Affine Macdonald conjectures and special values of Felder-Varchenko functions

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I. Affine Macdonald conjectures

II. Felder-Varchenko functions and hypergeometric functions

III. Integral evaluation of affine Macdonald polynomials

Macdonald polynomials

The Macdonald difference operators are

$$D_n^r(q^2, t^2) = t^{r(r-n)} \sum_{|I|=r} \prod_{i \in I, j \notin I} \frac{t^2 x_i - x_j}{x_i - x_j} T_{q^2, I}$$

with $T_{q^2,I} = \prod_{i \in I} T_{q^2,i}$ and

$$T_{q^2,i} f(x_1, \ldots, x_n) = f(x_1, \ldots, q^2 x_i, \ldots, x_n).$$

Macdonald polynomials $P_\lambda(x; q^2, t^2)$: eigenfn's of $D_n^r(q^2, t^2)$:

$$D_n^r(q^2, t^2) P_\lambda(x; q^2, t^2) = e_r(q^{2\lambda} t^{2\rho}) P_\lambda(x; q^2, t^2).$$

where $\rho = \left(\frac{n-1}{2}, \ldots, \frac{1-n}{2}\right)$ and e_r is the elem. sym. polynomial.

Etingof-Kirillov Jr. approach

Let L_λ be f.d. $U_q(\mathfrak{gl}_n)$-irrep for signature λ. For $m \in \mathbb{Z}_{\geq 0}$, define

$$W_{m-1} = \mathrm{Sym}^{n(m-1)}\mathbb{C}^n \otimes (\det)^{-(m-1)}.$$

For dominant λ, there is a unique intertwiner

$$\Phi_\lambda : L_{\lambda+(m-1)\rho} \to L_{\lambda+(m-1)\rho} \otimes W_{m-1}$$

so that $v_{HW} \mapsto v_{HW} \otimes w_0 + (\text{l.o.t.})$ with $W_{m-1}[0] \simeq \mathbb{C} \cdot w_0$.

Theorem (Etingof-Kirillov Jr.)

The Macdonald polynomial $P_\lambda(x; q^2, q^{2m})$ is given by

$$P_\lambda(x; q^2, q^{2m}) = \frac{\mathrm{Tr}(\Phi_\lambda x^h)}{\mathrm{Tr}(\Phi_0 x^h)}.$$

Note: Interpret traces of Φ_λ as scalars via $W_{m-1}[0] \simeq \mathbb{C} \cdot w_0$.

Affine Macdonald polynomials (I)

Affine setting: Replace $U_q(\mathfrak{gl}_n)$ by $U_q(\widehat{\mathfrak{sl}}_n)$ / $U_q(\widetilde{\mathfrak{sl}}_n)$ with

- Cartan: $\widetilde{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C} \cdot c \oplus \mathbb{C} \cdot d$
- dual Cartan: $\widetilde{\mathfrak{h}}^* := \mathfrak{h}^* \oplus \mathbb{C} \cdot \Lambda_0 \oplus \mathbb{C} \cdot \delta$
- $\widetilde{\rho} := \rho + h^\vee \Lambda_0$ ($h^\vee$ = dual Coxeter number)

Consider the $U_q(\widehat{\mathfrak{sl}}_n)$ representations:

- Verma module $M_{\mu+k\Lambda_0}$, integrable irrep $L_{\mu+k\Lambda_0}$
- Evaluation module $V(z)$ via $U_q(\widehat{\mathfrak{sl}}_n) \overset{\mathrm{ev}_z}{\to} U_q(\mathfrak{sl}_n)$

View h.w. $U_q(\widehat{\mathfrak{sl}}_n)$-rep as graded $U_q(\widetilde{\mathfrak{sl}}_n)$-rep by letting q^d act by 1 on h.w. vector

Affine Macdonald polynomials (II)

Proposition

For $W_{m-1}[0] \simeq \mathbb{C} \cdot w_0$, there is a unique intertwiner

$$\Upsilon_{\mu,k,m}(z) : L_{\mu+k\Lambda_0+(m-1)\widetilde{\rho}} \to L_{\mu+k\Lambda_0+(m-1)\widetilde{\rho}} \otimes W_{m-1}(z)$$

with $\Upsilon_{\mu,k,m}(z) v_{HW} = v_{HW} \otimes w_0 + \text{(l.o.t.)}$.

