A generalization of the q-Painlevé VI equation from a viewpoint of a basic hypergeometric solution

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Abstract

The aim of this poster is to give a generalization of Jimbo-Sakai's q-Painlevé VI equation ([1]) from a viewpoint of Heine's basic hypergeometric function.

$$\begin{array}{ccc} \boxed{q\text{-}P_{\rm VI}} & \longrightarrow & \boxed{\text{Generalized } q\text{-}P_{\rm VI}} \\ \text{(sol'n)} \Downarrow & & \Downarrow \text{(sol'n)} \\ \boxed{\text{Heine's } {}_2\phi_1} & \longrightarrow & \boxed{\text{Heine's } {}_{n+1}\phi_n} \end{array}$$

Lax Formulation

In the following we use the notations

$$\overline{x(t)} = x(qt), \quad \underline{x(t)} = x(q^{-1}t) \quad (t, q \in \mathbb{C}; |q| < 1).$$

We consider a system of linear q-difference equations

$$\Psi(q^{-1}z, t) = M(z, t)\Psi(z, t), \quad \Psi(z, q^{-1}t) = B(z, t)\Psi(z, t), \tag{1}$$

with $(2n+2) \times (2n+2)$ matrices

$$M(z,t) = \begin{bmatrix} a_1 & y_0 - y_1 & -1 & & & & \\ & b_1 & x_1 - x_2 & -1 & & & \\ & & & \ddots & & & \\ & & & & y_{n-1} - y_n & -1 & \\ & & & & b_n & x_n - x_{n+1} & -1 \\ -tz & & & & & a_{n+1} & y_n - y_{n+1} \\ (x_0 - x_1)z & -z & & & & & b_{n+1} \end{bmatrix},$$

$$B(z,t) = \begin{bmatrix} \frac{a_1}{1+\underline{x_1}y_0} & -y_1 & -1 & & & & \\ & 1 + \underline{x_1}y_1 & \underline{x_1} & 0 & & & \\ & & & \ddots & & & \\ & & & & -y_n & -1 & \\ & & & & 1 + \underline{x_n}y_n & \underline{x_n} & 0 \\ -tz & & & & & \frac{a_{n+1}}{1+\underline{x_{n+1}}y_n} & -y_{n+1} \\ t\underline{x_{n+1}}z & 0 & & & & & 1 + \underline{x_{n+1}}y_{n+1} \end{bmatrix}.$$

We assume that

$$\prod_{i=1}^{n+1} a_i \frac{1 + \underline{x_i}y_i}{1 + \underline{x_i}y_{i-1}} = q^{-n/2}, \quad b_0 = qb_{n+1}, \quad x_0 = tx_{n+1}, \quad y_0 = \frac{q}{t}y_{n+1},$$

Theorem 1 ([4]) *The compatibility condition of system (1) implies a system of non-linear q-difference equations*

$$\begin{cases} x_{i-1} - x_i = \dfrac{b_{i-1}\underline{x_{i-1}}}{1 + \underline{x_{i-1}}y_{i-1}} - \dfrac{a_i\underline{x_i}}{1 + \underline{x_i}y_{i-1}} \\ \underline{y_{i-1}} - \underline{y_i} = \dfrac{a_i y_{i-1}}{1 + \underline{x_i}y_{i-1}} - \dfrac{b_i y_i}{1 + \underline{x_i}y_i} \end{cases} \quad (i = 1, \ldots, n+1). \tag{2}$$

