Additional Exercise Problems

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A.5. The *Chebyshev polynomials of the first kind* T_n are defined by

$$T_n(\cos\theta) = \cos n\theta.$$

The first Chebyshev polynomials are

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x.$$

Derive the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ and deduce the identity

$$\sum_{n\geq 0} T_n(x)t^n = \frac{1-tx}{1-2tx+t^2}.$$

A.6. Let $a_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ and $b_n = 2^n n!$. Show that the respective *exponential* generating functions $a(z)$ and $b(z)$ of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ satisfy the simple relation $b(z) = a^2(z)$.

A.7. A sampling of a space probe revealed that on Mars the organic structure DNA is built from five different components, symbolically denoted by k, y, j, i, w, instead of the four components of the DNA on Earth. The four pairs ji, jw, wi and ww never appear as a subsequence of any two consecutive components in a martian DNA sequence but all other sequences that do not contain forbidden pairs are possible. (For instance, $yyjik$ is forbidden but $yyijk$ is allowed.) How many different martian DNA-sequences of length n are possible? (For $n = 2$ the answer is 21; we assume that the left and right ends of a sequence can be distinguished.)
[*Hint*: Denote the number of martian DNA-sequences of length n that do *not* end with j or w by a_n, and denote those sequences that end with j or w by b_n. Derive a (coupled) system of two linear recurrences for a_n and b_n, and solve it!]

A.8. A *ladder* is a graph $L_n = (V_n, E_n)$ on $2n$ vertices $V_n = \{1, 2, \ldots, 2n\}$, which are connected by $3n-2$ edges, geometrically–graphically "in form of a ladder". (Say, $E_n = \{(1,2), (3,4), \ldots, (2n-1, 2n), (1,3), (3,5), \ldots, (2n-3, 2n-1), (2,4), (4,6), \ldots, (2n-2, 2n)\}$.) Use the "standard recipe" (deriving a recursion, determining the generating function, etc.) to find the number of *spanning trees* of L_n.

A.9. *Motzkin paths* are paths in the integer lattice $\mathbb{Z}^2$, which start in $(0,0)$, remain non-negative (i.e., they always have y-coordinate ≥ 0), end on one the x-axis, and consist of the following steps: $(1,1)$, $(1,-1)$, $(1,0)$ (i.e., one is dealing with Dyck paths with additional horizontal steps $(1,0)$). The number of Motzkin paths with n steps is called n-th *Motzkin number* and is denoted (here) by M_n. The sequence of Motzkin numbers starts as follows: $M_0 = 1$, $M_1 = 1$, $M_2 = 2$, $M_3 = 4$. Compute further elements of this sequence. Find a recursion for M_n and determine the generating function!