- Construction also works for

$$L_{\mu+k\Lambda_0+(m-1)\widetilde{\rho}} \mapsto M_{\mu+k\Lambda_0} \qquad W_{m-1}(z) \mapsto \text{product of eval reps}$$

- Multiple eval reps $\implies$ sol. of q-KZ, q-KZB ([FR], [ESV])
- Both cases $\implies$ eigenfn of affine MR operators ([ESV])
  - Formal difference operators, e.g. power series in finite difference operators of growing order

Affine Macdonald polynomials (III)

Define the trace function

$$\chi_{\mu,k,m}(q,\lambda,\omega) := \mathrm{Tr}\Big(\Upsilon_{\mu,k,m}(z) q^{2\lambda+2\omega d}\Big).$$

- Scalar function via $W_{m-1}[0] \simeq \mathbb{C} \cdot w_0$
- Each diagonal matrix element is independent of z
- Formal series in $q^{-2\omega}$, convergence known in some cases

Definition (Etingof-Kirillov Jr. 1995)

The **affine Macdonald polynomial** for $\widehat{\mathfrak{sl}}_n$ at $t = q^m$ is

$$J_{\mu,k,m}(q,\lambda,\omega) := \frac{\chi_{\mu,k,m}(q,\lambda,\omega)}{\chi_{0,0,m}(q,\lambda,\omega)}.$$

Macdonald conjectures

Denominator conjecture:

$$\mathrm{Tr}(\Phi_0 q^{2\lambda}) = q^{2(m-1)(\rho,\lambda)} \prod_{i=1}^{m-1} \prod_{\alpha>0} (1 - q^{-2(\alpha,\lambda)+2i})^{\mathrm{mult}(\alpha)}$$

Evaluation conjecture:

$$P_\mu(q^{2m\rho}; q^2, q^{2m}) = q^{2m(\rho,\mu)} \prod_{i=0}^{m-1} \prod_{\alpha>0} \frac{(1 - q^{-2(\alpha,\mu+m\rho)-2i})^{\mathrm{mult}(\alpha)}}{(1 - q^{-2(\alpha,m\rho)-2i})^{\mathrm{mult}(\alpha)}}$$

In the rest of this talk, we modify this to the affine setting.
Note: Naive generalization is wrong. **New factors** appear!

Affine denominator conjecture (I)

Theorem (Rains-S.-Varchenko)

For $n = 2$ and $m = 2$, we have

$$\chi_{0,0,2}(q, \lambda, \omega) = q^{\lambda} \frac{(q^{-2\omega+2}; q^{-2\omega})^2}{(q^{-2\omega+4}; q^{-2\omega})} (q^{-2\lambda+2}; q^{-2\omega})(q^{2\lambda+2} q^{-2\omega}; q^{-2\omega}).$$

Theorem (Etingof-Kirillov Jr. 1995)

The affine Macdonald denominator is given by

$$\chi_{0,0,m}(q, \lambda, \omega) = q^{2(m-1)(\rho,\lambda)} \prod_{i=1}^{m-1} \prod_{\alpha>0} (1 - q^{-2(\alpha,\lambda+\omega d)+2i}) \cdot f_{n,m}(q, q^{-2\omega}),$$

where $f_{n,m}(q, q^{-2\omega})$ has unit constant term in $q^{-2\omega}$-expansion.

Affine denominator conjecture (II)

Conjecture (Rains-S.-Varchenko)

The affine Macdonald denominator is given by

$$\chi_{0,0,m}(q,\lambda,\omega) = q^{2(m-1)(\rho,\lambda)} \prod_{i=1}^{m-1}\prod_{\alpha>0}(1-q^{-2(\alpha,\lambda+\omega d)+2i})^{\mathrm{mult}(\alpha)} \cdot \Delta_m(q,q^{-2\omega})$$

where

$$\Delta_m(q,q^{-2\omega}) := \frac{\prod_{i=1}^{m-1}(q^{-2\omega+2i};q^{-2\omega})}{\prod_{i=1}^{m-1}(q^{-2\omega+2ni};q^{-2\omega})}.$$

- For $n=2$, $m=2$, proven in our theorem.
- We verified by computer in Magma to first order in $q^{-2\omega}$ for $n=2$, $m\le 15$ and $n=3$, $m\le 3$.
- Classical limit: $\Delta_m(q,q^{-2\omega})\to 1$, Etingof-Kirillov Jr. 1995.

