A determinant formula associated with the elliptic hypergeometric integrals of type BC_n

Masahiko ITO (Tokyo Denki University)

(based on joint work with Masatoshi Noumi)

2017 March 23

Workshop on Elliptic Hypergeometric Functions in Combinatorics, Integrable Systems and Physics (ESI, Vienna)

Background BC_n-type Jackson integrals (q-series)

Fix $q \in \mathbb{C}^*$ as $|q| < 1$. For $\xi = (\xi_1, \ldots, \xi_n) \in (\mathbb{C}^*)^n$ and $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$, we set

$$\xi q^\nu := (\xi_1 q^{\nu_1}, \ldots, \xi_n q^{\nu_n}) \in (\mathbb{C}^*)^n.$$

For a function $\varphi(z) = \varphi(z_1, \ldots, z_n)$ on $(\mathbb{C}^*)^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in (\mathbb{C}^*)^n$, we define

$$\langle \varphi, \xi \rangle = \int_0^{\xi\infty} \varphi(z)\Phi(z)\Delta(z)\frac{d_q z_1}{z_1}\cdots\frac{d_q z_n}{z_n}$$
$$:= (1-q)^n \sum_{\nu\in\mathbb{Z}^n} \varphi(\xi q^\nu)\Phi(\xi q^\nu)\Delta(\xi q^\nu),$$

where $q^{\alpha_m} = a_m, \quad q^\tau = t$ and

$$\Phi(z) := \prod_{i=1}^{n}\prod_{m=1}^{2s+2} z_i^{1/2-\alpha_m}\frac{(qa_m^{-1}z_i;q)_\infty}{(a_m z_i;q)_\infty}\prod_{1\le j<k\le n} z_j^{1-2\tau}\frac{(qt^{-1}z_j z_k^{\pm 1};q)_\infty}{(tz_j z_k^{\pm 1};q)_\infty},$$

$$\Delta(z) := \prod_{i=1}^{n}\frac{1-z_i^2}{z_i}\prod_{1\le j<k\le n}\frac{(1-z_j/z_k)(1-z_j z_k)}{z_j}.$$

We call $\langle \varphi, \xi \rangle$ the BC_n-type Jackson integrals if it converges. $\Phi(z)$ has the parameters t and a_i $(1 \le i \le 2s+2, \; s = 1, 2, \ldots)$.

We denote by

$$W_n = \{\pm 1\}^n \rtimes \mathfrak{S}_n$$

the Weyl group of type C_n acting on $(\mathbb{C}^*)^n$ through permutations and inversions of the coordinates z_i $(i = 1, \ldots, n)$.

For the W_n-symmetric holomorphic function $\varphi(z)$ on $(\mathbb{C}^*)^n$, we define

$$\langle\!\langle \varphi, z \rangle\!\rangle := \langle \varphi, z \rangle / \Theta(z),$$

where $$\Theta(z) := \prod_{i=1}^{n} \frac{z_i^s \theta(z_i^2; q)}{\prod_{m=1}^{2s+2} z_i^{\alpha_m} \theta(a_m z_i; q)} \prod_{1 \le j < k \le n} \frac{\theta(z_j z_k^{\pm 1}; q)}{z_j^{2\tau} \theta(t z_j z_k^{\pm 1}; q)}.$$

Then $\langle\!\langle \varphi, z \rangle\!\rangle$ is also W_n-symmetric holomorphic function of $z \in (\mathbb{C}^*)^n$. We call this function the regularization of $\langle \varphi, z \rangle$.

Remark ($s = 1$ case) [van Diejen, Publ. RIMS 33 (1997)]

$$\langle\!\langle 1, z \rangle\!\rangle = \prod_{k=1}^{n} (1-q) \frac{(q;q)_\infty (qt^{-k}; q)_\infty}{(qt^{-1}; q)_\infty} \frac{\prod_{1 \le i < j \le 4} (qt^{-(n-k)} a_i^{-1} a_j^{-1}; q)_\infty}{(qt^{-(n+k-2)} a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1}; q)_\infty},$$

which is equivalent to the q-Macdonald–Morris identity of type $(C_n^\vee, C_n)$ studied by Gustafson (1990).

