A Strong Law of Large Numbers for Positive Random Variables *

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Abstract

In the spirit of the famous Komlós (1967) theorem, every sequence of nonnegative, measurable functions $\{f_n\}_{n\in\mathbb{N}}$ on a probability space, contains a subsequence which—along with all its subsequences—converges a.e. in Cesàro mean to some measurable $f : \Omega \to [0,\infty]$. This result of von Weizsäcker (2004) is proved here from scratch, with minimal tools. The methodology we develop sharpens a result of Delbaen & Schachermayer (1994), replacing general convex combinations by Cesàro means; and leads to an elementary proof of the original Komlós theorem (which allows the functions $\{f_n\}_{n\in\mathbb{N}}$ to be real-valued, but imposes boundedness in $\mathbb{L}^1$).

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1 Introduction

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider independent, real-valued measurable functions $f_1, f_2, \cdots$ with the same distribution, and $\mathbb{E}(|f_1|) < \infty$. The celebrated Kolmogorov strong law of large numbers ([7], p. 126) states that the "sample average" $(f_1+\cdots+f_N)/N$ converges $\mathbb{P}-$a.e. to the "ensemble average" $\mathbb{E}(f_1) = \int_\Omega f_1 \,\mathrm{d}\mathbb{P}$, as $N \to \infty$.

A deep result of Komlós [16], already 55 years old but always very striking, says that such averaging occurs within *any* sequence $f_1, f_2, \cdots$ of measurable, real-valued functions satisfying $\sup_{n\in\mathbb{N}} \mathbb{E}(|f_n|) < \infty$. More precisely, there exist then a function f and a subsequence $f_{n_1}, f_{n_2}, \cdots$, such that $(f_{n_1} + \cdots + f_{n_K})/K$ converges to f, $\mathbb{P}-$a.e. as $K \to \infty$; and the same is true for any further subsequence of $\{f_{n_k}\}_{k\in\mathbb{N}}$.

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This result inspired further deep, foundational work in probability theory ([11], [6]), culminating with that of ALDOUS (1977) where exchangeability plays a crucial rôle. It, and its ramifications involving forward convex combinations in [9]–[10], have been very useful in the context of convex optimization; and more generally, when one seeks objects with specific properties, and tries to ascertain their existence using weak compactness arguments. Stochastic control, optimal stopping and hypothesis testing are examples of the former (e.g., [18], [15], [8], [19]); whereas the DOOB-MEYER and BICHTELER-DELLACHERIE theorems in stochastic analysis provide instances of the latter (e.g., [12], [2], [3]).

We develop here a very simple argument for the KOMLÓS theorem, starting with the important case of nonnegative $f_1, f_2, \cdots$ treated by VON WEIZSÄCKER (2004). The proof dispenses with boundedness in $\mathbb{L}^1$, at the cost of allowing the function f to take infinite values. When the sequence $\{f_n\}_{n\in\mathbb{N}}$ *is* bounded in $\mathbb{L}^1$, the method leads also to an elementary proof for the original KOMLÓS result.

2 Background

We place ourselves on a given, fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ of measurable, real-valued functions defined on it.

We say that this sequence *converges hereditarily in* CESÀRO *mean* to some measurable $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$, and write $f_n \xrightarrow[n\to\infty]{hC} f, \quad \mathbb{P}-\text{a.e.}$, if, for *every* subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of the original sequence, we have

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} f_{n_k} = f, \qquad \mathbb{P}-\text{a.e.} \tag{2.1}$$

Clearly then, every other such sequence $\{g_n\}_{n\in\mathbb{N}}$ which is BOREL-CANTELLI *equivalent* to $\{f_n\}_{n\in\mathbb{N}}$, in the sense $\sum_{n\in\mathbb{N}} \mathbb{P}(f_n \neq g_n) < \infty$, also has this property.

In 1967, KOMLÓS proved the following remarkable result. The argument in [16] is very clear, but also long and quite involved. Simpler proofs and extensions have appeared since (e.g., [21], [23]; [4]); we provide another such proof in § 5.5.