Affine evaluation conjecture

Theorem (Rains-S.-Varchenko)

For $|q| > 1$, $n = 2$, $m = 2$, and $\kappa = k + 4$, we have

$$J_{\mu,k,2}(q,2,4) = q^{2\mu}\frac{(q^{-2};q^{-2\kappa})}{(q^{-4};q^{-2\kappa})}\theta_0(q^{-2\mu-4};q^{-2\kappa})(q^{-2\mu-6};q^{-2\kappa})$$
$$\frac{(q^{2\mu+2}q^{-2\kappa};q^{-2\kappa})(q^{-2\kappa-2};q^{-2\kappa})(q^{-2\kappa};q^{-2\kappa})}{(q^{-4};q^{-2})(q^{-6};q^{-8})(q^{-2};q^{-8})}.$$

Conjecture (Rains-S.-Varchenko)

For $|q| > 1$, we have the affine Macdonald evaluation

$$J_{\mu,k,m}(q,m\rho,mn) = \prod_{i=1}^{m-1}\frac{(q^{-2i};q^{-2(k+mn)})}{(q^{-2ni};q^{-2(k+mn)})}q^{2m(\rho,\mu)}$$
$$\prod_{i=1}^{m-1}\frac{(q^{-2ni};q^{-2mn}))}{(q^{-2i};q^{-2mn})}\prod_{i=0}^{m-1}\prod_{\alpha>0}\frac{(1-q^{-2(\alpha,\mu+k\Lambda_0+m\widetilde{\rho})-2i})^{\mathrm{mult}(\alpha)}}{(1-q^{-2(\alpha,m\widetilde{\rho})-2i})^{\mathrm{mult}(\alpha)}}.$$

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Felder-Varchenko function

The Felder-Varchenko function is

$$u(\lambda, \mu, \tau, \sigma, \eta) := e^{-\frac{\pi i \lambda \mu}{2\eta}} \int_\gamma \Omega_{2\eta}(t; \tau, \sigma) \frac{\theta(t+\lambda; \tau)}{\theta(t-2\eta; \tau)} \frac{\theta(t+\mu; \sigma)}{\theta(t-2\eta; \sigma)} dt$$

for the phase function and elliptic gamma function

$$\Omega_{2\eta}(t; \tau, \sigma) := \frac{\Gamma(t+2\eta; \tau, \sigma)}{\Gamma(t-2\eta; \tau, \sigma)} \text{ and } \Gamma(z; \tau, \sigma) := \frac{(\tau+\sigma-z; \tau, \sigma)}{(z; \tau, \sigma)}.$$

- Similar form to Bethe ansatz solution to q-KZB equations
- Solution to q-KZB heat equation for 3-dim rep of $\mathfrak{sl}_2$
- Symmetric under $(\lambda, \tau) \leftrightarrow (\mu, \sigma)$

Hypergeometric theta function

Definition (Felder-Varchenko 2004)

The non-symmetric hypergeometric theta function is

$$\widetilde{\Delta}_{\mu,\kappa}(\lambda;\tau,\eta) := \sum_{j\in 2\kappa\mathbb{Z}+\mu} u(\lambda, 2\eta j, \tau, -2\eta\kappa, \eta) Q(2\eta j, -2\eta\kappa, \eta) e^{\pi i \frac{\tau+4\eta}{2\kappa} j^2}$$

for $Q(\mu;\sigma,\eta) := \frac{\theta(4\eta;\sigma)\theta'(0;\sigma)}{\theta(\mu-2\eta;\sigma)\theta(\mu+2\eta;\sigma)}$.