Set

$$Z_{s,n} = \{\mu = (\mu_1, \mu_2, \dots, \mu_s) \in \mathbb{N}^s\,;\, \mu_1 + \mu_2 + \cdots + \mu_s = n\},$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$. Then $|Z_{s,n}| = \binom{n+s-1}{n}$.

For $x = (x_1, x_2, \dots, x_s) \in (\mathbb{C}^*)^s$ and $\mu \in Z_{s,n}$ we set

$$(x)_{t,\mu} = (\underbrace{x_1, x_1t, \dots, x_1t^{\mu_1-1}}_{\mu_1}, \underbrace{x_2, x_2t, \dots, x_2t^{\mu_2-1}}_{\mu_2}, \dots, \underbrace{x_s, x_st, \dots, x_st^{\mu_s-1}}_{\mu_s})$$

$$\in (\mathbb{C}^*)^n.$$

From the viewpoint of the (Jackson) integral representation of $\langle\langle\varphi, z\rangle\rangle$, it is known that $\langle\langle\varphi, z\rangle\rangle$ satisfies a q-difference system of the rank $\binom{n+s-1}{n}$, and the q-difference system is independent of the choice of cycles z. [Aomoto−I, Adv. Math. 221 (2009)]. The set $\{\langle\langle\varphi, (x)_{t,\mu}\rangle\rangle \mid \mu \in Z_{s,n}\}$ is regarded as a basis of the solution space of the q-difference system. It means that $\langle\langle\varphi, z\rangle\rangle$ is expressed as a linear combination in terms of $\langle\langle\varphi, (x)_{t,\mu}\rangle\rangle$ $(\mu \in Z_{s,n})$. We call this the connection formula between $\langle\langle\varphi, z\rangle\rangle$ and $\langle\langle\varphi, (x)_{t,\mu}\rangle\rangle$ $(\mu \in Z_{s,n})$.

Theorem [I–Noumi, Adv. Math. 299 (2016)]. If $\varphi(z)$ is W_n-symmetric holomorphic function on $(\mathbb{C}^*)^n$, then $\langle\!\langle \varphi, z \rangle\!\rangle$ is expanded as

$$\langle\!\langle \varphi, z \rangle\!\rangle = \sum_{\lambda \in Z_{s,n}} c_\lambda \, \langle\!\langle \varphi, (x)_{t,\lambda} \rangle\!\rangle,$$

where
$$c_\lambda = \sum_{\substack{K_1 \sqcup \cdots \sqcup K_s \\ = \{1,2,\ldots,n\}}} \prod_{i=1}^{s} \prod_{k \in K_i} \prod_{\substack{1 \le j \le s \\ j \ne i}} \frac{\theta(x_j t^{\lambda_j^{(k-1)}} z_k^{\pm 1}; q)}{\theta(x_j t^{\lambda_j^{(k-1)}} (x_i t^{\lambda_i^{(k-1)}})^{\pm 1}; q)}.$$

Here $\lambda_i^{(k)} = |K_i \cap \{1, 2, \ldots, k\}|$ and the summation is taken over all index sets K_i $(i = 1, 2, \ldots, s)$ satisfying $|K_i| = \lambda_i$ and $K_1 \sqcup \cdots \sqcup K_s = \{1, 2, \ldots, n\}$.

Remark The connection coefficients c_λ as functions of z are characterized by the elliptic Lagrange interpolation functions of type BC_n, which will be explained later.

Remark ($n = 1$ case). When $n = 1$, the connection formula is written simply as

$$\langle\!\langle \varphi, z \rangle\!\rangle = \sum_{i=1}^{s} \langle\!\langle \varphi, x_i \rangle\!\rangle \prod_{\substack{1 \le j \le s \\ j \ne i}} \frac{\theta(x_j z^{\pm 1}; q)}{\theta(x_j x_i^{\pm 1}; q)}.$$

In particular, if $\varphi(z) = 1$, then the above formula coincides with Slater's very-well-poised ${}_{2r}\psi_{2r}$ hypergeometric transformation formula. [I–Sanada, Ramanujan J. 17 (2008)].