We define birational transformations $r_0, \ldots, r_{2n+1}, \pi$ by

$$r_{2j-2}(a_j) = b_{j-1}, \quad r_{2j-2}(b_{j-1}) = a_j, \quad r_{2j-2}(x_{j-1}) = x_{j-1}, \quad r_{2j-2}(y_{j-1}) = y_{j-1} - \frac{b_{j-1} - a_j}{x_{j-1} - x_j},$$
$$r_{2j-2}(a_i) = a_i, \quad r_{2j-2}(b_{i-1}) = b_{i-1}, \quad r_{2j-2}(x_{i-1}) = x_{i-1}, \quad r_{2j-2}(y_{i-1}) = y_{i-1} \quad (i \neq j),$$

for $j = 1, \ldots, n+1$,

$$r_{2j-1}(a_j) = b_j \quad r_{2j-1}(b_j) = a_j, \quad r_{2j-1}(x_j) = x_j - \frac{a_j - b_j}{y_{j-1} - y_j}, \quad r_{2j-1}(y_j) = y_j,$$
$$r_{2j-1}(a_i) = a_i, \quad r_{2j-1}(b_i) = b_i, \quad r_{2j-1}(x_i) = x_i, \quad r_{2j-1}(y_i) = y_i \quad (i \neq j).$$

for $j = 1, \ldots, n+1$, and

$$\pi(a_i) = q^{-\rho_1}b_i, \quad \pi(b_i) = q^{-\rho_1}a_{i+1}, \quad \pi(x_i) = q^{-2\rho_1}t^{\rho_1}\underline{y_i}, \quad \pi(y_i) = q^{\rho_1}t^{-\rho_1}\underline{x_{i+1}} \quad (i \neq n+1),$$
$$\pi(a_{n+1}) = q^{-\rho_1}b_{n+1}, \quad \pi(b_{n+1}) = q^{-\rho_1-1}a_1, \quad \pi(x_{n+1}) = q^{-2\rho_1}t^{\rho_1}\underline{y_{n+1}}, \quad \pi(y_{n+1}) = q^{\rho_1+1}t^{-\rho_1-1}\underline{x_1},$$
$$\pi(\rho_1) = -\rho_1 - \frac{1}{n+1}, \quad \pi(t) = \frac{q^2}{t},$$

where $a_1b_1 \ldots a_{n+1}b_{n+1} = q^{(n+1)\rho_1-n}$.

Theorem 2 ([4, 5]) *System (2) is invariant under actions of the transformations $r_0, \ldots, r_{2n+1}, \pi$. Furthermore a group of symmetries $\langle r_0, \ldots, r_{2n+1}, \pi \rangle$ is isomorphic to the extended affine Weyl group of type $A^{(1)}_{2n+1}$.*

We consider a system of linear q-difference equations

$$\overline{\mathbf{x}} = \left(A_0 + \frac{A_1}{1 - qt}\right)\mathbf{x}, \tag{3}$$

with $(n+1) \times (n+1)$ matrices

$$A_0 = \begin{bmatrix} b_1 & b_2 - a_2 & b_3 - a_3 & \ldots & b_n - a_n & b_{n+1} - a_{n+1} \\ & b_2 & b_3 - a_3 & \ldots & b_n - a_n & b_{n+1} - a_{n+1} \\ & & & \ddots & \vdots & \vdots \\ & & & & b_n - a_n & b_{n+1} - a_{n+1} \\ & O & & & b_n & b_{n+1} - a_{n+1} \\ & & & & & b_{n+1} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} a_1 - b_1 & a_2 - b_2 & \ldots & a_{n+1} - b_{n+1} \end{bmatrix}.$$

This system admits a solution given by

$$\mathbf{x} = t^{-\log_q a_1} \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{n+1} \end{bmatrix}, \quad \varphi_j = \prod_{i=1}^{j-1} \frac{b_i - a_1}{a_{i+1} - a_1} \cdot {}_{n+1}\phi_n \left[\begin{matrix} \frac{a_1}{qb_1}, \ldots, \frac{a_1}{qb_{j-1}}, \frac{a_1}{b_j}, \ldots, \frac{a_1}{b_n}, \frac{a_1}{b_{n+1}} \\ \frac{a_1}{qa_2}, \ldots, \frac{a_1}{qa_j}, \frac{a_1}{a_{j+1}}, \ldots, \frac{a_1}{a_{n+1}} \end{matrix}; q, q\frac{b_1 \ldots b_{n+1}}{a_1 \ldots a_{n+1}}t \right],$$

where

$${}_{n+1}\phi_n \left[\begin{matrix} \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \\ \beta_1, \ldots, \beta_n \end{matrix}; q, t \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1; q)_k \ldots (\alpha_n; q)_k (\alpha_{n+1}; q)_k}{(\beta_1; q)_k \ldots (\beta_n; q)_k (q; q)_k} t^k.$$