Theorem 2.1 (KOMLÓS (1967))**.** *If the sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^1$, i.e., satisfies $\sup_{n\in\mathbb{N}} \mathbb{E}(|f_n|) < \infty$, there exist an integrable $f : \Omega \to \mathbb{R}$ and a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$, which converges hereditarily in* CESÀRO *mean to f:*

$$f_{n_k} \xrightarrow[k\to\infty]{hC} f, \qquad \mathbb{P}-a.e. \tag{2.2}$$

This result was motivated by an earlier one in [20]. For the convenience of the reader, we provide in § 5.1 its proof (after [5], pp. 137-141), which proceeds by extracting a *martingale difference* subsequence. This crucial idea makes the connection to martingale theory, and appears in such a context for the first time in [16] (for related results, see [17]).

Theorem 2.2 (RÉVÉSZ (1965))**.** *If the sequence* $\{f_n\}_{n\in\mathbb{N}}$ *satisfies* $\sup_{n\in\mathbb{N}} \mathbb{E}(f_n^2) < \infty$, *there exist a function* $f \in \mathbb{L}^2$ *and a subsequence* $\{f_{n_k}\}_{k\in\mathbb{N}}$, *such that* $\sum_{k\in\mathbb{N}} a_k\big(f_{n_k} - f\big)$ *converges* $\mathbb{P}-$*a.e., for any sequence* $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *with* $\sum_{n\in\mathbb{N}} a_n^2 < \infty$.

In a related development, DELBAEN & SCHACHERMAYER ([9], Lemma A1.1; [10]) showed with very simple arguments that, from every sequence $\{f_n\}_{n\in\mathbb{N}}$ of nonnegative, measurable functions, a sequence of convex combinations $g_n \in \mathrm{conv}(f_n, f_{n+1} \cdots)$, $n \in \mathbb{N}$ of its elements can be extracted, which converges $\mathbb{P}-$a.e. to a measurable $f : \Omega \to [0,\infty]$. This result was called "a somewhat vulgar version of KOMLÓS's theorem" in [10], and is implied by Theorem 3.1 below. Indeed, the assertion of convergence is much more precise for CESÀRO averages, than it is for unspecified forward convex combinations.

In several contexts, including optimization problems treated via convex duality, nonnegativity is often no restriction at all, but rather the natural setting (e.g., [18]; [19]; [13]; [14], Chapter 3 and Appendix). Then, in the presence of convexity, Lemma A1.1 in [9], or Theorem 3.1 here, are very useful analogues of Theorem 2.1: they lead to limit functions f in convex sets (such as the positive orthant in $\mathbb{L}^0$, or the unit ball in $\mathbb{L}^1$) which are not compact in the usual sense, but are "convexly compact" as in ŽITKOVIĆ [25].

3 Results

The purpose of this note is to prove with elementary tools the following version of Theorem 2.1 and its companion result, Theorem 3.2 below.

Theorem 3.1. *Given a sequence* $\{f_n\}_{n\in\mathbb{N}}$ *of* nonnegative*, measurable functions on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, there exist a measurable function* $f : \Omega \to [0,\infty]$ *and a subsequence* $\big\{f_{n_k}\big\}_{k\in\mathbb{N}}$ *of the original sequence, such that* (2.2) *holds.*

We observe that the result imposes no condition on the functions $f_1, f_2, \cdots$, apart from measurability and nonnegativity. This comes at a price: the function f, constructed here carefully starting with (4.3) below, can take the value $+\infty$ on a set of positive measure. We formulate now our second result, a direct consequence of the first, recalling the notation $x^\pm = \max(\pm x, 0)$ for the positive and negative parts of a real number x.

Theorem 3.2. *Given a sequence* $\{f_n\}_{n\in\mathbb{N}}$ *of real-valued, measurable functions on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\sup_{n\in\mathbb{N}} \mathbb{E}\big(f_n^-\big) < \infty$, *there exist a measurable function* $f : \Omega \to (-\infty,\infty]$ *and a subsequence* $\big\{f_{n_k}\big\}_{k\in\mathbb{N}}$ *of the original sequence, such that* (2.2) *holds.*

Remark 3.3. The function f in Theorem 3.2 is integrable if, in addition to the conditions there, $\sup_{n\in\mathbb{N}} \mathbb{E}\big(f_n^+\big) < \infty$ (equivalently, $\sup_{n\in\mathbb{N}} \mathbb{E}\big(|f_n|\big) < \infty$) holds as well. Thus, Theorem 2.1 emerges as a consequence of Theorem 3.2.