Slater's very-well-poised ${}_{2r}\psi_{2r}$ transformation (1950)

If $|a^{r-1}q^{r-2}/b_3\cdots b_{2r}|<1$, then, Gasper & Rahman, BHS 2nd edn, p.143, (5.5.2)

$$
{}_{2r}\psi_{2r}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b_3, \ldots, b_{2r} \\ \sqrt{a}, \ \ -\sqrt{a}, \ \frac{aq}{b_3}, \ldots, \frac{aq}{b_{2r}} \end{matrix}; q, \frac{a^{r-1}q^{r-2}}{b_3\ldots b_{2r}}\right]
$$

$$
= \frac{\left(a_4, \ldots, a_r, \frac{q}{a_4}, \ldots, \frac{q}{a_r}, \frac{a_4}{a}, \ldots, \frac{a_r}{a}, \frac{aq}{a_4}, \ldots, \frac{aq}{a_r}; q\right)_\infty}{\left(\frac{q}{b_3}, \ldots, \frac{q}{b_{2r}}, \frac{aq}{b_3}, \ldots, \frac{aq}{b_{2r}}, \frac{a_4}{a_3}, \ldots, \frac{a_r}{a_3}, \frac{a_3q}{a_4}, \ldots, \frac{a_3q}{a_r}; q\right)_\infty}
$$

$$
\times \frac{\left(\frac{a_3q}{b_3}, \ldots, \frac{a_3q}{b_{2r}}, \frac{aq}{a_3b_3}, \ldots, \frac{aq}{a_3b_{2r}}, aq, \frac{q}{a}; q\right)_\infty}{\left(\frac{a_3a_4}{a}, \ldots, \frac{a_3a_r}{a}, \frac{aq}{a_3a_4}, \ldots, \frac{aq}{a_3a_r}, \frac{a_3^2q}{a}, \frac{aq}{a_3^2}; q\right)_\infty}
$$

$$
\times {}_{2r}\psi_{2r}\left[\begin{matrix} \frac{qa_3}{\sqrt{a}}, -\frac{qa_3}{\sqrt{a}}, \frac{a_3b_3}{a}, \ldots, \frac{a_3b_{2r}}{a} \\ \frac{a_3}{\sqrt{a}}, -\frac{a_3}{\sqrt{a}}, \frac{a_3q}{b_3}, \ldots, \frac{a_3q}{b_{2r}} \end{matrix}; q, \frac{a^{r-1}q^{r-2}}{b_3\ldots b_{2r}}\right]
$$

$$
+ \text{idem}(a_3; a_4, \ldots, a_r).
$$

The symbol "idem$(a_3; a_4, \ldots, a_r)$" after the expression above stands for the sum of the $r-3$ expressions obtained from the preceding expression by interchanging a_3 with each a_i $(4\le i\le r)$.

Set

$$B_{s,n}=\{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{Z}^n\,;\,s-1\ge\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n\ge 0\}$$

then $|B_{s,n}|=\binom{n+s-1}{n}$. We define the Schur functions of type C_n as

$$\chi_\lambda(z):=\det\left(z_i^{\lambda_j+n-j+1}-z_i^{-\lambda_j-(n-j+1)}\right)_{1\le i,j\le n}\Big/\Delta(z).$$

Theorem (determinant formula) [Aomoto−I (2009), I−Noumi (2016)]

For generic $x\in(\mathbb{C}^*)^s$, we have

$$\det\left(\langle\!\langle\chi_\lambda,(x)_{t,\mu}\rangle\!\rangle\right)_{\substack{\lambda\in B_{s,n}\\ \mu\in Z_{s,n}}}=\left((1-q)^n(q;q)_\infty^n\right)^{\binom{s+n-1}{n}}$$

$$\times\prod_{i=1}^{n}\left[\left(\frac{(qt^{-(n-i+1)};q)_\infty}{(qt^{-1};q)_\infty}\right)^s\frac{\prod_{1\le k<l\le 2s+2}(qt^{-(n-i)}a_k^{-1}a_l^{-1};q)_\infty}{(qt^{-(n+i-2)}a_1^{-1}a_2^{-1}\cdots a_{2s+2}^{-1};q)_\infty}\right]^{\binom{s+i-2}{i-1}}$$

$$\times\prod_{i=1}^{n}\left[\prod_{j=0}^{n-i}\prod_{1\le k<l\le s}e(t^jx_k,t^{n-i-j}x_l;q)\right]^{\binom{s+i-3}{i-1}},$$

where $e(u,v;q)=u^{-1}\theta(u/v;q)\theta(uv;q)$.