Proposition (Felder-Varchenko 2004)

For certain parameters $\widetilde{\Delta}_{\mu,\kappa}$ is holomorphic in λ and

$$\widetilde{\Delta}_{\mu,\kappa}(\lambda;\tau,\eta) = e^{\frac{2\pi i\eta}{\kappa}\mu^2} Q(2\eta\mu; -2\eta\kappa,\eta)\widetilde{I}_{\mu,\kappa}(\lambda;\tau,\eta)$$

for

$$\widetilde{I}_{\mu,\kappa}(\lambda;\tau,\eta) := e^{\pi i\tau\frac{\mu^2}{2\kappa} - \pi i\lambda\mu}(2\kappa\tau;2\kappa\tau)\int_\gamma \Omega_{2\eta}(t;\tau,-2\eta\kappa)\frac{\theta(t+\lambda;\tau)}{\theta(t-2\eta;\tau)}$$

$$\frac{\theta(t+2\eta\mu;-2\eta\kappa)}{\theta(t-2\eta;-2\eta\kappa)}\theta_0(\frac{1}{2}+\mu\tau+\kappa\tau-\kappa\lambda+2t;2\kappa\tau)dt.$$

Elliptic Macdonald polynomials

Definition (Felder-Varchenko 2004)

Define the symmetrized version of $\widetilde{\Delta}_{\mu,\kappa}$ by

$$\Delta_{\mu,\kappa}(\lambda;\tau,\eta) := \widetilde{\Delta}_{\mu,\kappa}(\lambda;\tau,\eta) - \widetilde{\Delta}_{\mu,\kappa}(-\lambda;\tau,\eta).$$

The **elliptic Macdonald polynomial** at $t = q^2$ is

$$P_{\mu,\kappa}(\lambda;\tau,\eta) := e^{-\pi i \frac{4\eta+\tau}{2\kappa}(\mu+2)^2 + \pi i 3\tau/4} \frac{\Delta_{\mu+2,\kappa}(\lambda;\tau,\eta)}{\theta(\lambda-2\eta;\tau)\theta(\lambda;\tau)\theta(\lambda+2\eta;\tau)}.$$

This is a difference of two explicit hypergeometric integrals.

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Integrals for affine Macdonald polynomials

Theorem (S. 2016)

For $|q| > 1$, $|q^{-2\omega}| < |q^{-6}|$, and $|q^{-2\omega}|$ close to 0, we have

$$J_{\mu,k,2}(q, \lambda, \omega) = P_{\mu,\kappa}(2\eta\lambda; -2\eta\omega, \eta) \cdot \text{(explicit product)}$$

- Hypergeom. integral for affine Macdonald poly at $m = 2$
- Implies convergence of $J_{\mu,k,2}(q, \lambda, \omega)$ for numerical params
- Conjectured by Felder-Varchenko 2004

Proof sketch:

1. Relate trace over Verma to Felder-Varchenko function
   - Use q-Wakimoto bosonization of Matsuo 1994
   - Conjectured by Etingof-Varchenko 1999
2. Affine BGG resolution gives theta series representation for elliptic Macdonald polynomial / hypergeom. theta function

Evaluating the integrals (I)

Guess answers from $SL(3,\mathbb{Z})$ modular relations:

$$P_{0,4}(\lambda;\tau,\eta)S^{-}(\tau,\eta)P_{0,4}(\lambda;-1/\tau,\eta/\tau)^{-1}$$
$$=4\sqrt{2}\pi i\tau\exp\left(\pi i\frac{4+216\eta^2-42\eta(\tau-1)+3\tau+4\tau^2}{12\tau}\right)$$

from Felder-Varchenko 2004 for

$$S^{-}(\tau,\eta):=-2\frac{\theta(1/2;\tau/8\eta)\theta'(0;\tau/8\eta)}{\theta(3/4;\tau/8\eta)\theta(1/4;\tau/8\eta)}u\Big(\frac{1}{2},\frac{1}{2},\frac{1}{8\eta},\frac{\tau}{8\eta},-\frac{1}{8}\Big).$$

Ansatz: relation follows from $SL(3,\mathbb{Z})$ modular relation

$$\Gamma(\frac{z}{\sigma};\frac{\tau}{\sigma},-\frac{1}{\sigma})=e^{\pi iR(z;\tau,\sigma)}\Gamma(\frac{z-\sigma}{\tau},-\frac{1}{\tau},-\frac{\sigma}{\tau})\Gamma(z;\tau,\sigma)$$

for explicit $R(z;\tau,\sigma)$ and $\theta(z/\tau;-1/\tau)=-i\sqrt{-i\tau}e^{\pi iz^2/\tau}\theta(z;\tau)$.
Numerically verified our guess in Mathematica.

