$. For $p \in (1/2, 1)$ it follows from the asymptotics that

$$0 < a_n \leq \frac{1}{2}, \quad \frac{1}{8} - \delta \leq b_n \leq \frac{1}{8}$$

(49)

for all $n$ sufficiently large. In fact, these inequalities are also true for $p = 1/2$ as can be seen from (22).

Instead of (24) we now consider

$$H_n(x) = V(y \exp(-a_n x - b_n)), \quad x \in \mathbb{R}.$$ 

(50)

If we are in the right tail and $z_{n,k} \geq 0$ then (49) yields $H_n(z_{n,k}) \leq H(z_{n,k})$ and the uniform inegrability follows just as in the symmetric case.

If we are in the left tail and $z_{n,k} \leq 0$ then (49) yields $H_n(z_{n,k}) \geq \tilde{H}(z_{n,k})$, where $\tilde{H}(x) = V(\tilde{y}e^{-x/2 - 1/8})$ with $\tilde{y} = ye^\delta$. Due to the convexity of $V$ we have $v(\tilde{y}) > -\infty$ and the uniform inegrability follows just as in the symmetric case.

□
Proof of Proposition 2 (asymmetric case). Following the steps from the symmetric case we now get
\[
\log f_{n,k} = \log(n!) - \log(k!) - \log((n-k)!) + k \log p + (n-k) \log(1-p).
\] (51)
and
\[
\log \phi(z_{n,k+1}) = \frac{1}{2}(-\log(2) - \log(\pi)) - \frac{(k-np+1)^2}{2n(1-p)p}.
\] (52)
the key inequality (41) becomes
\[
\log \left( \frac{\sqrt{np(1-p)f_{n,k}}}{\phi(z_{n,k+1})} \right) \leq g_n \left( \frac{k}{n} \right),
\] (53)
with \(g_n(w) = \alpha(w)n + \beta_n(w)\), where \(w \in [p, 1]\), and
\[
\alpha(w) = -w \log w - (1-w) \log(1-w) + \frac{w^2}{2p(1-p)} + \left( \log \frac{p}{1-p} - \frac{1}{1-p} \right) w + \frac{p}{2(1-p)} + \log(1-p)
\] (54)
and
\[
\beta_n(w) = -\frac{1}{2} \log (w(1-w)) + \frac{w}{p(1-p)} - \log 2 - \frac{1}{1-p} + \left( \frac{1}{12} + \frac{1}{2p(1-p)} \right) \frac{1}{n}.
\] (55)
Again we have \(\alpha(p) = \alpha'(p) = \alpha''(p) = 0\). As regards the third derivative we find \(\alpha'''(w) = \frac{1-2p}{(p-1)^2} < 0\) for \(w \in (p, 1)\), and thus \(\alpha'''(w), \alpha''(w), \alpha'(w), \text{ and } \alpha(w)\) again are strictly negative and decreasing for \(w \in (p, 1)\).

We have \(\beta_n(p) = \frac{1}{12} \left( \frac{6}{np-np^2} + \frac{1}{n} - 6 \log(-4(p-1)p) \right)\) and \(\beta'_n(w) = \frac{1}{p-p^2} + \frac{1}{2w} > 0\) for \(w \in (p, 1)\), thus \(\beta_n(w)\) is strictly positive and strictly increasing for \(w \in (p, 1)\).

Similarly as in the proof of Proposition 3, fix \(\vartheta \in (p, 1)\). For \(w \in [p, \vartheta]\) we have \(g_n(w) \leq \beta_n(\vartheta)\). As \(\lim_{n \to \infty} \beta_n(\vartheta) = -1/2 \log(\vartheta(1-\vartheta)) + \frac{h}{p(1-p)} - \frac{1}{p} \log 2\) it follows that \(g_n(w)\) is bounded from above for the interval under consideration. For \(w \in [\vartheta, 1-1/n]\) we have \(g_n(w) \leq \alpha(\vartheta)n + \beta_n(1-1/n)\). Now \(\alpha(\vartheta) < 0\) and \(\beta_n(1-1/n) \sim 1/2 \log n\) as \(n \to \infty\) and thus \(g_n(w)\) is negative for \(w \in [\vartheta, 1-1/n]\) and sufficiently large \(n\).

References