Theorem 3.1 is not new. It was established by VON WEIZSÄCKER in [24] using the KOMLÓS Theorem 2.1, and was studied further in [23]; see also § 5.2.3 of [13]. Our proofs for Theorems 3.1, 3.2 appear in Section 5; they are new and, we believe, not without methodological/paedagogical merit. They proceed from scratch, and lead to the original KOMLÓS theorem as well (cf. Remark 3.3) using tools minimal and simple.

4 Preparation

We place ourselves in the setting of Theorem 3.1. In the arguments that follow we shall pass often to subsequences, and to diagonal subsequences, of a given $\{f_n\}_{n\in\mathbb{N}}$. To simplify typography, we denote frequently such subsequences by the same symbols, $\{f_n\}_{n\in\mathbb{N}}$.

For each integer $k \in \mathbb{N}$, we introduce now the truncated functions

$$f_n^{(k)} := \mathbf{1}_{[k-1,k)}(f_n) \cdot f_n\,, \qquad n \in \mathbb{N} \tag{4.1}$$

and note the partition of unity $\sum_{k\in\mathbb{N}} f_n^{(k)} = f_n\,,\ \forall\, n \in \mathbb{N}\,.$

Lemma 4.1. *For the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in Theorem 3.1, there exists a subsequence, denoted by the same symbols and such that the functions of* (4.1) *converge for every $k \in \mathbb{N}$ to an appropriate measurable function $f^{(k)} : \Omega \to [0,\infty)$, in the sense*

$$f_n^{(k)} \xrightarrow[n\to\infty]{hC} f^{(k)}, \qquad \mathbb{P} - a.e. \tag{4.2}$$

Proof (after [5], pp. 145–146): For arbitrary, fixed $k \in \mathbb{N}$, the sequence $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ of (4.1) is bounded in $\mathbb{L}^\infty$, thus also in $\mathbb{L}^2$. Theorem 2.2 provides a function $f^{(k)} \in \mathbb{L}^2$ and a subsequence $\{f_{n_j}^{(k)}\}_{j\in\mathbb{N}}$ of $\{f_n^{(k)}\}_{n\in\mathbb{N}}$, such that $\sum_{j\in\mathbb{N}} \big(f_{n_j}^{(k)} - f^{(k)}\big)/j$ converges $\mathbb{P}$–a.e. The KRONECKER Lemma ([7], p. 123) gives

$$0 = \lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} \big(f_{n_j}^{(k)} - f^{(k)}\big) = \lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} f_{n_j}^{(k)} - f^{(k)}, \qquad \mathbb{P} - \text{a.e.}$$

for the sequence $\{f_{n_j}^{(k)}\}_{j\in\mathbb{N}}$ and all its subsequences. We pass now to a diagonal subsequence, denoted $\{f_n\}_{n\in\mathbb{N}}$ again, and such that (4.2) holds for *every* $k \in \mathbb{N}$. $\square$

With these ingredients, we introduce the measurable function $f : \Omega \to [0,\infty]$ via

$$f := \sum_{k\in\mathbb{N}} f^{(k)}, \qquad \text{and consider the set} \quad A_\infty := \{f = \infty\}. \tag{4.3}$$

With the help of FATOU's Lemma, and the notation of (4.1)–(4.3), we obtain then

$$\varliminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n \geq f\,, \qquad \mathbb{P} - \text{a.e.} \tag{4.4}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n = \infty = f\,, \qquad \mathbb{P} - \text{a.e.} \quad \text{on} \quad A_\infty \tag{4.5}$$

from Lemma 4.1, for a suitable subsequence (denoted by the same symbols) of the original sequence $\{f_n\}_{n\in\mathbb{N}}$ and for all further subsequences of this subsequence.

The inequality in (4.4) can easily be strict. Consider, for instance, $f_n \equiv n$, so that $f_n^{(k)} = 0$ holds in (4.1) for every fixed $k \in \mathbb{N}$ and all $n \in \mathbb{N}$ sufficiently large. This leads to $f^{(k)} = 0$ in (4.2), thus $f = 0$ in (4.3); but $\frac{1}{N}\sum_{n=1}^{N} f_n \to \infty$ as $N \to \infty$.

