LAGRANGE INVERSION

Theorem 1. *Let*

$$f(z) = f_1 z + f_2 z^2 + \cdots, \qquad f(0) = 0, \quad f_1 \neq 0, \quad \textit{and}$$
$$g(z) = \sum_k c_k f^k(z).$$

Then

$$c_n = \frac{1}{n}[[z^{-1}]]\, g'(z) f^{-n}(z), \qquad n \neq 0, \tag{i}$$
$$c_n = [[z^{-1}]]\, g(z)\, f'(z) f^{-n-1}(z), \qquad n \in \mathbb{Z}. \tag{ii}$$

Let $\mathbb{e}^z (= \exp(z))$ denote the formal power series $\sum_{k\geq 0} \frac{z^k}{k!}$.

Proposition 2. *The following expansions (due to J.H. Lambert [1758]) hold (as formal power series, or analytically when $|z| < 1$):*

$$\mathbb{e}^{az} = \sum_{k=0}^{\infty} \frac{a(a+bk)^{k-1}}{k!} z^k \mathbb{e}^{-bzk}, \tag{L1}$$
$$\frac{\mathbb{e}^{az}}{1-bz} = \sum_{k=0}^{\infty} \frac{(a+bk)^k}{k!} z^k \mathbb{e}^{-bzk}, \tag{L2}$$
$$(1+z)^a = \sum_{k=0}^{\infty} \frac{a}{a+bk} \binom{a+bk}{k} z^k (1+z)^{-bk}, \tag{L3}$$
$$\frac{(1+z)^a}{1-\frac{bz}{1+z}} = \sum_{k=0}^{\infty} \binom{a+bk}{k} z^k (1+z)^{-bk}. \tag{L4}$$

For $b = 0$, (L1) and (L2) reduce to Euler's expansion for the exponential function, and (L3) and (L4) to Newton's binomial series expansion.

Proof of Prop. 2. We apply Theorem 1 with different instances of $f(z)$ and $g(z)$.

To show (L1), we take

$$f(z) = z\mathbb{e}^{-bz} \qquad \text{and} \qquad g(z) = \mathbb{e}^{az}.$$

Then $c_0 = 1$, and using (i) we have for $n > 0$

$$\begin{aligned} c_n &= \frac{1}{n}[[z^{-1}]]\, g'(z) f^{-n}(z) \\ &= \frac{1}{n}[[z^{-1}]]\, a\mathbb{e}^{az} z^{-n} \mathbb{e}^{bnz} \\ &= \frac{a}{n}[[z^{n-1}]]\, \mathbb{e}^{(a+bn)z} \\ &= \frac{a}{n} \frac{(a+bn)^{n-1}}{(n-1)!}, \end{aligned}$$

as claimed.

To show (L2), we take

$$f(z) = z\mathbb{e}^{-bz} \qquad \text{and} \qquad g(z) = \frac{\mathbb{e}^{az}}{1-bz}.$$

Then using (ii) we have for $n \geq 0$

$$\begin{aligned}
c_n &= [[z^{-1}]]\, g(z)f'(z)f^{-n-1}(z) \\
&= [[z^{-1}]]\, \frac{\mathbb{e}^{az}}{1-bz}(1-bz)\mathbb{e}^{-bz}z^{-n-1}\mathbb{e}^{b(n+1)z} \\
&= [[z^{n}]]\, \mathbb{e}^{(a+bn)z} \\
&= \frac{(a+bn)^n}{n!},
\end{aligned}$$

as claimed.

To show (L3), we take

$$f(z) = z(1+z)^{-b} \qquad \text{and} \qquad g(z) = (1+z)^{a}.$$

Then $c_0 = 1$, and using (i) we have for $n > 0$

$$\begin{aligned}
c_n &= \frac{1}{n}[[z^{-1}]]\, g'(z)f^{-n}(z) \\
&= \frac{1}{n}[[z^{-1}]]\, a(1+z)^{a-1}z^{-n}(1+z)^{bn} \\
&= \frac{a}{n}[[z^{n-1}]]\, (1+z)^{a+bn-1} \\
&= \frac{a}{n}\binom{a+bn-1}{n-1} \\
&= \frac{a}{a+bn}\binom{a+bn}{n},
\end{aligned}$$

as claimed.

To show (L4), we take

$$f(z) = z(1+z)^{-b} \qquad \text{and} \qquad g(z) = \frac{(1+z)^{a}}{1-\frac{bz}{1+z}}.$$

Then using (ii) we have for $n \geq 0$

$$\begin{aligned}
c_n &= [[z^{-1}]]\, g(z)f'(z)f^{-n-1}(z) \\
&= [[z^{-1}]]\, \frac{(1+z)^{a}}{1-\frac{bz}{1+z}}\left(\frac{-bz(1+z)^{-b}}{1+z} + (1+z)^{-b}\right)z^{-n-1}(1+z)^{b(n+1)} \\
&= [[z^{n}]]\, (1+z)^{a+bn} \\
&= \binom{a+bn}{n},
\end{aligned}$$

as claimed. $\square$