The aim of this talk is to present an elliptic analog of this formula.

BC_n elliptic hypergeometric integrals

Fix $p, q \in \mathbb{C}^*$ as $|p| < 1$ and $|q| < 1$. We define the W_n-symmetric meromorphic function $\Phi(z)$ as follows:

$$\Phi(z) = \prod_{i=1}^{n} \frac{\prod_{k=1}^{2r+4} \Gamma(a_k z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \le i < j \le n} \frac{\Gamma(t z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)},$$

where $\Gamma(u; p, q)$ $(u \in \mathbb{C}^*)$ is the Ruijsenaars elliptic gamma function. $\Phi(z)$ has the parameters t and a_i $(1 \le i \le 2r+4,\ r = 1, 2, \ldots)$.

For holomorphic functions $f(z)$ and $g(z)$ on $(\mathbb{C}^*)^n$ we define

$$\langle f, g \rangle = \int_{\mathbb{T}^n} f(z)\, g(z)\, \Phi(z)\, \omega_n(z), \qquad \omega_n(z) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n},$$

where $\mathbb{T}^n = \{z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n \,|\, |z_i| = 1 \quad (i = 1, \ldots, n)\}$.

We call the integral $\langle f, g \rangle$ the BC_n-type elliptic hypergeometric integrals.

Theorem of this talk Fix $t \in \mathbb{C}^*$ and $a_i \in \mathbb{C}^*$ $(1 \le i \le 2r+4)$ as $|t| < 1$ and $|a_i| < 1$. Under the condition $a_1 \cdots a_{2r+4} t^{2n-2} = pq$, we have the determinant formula as

$$\det\Big(\langle E_\mu(x;z;p), E_\nu(y;z;q)\rangle\Big)_{\mu,\nu\in Z_{r,n}} = \left(\frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n}\right)^{\binom{n+r-1}{n}}$$

$$\times \prod_{i=1}^{n} \frac{\left[\left(\dfrac{\Gamma(t^{n-i+1};p,q)}{\Gamma(t;p,q)}\right)^r \displaystyle\prod_{1\le k<l\le 2r+4} \Gamma(t^{n-i}a_k a_l;p,q)\right]^{\binom{i+r-2}{i-1}}}{\left[\displaystyle\prod_{j=0}^{n-i} \prod_{1\le k<l\le r} e(t^j x_k, t^{n-i-j} x_l; p)\, e(t^j y_k, t^{n-i-j} y_l; q)\right]^{\binom{i+r-3}{i-1}}},$$

where $e(u,v;p) = u^{-1}\theta(u/v;p)\theta(uv;p)$.

Remark. If $r = 1$, then under the condition $a_1 \cdots a_6 t^{2n-2} = pq$,

$$\langle 1, 1\rangle = \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \prod_{i=1}^{n} \left(\frac{\Gamma(t^{n-i+1};p,q)}{\Gamma(t;p,q)} \prod_{1\le k<l\le 6} \Gamma(t^{n-i}a_k a_l;p,q)\right),$$

which is the BC_n elliptic Selberg integral established by van Diejen-Spiridonov, Spiridonov, Rains.

Definition of the functions $E_\mu(x;z;p)$

We denote by T_{p,z_i} the p-shift operator with respect to z_i:

$$T_{p,z_i}f(z_1,\ldots,z_n) = f(z_1,\ldots,pz_i,\ldots,z_n) \quad (i=1,\ldots,n).$$

For each $r=1,2,\ldots$ we introduce the $\mathbb{C}$-vector space

$$\mathcal{H}^{(p)}_{r-1,n} = \{f(z)\in\mathcal{O}((\mathbb{C}^*)^n)^{W_n} \,|\, T_{p,z_i}f(z) = f(z)(pz_i^2)^{-r+1} \ (i=1,\ldots,n)\}$$

of all W_n-symmetric holomorphic functions on $(\mathbb{C}^*)^n$ with quasi-periodicity *of degree* $r-1$. In [Rains, Ann. of Math. (2) 171 (2010)], $\mathcal{H}^{(p)}_{r-1,n}$ is called the space of the BC_n-symmetric theta functions of degree $r-1$. It is known that the vector space $\mathcal{H}^{(p)}_{r-1,n}$ has the dimension $\binom{n+r-1}{n}$,

$$\dim_{\mathbb{C}}\mathcal{H}^{(p)}_{r-1,n} = |Z_{r,n}| = \binom{n+r-1}{n}.$$

Recall $\quad Z_{r,n} = \{\mu = (\mu_1, \mu_2, \dots, \mu_r) \in \mathbb{N}^r ; \ \mu_1 + \mu_2 + \cdots + \mu_r = n\}$.