This preparation allows us to formulate a somewhat more technical and precise version of Theorem 3.1, as follows. The familiar convention $\infty \cdot 0 = 0$ is employed throughout.

Proposition 4.2. *Fix a sequence $\{f_n\}_{n\in\mathbb{N}}$ of nonnegative, measurable functions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and recall the notation* (4.1)–(4.3). *There exist then a subsequence, denoted again as $\{f_n\}_{n\in\mathbb{N}}$, and a set $A \supseteq A_\infty$, such that*

$$f_n \xrightarrow[n\to\infty]{hC} f^A := \max\big(f,\, \infty\cdot \mathbf{1}_A\big)\,, \qquad \mathbb{P}-a.e. \tag{4.6}$$

It is clear that Theorem 3.1 will be established, once Proposition 4.2 is. When $C := \sup_{n\in\mathbb{N}} \mathbb{E}(f_n) < \infty$ holds, f in (4.3) is integrable, namely $\mathbb{E}(f) \le C$ from (4.4) and Fatou. In particular, f is then real-valued, thus $f^A \equiv f$ in (4.6).

5 Proofs

We shall need a couple of auxiliary results. First, and always with the notation of (4.1)–(4.3), we note the following consequence of monotone and dominated convergence.

Lemma 5.1. *Suppose the set $D \subseteq \Omega\backslash A_\infty = \{f<\infty\}$ satisfies $\mathbb{E}\big(f\,\mathbf{1}_D\big) < \infty$. Then, for any $\varepsilon \in (0,1)$, there exists, after passing to a suitable subsequence, an integer $K \in \mathbb{N}$ with*

$$\lim_{n\to\infty} \mathbb{E}\big[\, f_n\, \mathbf{1}_{\{K \le f_n < L\}\cap D}\,\big] = \lim_{n\to\infty} \mathbb{E}\Big[\, f_n^{[K,L)}\, \mathbf{1}_D\,\Big] < \varepsilon\,, \quad \forall\;\; L = K+1, K+2, \cdots. \tag{5.1}$$

We are using, here and throughout, the notation

$$f_n^{[K,L)} := \sum_{k=K+1}^{L} f_n^{(k)} = f_n\,\mathbf{1}_{[K,L)}(f_n)\,, \qquad f_n^{[K,\infty)} := \sum_{k\ge K+1} f_n^{(k)} = f_n\,\mathbf{1}_{[K,\infty)}(f_n)\,, \tag{5.2}$$

and in an analogous manner $f^{[K,L)} := \sum_{k=K+1}^{L} f^{(k)}$, $f^{[K,\infty)} := \sum_{k\ge K+1} f^{(k)}$.

Secondly, we recall (4.5) and observe the following dichotomy.

Lemma 5.2. *In the setting of Proposition 4.2, consider any measurable set $B \supseteq A_\infty$ such that the property $f_n \xrightarrow[n\to\infty]{hC} \infty$ of* (4.5) *holds $\mathbb{P}-$a.e. on B. Then, either*

(i) there exist a set $C \supseteq B$ with $\mathbb{P}(C) > \mathbb{P}(B)$ and a subsequence, still denoted $\{f_n\}_{n\in\mathbb{N}}$, such that

$$f_n \xrightarrow[n\to\infty]{hC} \infty \qquad \textit{holds} \quad \mathbb{P}-a.e.\ \textit{on} \quad C\,; \qquad \textit{or} \tag{5.3}$$

(ii) the Cesàro *convergence $f_n \xrightarrow[n\to\infty]{hC} f < \infty$ holds $\mathbb{P}-a.e.$ on $\Omega\setminus B \subseteq \{f<\infty\}$.*

Under *Case (ii),* the set $B \supseteq A_\infty$ is maximal for the $\mathbb{P}-$a.e. property $f_n \xrightarrow{hC} \infty$: it cannot be "inflated" to a set $C \supseteq B$, which satisfies (5.3) and has bigger measure. This leads then to Proposition 4.2, and thus to Theorem 3.1 as well.