Evaluating the integrals (II)

Denominator conj: want $I(\lambda;\tau,\eta) = \widetilde{I}(\lambda;\tau,\eta) - \widetilde{I}(-\lambda;\tau,\eta)$ for

$$\widetilde{I}(\lambda;\tau,\eta) = e^{-3\pi i\lambda} \int_\gamma \frac{\Gamma(t-2\eta;\tau,8\eta)}{\Gamma(t-2\eta;\tau,8\eta)}$$
$$\frac{\theta_0(t+\lambda;\tau)}{\theta_0(t+2\eta;\tau)} \frac{\theta_0(t-4\eta;8\eta)}{\theta_0(t+2\eta;8\eta)} \theta_0(2t+6\tau-4\lambda+\frac{1}{2};8\tau)dt.$$

Main idea: manipulate to reveal the elliptic beta integral.

Proposition (elliptic beta integral, Spiridonov 2001)

Given $\tau,\sigma,s_1,\ldots,s_6$ with $\mathrm{Im}(\tau),\mathrm{Im}(\sigma),\mathrm{Im}(s_i)>0$ and $\sum_i s_i = \tau+\sigma$, we have

$$\int_{[-1/2,1/2]} \frac{\prod_{i=1}^6 \Gamma(\pm t+s_i;\tau,\sigma)}{\Gamma(\pm 2t;\tau,\sigma)} dt = \frac{2\prod_{i\neq j}\Gamma(s_i+s_j;\tau,\sigma)}{(\tau;\tau)(\sigma;\sigma)}.$$

Evaluating the integrals (III)

1. Split integrand into (a)-symmetric parts under $t \leftrightarrow -t$:

$$\widetilde{I}(\lambda;\tau,\eta) = e^{-12\pi i\eta}\int_\gamma \widetilde{J}_1(t;\tau,\eta)\widetilde{J}_2(t;\tau,\eta)dt$$

   with

$$\widetilde{J}_1(t;\tau,\eta) := \Gamma(\pm t - 2\eta;\tau,8\eta)\theta_0(t+8\eta;8\eta)$$
$$\widetilde{J}_2(t;\lambda,\tau) := e^{-3\pi i\lambda}\theta_0(t+\lambda;\tau)\theta_0(2t+6\tau-4\lambda+1/2;8\tau).$$

2. Since γ is invariant under $t \leftrightarrow -t$, can average. Get:

$$\frac{1}{2}\int_\gamma \widetilde{J}_1(t;\tau,\eta)\Big(\widetilde{J}_2(t;\lambda,\tau) - \widetilde{J}_2(t;-\lambda,\tau) + \widetilde{J}_2(-t;\lambda,\tau) - \widetilde{J}_2(-t;-\lambda,\tau)\Big)dt.$$

Evaluating the integrals (IV)

3. Simplify resulting sum using periodicity properties:

$$
\begin{aligned}
&C \cdot [\widetilde{J}_2(t;\lambda,\tau) - \widetilde{J}_2(t;-\lambda,\tau) + \widetilde{J}_2(-t;\lambda,\tau) - \widetilde{J}_2(-t;-\lambda,\tau)] \\
&= \frac{\theta_0(2\lambda+1/2;2\tau)}{\theta_0(\tau+1/2;2\tau)} e^{-2\pi i t}\theta_0(t+\tau-1/2;2\tau)\theta_0(t+1/2;2\tau)^2 \\
&\quad - \frac{\theta_0(\lambda+1/2;\tau)^2}{\theta_0(\tau;2\tau)} e^{-2\pi i t}\theta_0(t+\tau+1/2;2\tau)\theta_0(t;2\tau)^2.
\end{aligned}
$$

