Additional Exercise Problems

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Let $\mathrm{e}^z (= \exp(z))$ denote the formal power series $\sum_{k\ge 0} \frac{z^k}{k!}$.

A.1. Use suitable special cases of the Lagrange inversion formula to prove the following identities:

$$\frac{(1+z)^r \mathrm{e}^{az}}{1-bz-\frac{sz}{1+z}} = \sum_{k\ge 0}\left(\sum_{j=0}^{k}\binom{r+sk}{j}\frac{(a+bk)^{k-j}}{(k-j)!}\right) z^k (1+z)^{-sk}\mathrm{e}^{-bkz}, \tag{1}$$

$$\log(1+z) = \sum_{k\ge 1}\frac{1}{ak}\binom{ak}{k} z^k (1+z)^{-ak}, \tag{2}$$

$$\frac{\log(1+z)}{1-\frac{az}{1+z}} = \sum_{k\ge 1}\left(\sum_{j=1}^{k}\frac{(-1)^{j-1}}{j}\binom{ak}{k-j}\right) z^k (1+z)^{-ak} \tag{3}$$

$$= \sum_{k\ge 1}\left(\sum_{j=1}^{k}\frac{1}{aj}\binom{aj}{j}\binom{a(k-j)}{k-j}\right) z^k (1+z)^{-ak}. \tag{4}$$

A.2. Solve the recurrence

$$\begin{aligned} &g_0 = 0, \qquad g_1 = 1,\\ &g_n = -2n g_{n-1} + \sum_{k=0}^{n}\binom{n}{k} g_k g_{n-k}, \qquad \text{for} \quad n > 1, \end{aligned}$$

using the exponential generating function $G(z) := \sum_{k\ge 0} g_k \frac{z^k}{k!}$.

Is there a (nice) explicit product formula for g_n?

A.3. Show (possibly with the help of generating functions) the following identities for the Fibonacci numbers (defined by $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, for $n \ge 2$):

$$\sum_{k=0}^{n} F_k F_{n-k} = \frac{(n+1)F_{n+1} + 2(n+2)F_n}{5}, \tag{5}$$

$$\sum_{k\ge 0} F_{mk} z^k = \frac{1 - F_{m-2} z}{1 - (F_{m-2}+F_m)z + (-1)^m z^2}. \tag{6}$$

where $m \ge 1$. (The formula in the case $m = 1$ is known from the lecture course Discrete Mathematics. We implicitly assume $F_{-1} = 0$, consistent with the recursion.) [*Hint:* Make use of the following ingredients (which have to be shown): $\phi^m + \hat\phi^m = F_{m-2} + F_m$ and $\phi^m \hat\phi^m = (-1)^m$, for $1 - z - z^2 = (1-\phi z)(1-\hat\phi z)$. Work out the complete details!]

A.4. How many possibilities are there to arrange the numbers $\{1, 2, \ldots, 2n\}$ in a $(2\times n)$-matrix such that the numbers appear in ascending order in each of the two rows (from left to right) and n columns (from top to bottom)? For instance, for $n = 5$

$$\begin{pmatrix} 1 & 2 & 4 & 5 & 8 \\ 3 & 6 & 7 & 9 & 10 \end{pmatrix}$$

is such a valid arrangement.