5.1 Proof of Theorem 2.2

Because $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^2$, we can extract a subsequence that converges to some $f \in \mathbb{L}^2$ weakly in $\mathbb{L}^2$. Thus, it suffices to prove the result for a sequence $\{g_n\}_{n\in\mathbb{N}}$ bounded in $\mathbb{L}^2$, and with $g_n \to 0$ weakly in $\mathbb{L}^2$. We take such a sequence, then, and approximate each g_n by a *simple* $h_n \in \mathbb{L}^2$ with $\|g_n - h_n\|_2 \le 2^{-n}$, $\forall\, n \in \mathbb{N}$. This gives, in particular,

$$\sum_{n\in\mathbb{N}} |g_n - h_n| < \infty\,, \quad \mathbb{P}-\text{a.e.}\,; \qquad h_n \to 0 \quad \text{weakly in } \mathbb{L}^2. \tag{5.4}$$

We construct now by induction a sequence $1 = n_1 < n_2 < \cdots$ of integers, such that

$$|\vartheta_k| < 2^{-k} \quad \text{holds } \ \mathbb{P}-\text{a.e., for } \ \vartheta_k := \mathbb{E}\big(h_{n_k} \,\big|\, h_{n_1}, \cdots, h_{n_{k-1}}\big)\,, \quad k = 2, 3, \cdots, \tag{5.5}$$

thus: $h_{n_1} = h_1$ is simple, and $\mathbb{E}(h_n|h_1) = \sum_{j=1}^J \gamma_j^{(n)} \mathbf{1}_{A_j}$ a linear combination of indicators with $\mathbb{P}(A_j) > 0$, $\gamma_j^{(n)} = \big(1/\mathbb{P}(A_j)\big) \cdot \mathbb{E}\big(h_n \mathbf{1}_{A_j}\big)$. This last expectation tends to zero as $n \to \infty$ from (5.4), for every fixed j; so we can choose $n_2 > n_1 = 1$ with $|\gamma_j^{(n_2)}| < 2^{-2}$, for $j = 1, \cdots, J$; i.e., $|\vartheta_2| < 2^{-2}$, $\mathbb{P}-$a.e. We can keep repeating this argument since, at each stage, $\big(h_{n_1}, \cdots, h_{n_{k-1}}\big)$ generates a finite partition of the space; and arrive at (5.5).

The sequence $\{h_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^2$, and thus the same is true of the martingale $X_n := \sum_{k=0}^n a_k \big(h_{n_k} - \vartheta_k\big)$, $n \in \mathbb{N}_0$ for any $\{a_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}$ with $\sum_{n\in\mathbb{N}} a_n^2 < \infty$. Martingale convergence theory ([7], p. 334) shows that the series $\sum_{k\in\mathbb{N}} a_k \big(h_{n_k} - \vartheta_k\big)$ converges $\mathbb{P}-$a.e. But we have also $\sum_{k\in\mathbb{N}} \big(|\vartheta_k| + |g_{n_k} - h_{n_k}|\big) < \infty$, $\mathbb{P}-$a.e. from (5.4)–(5.5), and deduce that $\sum_{k\in\mathbb{N}} a_k\, g_{n_k}$ converges $\mathbb{P}-$a.e., the claim of the theorem. $\square$

5.2 Proof of Lemma 5.1

We shall argue by contradiction, assuming the existence of an $\varepsilon \in (0,1)$ with the property that, for every $K \in \mathbb{N}$, there exists an integer $L > K$ such that

$$\mathbb{E}\Big[\sum_{k=K+1}^{L} f_n^{(k)}\, \mathbf{1}_D\Big] = \mathbb{E}\Big(f_n^{[K,L)}\, \mathbf{1}_D\Big) \geq \varepsilon \tag{5.6}$$

holds for infinitely many integers $n \in \mathbb{N}$. But this means that there is a subsequence, again denoted by $\{f_n\}_{n\in\mathbb{N}}$, *along which we have* (5.6) *for every* $n \in \mathbb{N}$. As a result, also

$$\mathbb{E}\Big[\sum_{k=K+1}^{L} \Big(\frac{1}{N}\sum_{n=1}^{N} f_n^{(k)}\Big)\, \mathbf{1}_D\Big] \geq \varepsilon \tag{5.7}$$

holds for every $N \in \mathbb{N}$. Now all the truncated functions $f_n^{(k)}$ as in (4.1), for $k = K+1, \ldots, L$ and $n \in \mathbb{N}$, take values "on the Procrustean bed" $\{0\} \cup [K, L)$; and $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f_n^{(k)} = f^{(k)}$ holds $\mathbb{P}-$a.e., on account of Lemma 4.1. Therefore, $\mathbb{E}\big[\sum_{k=K+1}^L f^{(k)}\, \mathbf{1}_D\big] \geq \varepsilon$ from bounded convergence and (5.7); and the nonnegativity of these $f^{(k)}$'s implies also

$$\mathbb{E}\Big(\sum_{k\geq K+1} f^{(k)}\, \mathbf{1}_D\Big) = \mathbb{E}\Big(f^{\,[K,\infty)}\, \mathbf{1}_D\Big) \geq \varepsilon\,, \qquad \forall \ \ K \in \mathbb{N}\,. \tag{5.8}$$