For $x = (x_1, x_2, \dots, x_r) \in (\mathbb{C}^*)^r$ and $\mu \in Z_{r,n}$ we set

$$(x)_{t,\mu} = (\underbrace{x_1, x_1 t, \dots, x_1 t^{\mu_1 - 1}}_{\mu_1}, \dots, \underbrace{x_r, x_r t, \dots, x_r t^{\mu_r - 1}}_{\mu_r}) \in (\mathbb{C}^*)^n.$$

Theorem ([I – Noumi, Adv. Math. 299 (2016)])

Suppose $x \in (\mathbb{C}^*)^r$ is generic. For the set $\{(x)_{t,\mu} \,|\, \mu \in Z_{r,n}\}$ of the reference points, there exists a unique $\mathbb{C}$-basis $\{E_\mu(x; z; p) \,|\, \mu \in Z_{r,n}\}$ of $\mathcal{H}^{(p)}_{r-1,n}$ satisfying the interpolation condition

$$E_\mu(x; (x)_{t,\nu}; p) = \delta_{\mu,\nu} \qquad (\mu, \nu \in Z_{r,n})$$

where $\delta_{\mu,\nu}$ is the Kronecker delta.

we call $E_\mu(x; z; p)$ $(\mu \in Z_{r,n})$ the elliptic Lagrange interpolation functions of type BC_n and call $\{E_\mu(x; z; p) \,|\, \mu \in Z_{r,n}\}$ the interpolation basis of $\mathcal{H}^{(p)}_{r-1,n}$, with respect to $x \in (\mathbb{C}^*)^r$.

Bilinear form associated with the elliptic hypergeometric integral

With respect to the two bases p, q, for the two vector spaces $\mathcal{H}^{(p)}_{r-1,n}$, $\mathcal{H}^{(q)}_{r-1,n}$ we define the $\mathbb{C}$-bilinear form

$$\langle\, ,\, \rangle : \mathcal{H}^{(p)}_{r-1,n} \times \mathcal{H}^{(q)}_{r-1,n} \to \mathbb{C}$$

by $$\langle f, g\rangle = \int_{\mathbb{T}^n} f(z)g(z)\Phi(z)\omega_n(z) \qquad (f \in \mathcal{H}^{(p)}_{r-1,n}, \quad g \in \mathcal{H}^{(q)}_{r-1,n}).$$

Fixing generic $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_r)$, we take the interpolation bases for these two vector spaces with respect to x and y respectively:

$$\mathcal{H}^{(p)}_{r-1,n} = \bigoplus_{\mu \in Z_{r,n}} \mathbb{C}\, E_\mu(x; z; p), \quad \mathcal{H}^{(q)}_{r-1,n} = \bigoplus_{\mu \in Z_{r,n}} \mathbb{C}\, E_\mu(y; z; q).$$

For each pair $(\mu, \nu) \in Z_{r,n} \times Z_{r,n}$, we introduce the elliptic hypergeometric integral

$$\langle E_\mu(x; z; p), E_\nu(y; z; q)\rangle = \int_{\mathbb{T}^n} E_\mu(x; z; p)\, E_\nu(y; z; q)\Phi(z)\omega_n(z) \quad (\mu, \nu \in Z_{r,n}),$$

and consider that $\binom{n+r-1}{n} \times \binom{n+r-1}{n}$ matrix $\Big(\langle E_\mu(x; z; p), E_\nu(y; z; q)\rangle\Big)_{\mu,\nu \in Z_{r,n}}$, which is the matrix representation of the bilinear form $\langle\, ,\, \rangle$ with respect to the interpolation bases.