Theorem 3 ([4]) *If, in system (2), we assume that*

$$y_i = 0 \quad (i = 1, \ldots, n+1), \quad \prod_{i=1}^{n+1} a_i = q^{-n/2},$$

then a vector of dependent variables $\mathbf{x} = {}^t[x_1, \ldots, x_{n+1}]$ satisfies system (3).

Reformulation

We want to regard system (2) as a generalization of q-$P_{\rm VI}$. However we have the following two problems.

- This system is probably reducible and reduces to a system of $2n$-th order.
- We do not express the backward q-shifts $(\underline{x_i}, \underline{y_i})$ as functions in (x_i, y_i).

Therefore we rewrite system (2) to a more suitable one.

Theorem 4 ([5]) *If, in system (2), we set*

$$f_i = t\frac{x_i - x_{i+1}}{tx_{n+1} - x_1}, \quad g_i = a_{i+1}\frac{\underline{x_{i+1}}(1 + \underline{x_i}y_i)}{\underline{x_i}(1 + \underline{x_{i+1}}y_i)} \quad (i = 1, \ldots, n),$$

then they satisfy a system of q-difference equations

$$f_i\overline{f}_i = qt\frac{F_iF_{i+1}\overline{g}_0(b_i - \overline{g}_i)(\overline{g}_i - a_{i+1})}{F_{n+1}F_1\overline{g}_i(b_0 - \overline{g}_0)(\overline{g}_0 - a_1)}, \quad g_i\overline{g}_i = \frac{F_{i+1}G_i}{F_iG_{i+1}} \quad (i = 1, \ldots, n), \tag{4}$$

where

$$g_0 = \frac{1}{q^{(n-2)/2}t}\prod_{i=1}^{n}\frac{1}{g_i}, \quad F_i = \sum_{j=1}^{i-1} f_j + t\sum_{j=i}^{n} f_j + t,$$
$$G_i = \sum_{j=i}^{n}\prod_{k=i}^{j-1} b_ka_{k+1}\frac{\prod_{l=j+1}^{n} g_l}{\prod_{l=1}^{j-1} g_l}f_j + q^{n/2}t\prod_{k=i}^{n} b_ka_{k+1} + q^nt\sum_{j=1}^{i-1}\frac{b_{n+1}a_1\prod_{k=1}^{n} b_ka_{k+1}}{\prod_{k=j}^{i-1} b_ka_{k+1}}\frac{\prod_{l=j+1}^{n} g_l}{\prod_{l=1}^{j-1} g_l}f_j.$$

In the last we give some remarks.

- System (4) is equivalent to q-$P_{\rm VI}$ in the case of $n = 1$.
- The action of the transformations $r_0, \ldots, r_{2n+1}, \pi$ can be restricted to system (4); see [5].
- System (2) or (4) is essentially equivalent to Sakai's q-Garnier system ([3]), which was shown by Y. Yamada; see [2].

References

[1] M. Jimbo and H. Sakai, *A q-analog of the sixth Painlevé equation*, Let. Math. Phys. **38** (1996) 145–154.

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[4] T. Suzuki, *A q-analogue of the Drinfeld-Sokolov hierarchy of type A and q-Painlevé system*, AMS Contemp. Math. **651** (2015) 25–38.

[5] T. Suzuki, *A reformulation of the generalized q-Painlevé VI system with $W(A^{(1)}_{2n+1})$ symmetry*, J. Integrable Syst. **2** (2017) xyw017.