The nonnegativity gives also $\lim_{K\to\infty} \uparrow \sum_{k=1}^K f^{(k)} = f$, both $\mathbb{P}-$a.e. and in $\mathbb{L}^1$ on D. By monotone convergence, $\mathbb{E}\big[f^{\,[K,\infty)}\, \mathbf{1}_D\big] < \varepsilon/2$ holds for all $K \in \mathbb{N}$ large enough. But this contradicts (5.8), and we are done. $\square$

5.3 Proof of Lemma 5.2

We fix $j \in \mathbb{N}$, define

$$D_j := \{f \le j\} \backslash B\,, \qquad E_n^{[K,\infty)} := \big\{ f_n^{[K,\infty)} \ge K \big\} \cap D_j = \big\{ f_n \ge K \big\} \cap D_j\,, \tag{5.9}$$

and distinguish two contingencies:

Contingency A: $\alpha := \lim_{K\to\infty} \overline{\lim}_{n\to\infty} \mathbb{P}\big(E_n^{[K,\infty)}\big) > 0\,.$

Contingency B: $\alpha = 0\,.$

• In **Contingency A**, we pass to a subsequence $\{f_n\}_{n\in\mathbb{N}}$ with $\mathbb{P}\big(E_n^{[n^2,\infty)}\big) \ge \alpha/2\,, \forall\, n \in \mathbb{N}$; consider indicators $g_n := \mathbf{1}_{\mathbb{E}_n^{[n^2,\infty)}}$, $n \in \mathbb{N}$; and find a subsequence, still denoted $\{g_n\}_{n\in\mathbb{N}}$, that satisfies

$$g_n \xrightarrow[n\to\infty]{hC} g, \qquad \mathbb{P}-\text{a.e.} \tag{5.10}$$

for some measurable $g : \Omega \to [0,1]$ with $\mathbb{E}(g) \ge \alpha/2$, by bounded convergence. We are appealing here also to Theorem 2.2 and Kronecker, as in the proof of Lemma 4.1. We obtain in this manner $f_n \xrightarrow[n\to\infty]{hC} \infty$, $\mathbb{P}-$a.e. on $\{g > 0\}$.

The set $\{g > 0\} \subseteq \Omega\backslash B$ has measure $\mathbb{P}\big(g > 0\big) = \mathbb{E}[\mathbf{1}_{\{g>0\}}] \ge \mathbb{E}[g] \ge \alpha/2$; thus we are under *Case (i)* of Lemma 5.2, with $C := \{g > 0\} \cup B$ and $\mathbb{P}(C) > \mathbb{P}(B)$.

• Now we pass to **Contingency B**. We fix $\varepsilon > 0$ and $D_j = \{f \le j\}\backslash B$, and apply Lemma 5.1 to find $K \in \mathbb{N}$ for which (5.1) holds. We construct by induction a subsequence, this time indexed by $\{n_m\}_{m\in\mathbb{N}}$, and an increasing sequence $\{K_m\}_{m\in\mathbb{N}}$ of integers, as follows:

Start with $n_1 = 1, K_1 = K$, and suppose $n_1, \ldots, n_m$ as well as $K_1, \ldots, K_m$ have been constructed. Select, by the premise of Contingency B, an integer $K_{m+1} > K_m$, such that

$$\overline{\lim_{n\to\infty}}\, \mathbb{P}\big(E_n^{[K_{m+1},\infty)}\big) < 2^{-m}\,. \tag{5.11}$$