Remark. (Proof for the connection formula of the Jackson integral case)
Since we see $\langle\!\langle \varphi, z \rangle\!\rangle \in \mathcal{H}^{(q)}_{s-1,n}$, we have $\langle\!\langle \varphi, z \rangle\!\rangle = \sum_{\mu \in Z_{s,n}} d_\mu E_\mu(x; z; q)$. From the interpolation property $E_\mu(x; (x)_{t,\nu}; p) = \delta_{\mu,\nu}$ we have $d_\mu = \langle\!\langle \varphi, (x)_{t,\mu} \rangle\!\rangle$. □

In the previous setting, the connection coefficients c_λ as functions of z coincide with $E_\lambda(x; z; q)$:

$$c_\lambda = E_\lambda(x; z; q).$$

Explicit expression of $E_\mu(x; z; p)$ Set $e(u, v; p) := u^{-1}\theta(uv; p)\theta(uv^{-1}; p)$. When $n = 1$, the interpolation functions are parametrized by the canonical basis $\epsilon_1, \ldots, \epsilon_r$ of $\mathbb{N}^r$, and given explictly as

$$E_{\epsilon_k}(x; u; p) = \prod_{\substack{1 \le l \le r \\ l \ne k}} \frac{e(u, x_l; p)}{e(x_k, x_l; p)} = \prod_{\substack{1 \le l \le r \\ l \ne k}} \frac{\theta(x_l/u; p)\theta(x_l u; p)}{\theta(x_l/x_k; p)\theta(x_l x_k; p)}.$$

Recursion formula Suppose $n = k + l$. For $z = (z_1, \dots, z_n) \in (\mathbb{C}^*)^n$ we set $z' = (z_1, \dots, z_k) \in (\mathbb{C}^*)^k$ and $z'' = (z_{k+1}, \dots, z_n) \in (\mathbb{C}^*)^l$, so that $z = (z', z'')$.

Recursion formula $E_\lambda(x; z; p) = \displaystyle\sum_{\substack{\mu \in Z_{r,k},\ \nu \in Z_{r,l} \\ \mu+\nu=\lambda}} E_\mu(x; z'; p) E_\nu(xt^\mu; z''; p),$

where $xt^\mu = (x_1 t^{\mu_1}, \dots, x_r t^{\mu_r})$ for $x = (x_1, \dots, x_r) \in (\mathbb{C}^*)^r$.

Explicit expression Repeated use of the above recursion formula, we have

$$E_\lambda(x; z; p)$$

$$= \sum_{\substack{(i_1,\dots,i_n)\in\{1,\dots,r\}^n \\ \epsilon_{i_1}+\cdots+\epsilon_{i_n}=\lambda}} E_{\epsilon_{i_1}}(x; z_1; p) E_{\epsilon_{i_2}}(xt^{\epsilon_{i_1}}; z_2; p) E_{\epsilon_{i_3}}(xt^{\epsilon_{i_1}+\epsilon_{i_2}}; z_3; p) \cdots$$
$$\cdots E_{\epsilon_{i_n}}(xt^{\epsilon_{i_1}+\cdots+\epsilon_{i_{n-1}}}; z_n; p),$$

which reduces to $E_{\epsilon_i}(x; u; p) = \displaystyle\prod_{\substack{1\le j\le r \\ j\ne i}} \frac{e(u, x_j; p)}{e(x_i, x_j; p)}$ of the case $n = 1$.

The proof of the recursion formula is due to the following:

Dual Cauchy formula The interpolation functions $E_\lambda(x;z;p)$ satisfy

Dual Cauchy Kernel $$\prod_{i=1}^{n}\prod_{j=1}^{r-1} e(z_i, y_j; p) = \sum_{\lambda \in Z_{r,n}} E_\lambda(x;z;p) F_\lambda(x;y;p),$$

where

$$F_\mu(x;y) = \prod_{i=1}^{r}\prod_{j=1}^{r-1} e(x_i, y_j; p)_{t,\mu_i} \quad \text{for} \quad x \in (\mathbb{C}^*)^r,\ y \in (\mathbb{C}^*)^{r-1},$$

and

$$e(u,v;p)_{t,i} = e(u,v;p)e(ut,v;p)\cdots e(ut^{i-1},v;p) \quad (i = 0,1,2,\ldots).$$

The functions $F_\mu(x;y)$ satisfy the property $\boxed{F_\mu(x;y;p)F_\nu(xt^\mu;y;p) = F_{\mu+\nu}(x;y;p)}$, which is used effectively for the proof of the recursion formula.