4. Evaluate λ-indep integrals via elliptic beta integral with modular parameters $(2\tau, 8\eta)$:

$$
\int_\gamma \Gamma(\pm t - 2\eta;\tau,8\eta)\theta_0(t+8\eta;8\eta) \\
e^{-4\pi i t}\theta_0(t+\tau-1/2;2\tau)\theta_0(t+1/2;2\tau)^2 dt.
$$

Summary

This talk:

1. Formulate affine analogues of Macdonald denominator and evaluation conjectures (with new factors in affine setting).
2. In certain cases, link affine Macdonald polynomials to certain theta hypergeometric integrals.
3. Prove affine Macdonald conjectures in these cases by hypergeometric integral evaluations.

References:

- Y. S., Traces of intertwiners for quantum affine $\mathfrak{sl}_2$ and Felder-Varchenko functions. *Commun. Math. Phys.* 347 (2016), 573-653. arXiv:1508.03918.
- E. Rains, Y. S., and A. Varchenko, Affine Macdonald conjectures and special values of Felder-Varchenko functions, submitted, 2016. arXiv:1610.01917

Limiting cases of the conjectures (I)

4-D parameter space for affine Macdonalds: q, $t = q^m$, k, ω

1. **Trig** ($q^{-2\omega} \to 0$): Indep of k, get usual Macdonald theory
2. **Classical:** ($q = e^{\varepsilon}$, $\lambda = \varepsilon^{-1}\Lambda$, $\omega = \varepsilon^{-1}\Omega$, $\varepsilon \to 0$)
   - Get affine Jack polynomials studied by Etingof-Kirillov Jr.
   - Denominator conjecture behaves well and was shown
   - Evaluation conjecture is a complicated asymptotic
3. **Critical:** ($\kappa = k + mn \to 0$)
   - Expect elliptic Macdonald-Ruijsenaars system
   - For $n = 2$, $m = 2$, and the un-symmetrized case, limit of FV functions is Bethe ansatz eigenfunction of q-Lamé system found by Felder-Varchenko, i.e. eigenfunction of

$$\frac{\theta_0(\lambda - 1; \tau)}{\theta_0(\lambda; \tau)} T_{\lambda,1} + \frac{\theta_0(\lambda + 1; \tau)}{\theta_0(\lambda; \tau)} T_{\lambda,-1}.$$

Limiting cases of the conjectures (II)

4. **Affine Hall:** ($q \to 0$ with q^m, q^k, q^ω constant)
   We can write $\Delta_m(q, q^{-2\omega}) = \Delta(q, q^m, q^{-\omega})$ for

$$\Delta(q,t,p) := \frac{(p^2q^2;p^2,q^2)(p^2t^{2n};p^2,q^{2n})}{(p^2t^2;p^2q^2)(p^2q^{2n};p^2,q^{2n})}$$

   so that

$$\lim_{q\to 0} \Delta(q,t,p) = \frac{(p^2t^{2n};p^2)}{(p^2t^2;p^2)} =: \Delta^{\mathrm{Mac}}(t,p).$$

   Braverman-Kazhdan-Patnaik found correction $\Delta^{\mathrm{Mac}}(t,p)$ in p-adic loop groups and affine Hall-Littlewood polynomials.

Proposition (Macdonald 2003)

If W_{aff} is the affine Weyl group for $\widehat{\mathfrak{sl}}_n$, we have

$$\Delta^{\mathrm{Mac}}(t,p) = \frac{1}{W_{\mathrm{aff}}(t^2)} \sum_{w \in W_{\mathrm{aff}}} w \cdot \left(\prod_{\alpha > 0} \frac{(1 - t^{2(\alpha,\lambda)+2} p^{2(\alpha,d)})^{\mathrm{mult}(\alpha)}}{(1 - t^{2(\alpha,\lambda)} p^{2(\alpha,d)})^{\mathrm{mult}(\alpha)}} \right)$$

with $W_{\mathrm{aff}}(t^2) = \sum_{w \in W_{\mathrm{aff}}} t^{2\ell(w)}$.