Using (5.1)*, select now an integer $n_{m+1} > n_m$ so large, that $\mathbb{E}\big(f_{n_{m+1}}^{[K,K_{m+1})}\,\mathbf{1}_{D_j}\big) < \varepsilon$ holds, therefore also*

$$\mathbb{E}\big(f_{n_{m+1}}^{[0,K_{m+1})}\,\mathbf{1}_{D_j}\big) \;<\; \mathbb{E}\big(f_{n_{m+1}}^{[0,K)}\,\mathbf{1}_{D_j}\big) + \varepsilon\,, \tag{5.12}$$

completing the induction. Now, from (5.11), (5.9) and Borel-Cantelli, we have $\mathbb{P}-$a.e.

$$\lim_{M\to\infty} \left(\frac{1}{M} \sum_{m=1}^{M} f_{n_m}^{[0,K_m)} \cdot \mathbf{1}_{D_j} \right) \;=\; \lim_{M\to\infty} \left(\frac{1}{M} \sum_{m=1}^{M} f_{n_m} \cdot \mathbf{1}_{D_j} \right). \tag{5.13}$$

On account of (5.12), the expectation of the function on the left-hand side in (5.13) satisfies

$$\begin{aligned} \overline{\lim_{M\to\infty}}\, \mathbb{E}\left[\left(\frac{1}{M} \sum_{m=1}^{M} f_{n_m}^{[0,K_m)} \right) \mathbf{1}_{D_j} \right] \;&\le\; \overline{\lim_{M\to\infty}}\, \mathbb{E}\left[\left(\frac{1}{M} \sum_{m=1}^{M} f_{n_m}^{[0,K)} \right) \mathbf{1}_{D_j} \right] + \varepsilon \\ &\le \mathbb{E}(f\, \mathbf{1}_{D_j}) + \varepsilon\,. \end{aligned} \tag{5.14}$$

At this point, we need to let $\varepsilon \downarrow 0$, $j \to \infty$. We pass to the limit $\varepsilon \downarrow 0$ first, with $j \in \mathbb{N}$ fixed, and find a diagonal subsequence, still denoted by $\{f_n\}_{n\in\mathbb{N}}$, with

$$f_n \xrightarrow[n\to\infty]{hC} f, \qquad \mathbb{P}-\text{a.e. on } D_j\,; \tag{5.15}$$

details for this argument are supplied right below. Thus, on account of (5.13), (5.14):

$$\mathbb{E}\Big[\Big(\lim_{N\to\infty}\Big(\frac{1}{N}\sum_{n=1}^{N} f_n\Big)-f\Big)\mathbf{1}_{D_j}\Big]=0\,,\qquad \forall\ \ j\in\mathbb{N}\,. \tag{5.16}$$

The next step is to let $j\to\infty$. We do this again by extracting subsequences, successively for each $j\in\mathbb{N}$, then passing to a diagonal subsequence. In this manner, we obtain (5.16) with D_j replaced there by the set $D:=\bigcup_{j\in\mathbb{N}} D_j=\{f<\infty\}\backslash B$.

We invoke at this point (4.4), which gives $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_n=f$, $\mathbb{P}-$a.e. on this D, and deduce that we are in *Case (ii)* of Lemma 5.2. □

Proof of (5.15): Let us reprise what has been done so far, for $\varepsilon^{(1)}=\varepsilon>0$: We have seen that the subsequence $\{f_{n_m}\mathbf{1}_{D_j}\}_{m\in\mathbb{N}}$ of $\{f_n\mathbf{1}_{D_j}\}_{n\in\mathbb{N}}$ is Borel-Cantelli equivalent (cf. section 2) to $h^{(1)}_m := f^{[0,K_m)}_{n_m}\mathbf{1}_{D_j}$, $m\in\mathbb{N}$; i.e., $\sum_{m\in\mathbb{N}}\mathbb{P}\big(f_{n_m}\mathbf{1}_{D_j}\neq f^{[0,K_m)}_{n_m}\mathbf{1}_{D_j}\big)<\infty$. And for a given $\varepsilon^{(1)}>0$, we have found an integer $K_1\in\mathbb{N}$ with the property $\mathbb{E}\big[\,h^{(1)}_m\mathbf{1}_{\{h^{(1)}_m\geq K_1\}\cap D_j}\big]<\varepsilon^{(1)}$ for all $m\in\mathbb{N}$ large enough.

Repeating this argument with $\varepsilon^{(2)}>0$, we extract a subsequence $\{f_{n_{m_\ell}}\mathbf{1}_{D_j}\}_{\ell\in\mathbb{N}}$ of $\{f_{n_m}\mathbf{1}_{D_j}\}_{n\in\mathbb{N}}$ and find $K_2\in\mathbb{N}$, and a sequence $\big\{h^{(2)}_\ell\big\}_{\ell\in\mathbb{N}}$ which is Borel-Cantelli equivalent to $\{f_{n_{m_\ell}}\mathbf{1}_{D_j}\}_{\ell\in\mathbb{N}}$, with

$$\mathbb{E}\Big[\,h^{(2)}_\ell\,\mathbf{1}_{\{h^{(2)}_\ell\geq K_2\}\cap D_j}\Big]<\varepsilon^{(2)}\quad\text{for all }\ \ell\in\mathbb{N}\ \text{ sufficiently large.}$$

Continuing in a similar manner, then passing to a diagonal subsequence, we obtain a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that, for every term $\varepsilon^{(i)}>0$ in a sequence with $\varepsilon^{(i)}\downarrow 0$, there exist $K_i\in\mathbb{N}$ and a sequence $\big\{h^{(i)}_n\big\}_{n\in\mathbb{N}}$, Borel-Cantelli equivalent to $\{f_n\}_{n\in\mathbb{N}}$, with

$$\mathbb{E}\Big[\,h^{(i)}_n\,\mathbf{1}_{\{h^{(i)}_n\geq K_i\}\cap D_j}\Big]<\varepsilon^{(i)}\qquad\text{for all }\ n\in\mathbb{N}\ \text{ sufficiently large.}$$

But this implies $f_n\xrightarrow[n\to\infty]{hC} f$, $\ \mathbb{P}-$ a.e. on D_j, as claimed in (5.15). □

5.4 Proofs of Proposition 4.2 and Theorem 3.1

On the strength of Lemma 5.2 we construct, by exhaustion or transfinite induction arguments and as long as we are under the dispensation of its *Case (i),* an increasing sequence $B_1\subseteq B_2\subseteq\ldots$ of sets as postulated there, whose union $B_\infty:=\bigcup_{j\in\mathbb{N}}B_j$ is maximal with the property (5.3) for an appropriate subsequence.

But such maximality means that, on the complement $\Omega\backslash B_\infty$ of this set, we must be in the realm of *Case (ii).* This establishes the Proposition, thus also Theorem 3.1. □

5.5 Proofs of Theorems 3.2 and 2.1

The sequence $\{f^-_n\}_{n\in\mathbb{N}}$ satisfies the conditions of Theorem 3.1, and $\sup_{n\in\mathbb{N}}\mathbb{E}\big(f^-_n\big)<\infty$. Thus, from Theorem 3.1 and Fatou we obtain, after passing to a subsequence,

$$f^-_{n_k}\xrightarrow[k\to\infty]{hC} f^{(-)},\qquad \mathbb{P}-\text{a.e.}$$

for some $f^{(-)} : \Omega \to [0, \infty)$ which is integrable: $\mathbb{E}\big(f^{(-)}\big) \leq \sup_{n\in\mathbb{N}} \mathbb{E}\big(f_n^-\big) < \infty$.

Passing yet again to a subsequence, still denoted $\{f_{n_k}\}_{k\in\mathbb{N}}$, we apply Theorem 3.1 to the positive parts $\big\{f_{n_k}^+\big\}_{k\in\mathbb{N}}$ and obtain $f_{n_k}^+ \xrightarrow[k\to\infty]{hC} f^{(+)}$, $\mathbb{P}-$ a.e., for some measurable $f^{(+)} : \Omega \to [0, \infty]$. The proof of Theorem 3.2 is completed by the observation

$$f_{n_k}^+ - f_{n_k}^- = f_{n_k} \xrightarrow[k\to\infty]{hC} f := f^{(+)} - f^{(-)}, \quad \mathbb{P} - \text{a.e.}$$

If $\sup_{n\in\mathbb{N}} \mathbb{E}\big(f_n^+\big) < \infty$ also holds, we have as before $\mathbb{E}\big(f^{(+)}\big) \leq \sup_{n\in\mathbb{N}} \mathbb{E}\big(f_n^+\big) < \infty$, the just defined function f is integrable, and Theorem 2.1 follows. □

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