APPLICATIONS OF THE ELLIPTIC HYPERGEOMETRIC INTEGRALS

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ELLIPTIC HYPERGEOMETRIC INTEGRALS

Definition (V.S., 2003), univariate case: contour integrals

$$I = \int \rho(z)\frac{dz}{z}, \qquad \rho(qz) = h(z;p)\rho(z), \quad h(pz) = h(z),$$

where $h(z)$ is an elliptic function. **Theorem**:

$$h(z) = \prod_{k=1}^{m} \frac{\theta(t_k z;p)}{\theta(w_k z;p)}, \qquad \prod_{k=1}^{m} t_k = \prod_{k=1}^{m} w_k,$$

$$\theta(z;p) = (z;p)_\infty (pz^{-1};p)_\infty, \qquad (z;p)_\infty = \prod_{j=0}^{\infty}(1-zp^j).$$

Since the equation

$$\Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q)$$

is satisfied by

$$\Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k}, \qquad |p|, |q| < 1,$$

$\Rightarrow$

$$I(t,w;p,q) = \int \prod_{k=1}^{m} \frac{\Gamma(t_k z;p,q)}{\Gamma(w_k z;p,q)} \frac{dz}{z}, \qquad \prod_{k=1}^{m} t_k = \prod_{k=1}^{m} w_k.$$

NB: in $d = 4$ QFT $I(t,w;p,q) \simeq$ superconformal indices $\Rightarrow$ $h(pz) = h(z)$ is equivalent to the absence of gauge anomalies.

A QUANTUM MECHANICAL n-BODY PROBLEM

Hamiltonian of the van Diejen (1994) (a generalized elliptic Ruijsenaars) model

$$\mathcal{D} = \sum_{j=1}^{n} \Big(A_j(\underline{z})(T_j - 1) + A_j(\underline{z}^{-1})(T_j^{-1} - 1) \Big),$$

where $T_j f(\ldots, z_j, \ldots) = f(\ldots, q z_j, \ldots)$, and

$$A_j(\underline{z}) = \frac{\prod_{m=1}^{8} \theta(t_m z_j; p)}{\theta(z_j^2, q z_j^2; p)} \prod_{\substack{k=1 \\ \neq j}}^{n} \frac{\theta(t z_j z_k^{\pm 1}; p)}{\theta(z_j z_k^{\pm 1}; p)}$$

with the constraint $t^{2n-2} \prod_{m=1}^{8} t_m = p^2 q^2$.

The standard eigenvalue problem $\mathcal{D}\psi(\underline{z}) = \lambda \psi(\underline{z})$. For $n = 1$:

$$\frac{\prod_{j=1}^{8} \theta(t_j z; p)}{\theta(z^2, q z^2; p)} (f(qz) - f(z)) + \frac{\prod_{j=1}^{8} \theta(t_j z^{-1}; p)}{\theta(z^{-2}, q z^{-2}; p)} (f(q^{-1} z) - f(z)) = \lambda f(z).$$

For a special choice of the parameters and λ one gets the elliptic hypergeometric equation (V.S., 2004) solved by an elliptic analogue of the Euler-Gauss hypergeometric function:

$$V(\underline{t}; p, q) = \int_{\mathbb{T}} \frac{\prod_{j=1}^{8} \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}, \qquad \prod_{k=1}^{8} t_k = p^2 q^2.$$

For general rank n model, introduce the scalar product

$$\langle \varphi, \psi \rangle = \int_{\mathbb{T}^n} \Delta(\underline{z}, \underline{t}) \varphi(\underline{z}) \psi(\underline{z}) \, \frac{dz}{z},$$

such that $\langle \varphi, \mathcal{D}\psi \rangle = \langle \mathcal{D}\varphi, \psi \rangle$. Then the eigenfunction $\psi(\underline{z}) = 1$ (with zero eigenvalue) has the norm

$$\|1\|^2 = \int_{\mathbb{T}^n} \prod_{1 \le j < k \le n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1})}{\Gamma(z_j^{\pm 1} z_k^{\pm 1})} \prod_{j=1}^{n} \frac{\prod_{m=1}^{8} \Gamma(t_m z_j^{\pm 1})}{\Gamma(z_j^{\pm 2})} \frac{dz_j}{z_j},$$

where $t^{2n-2} \prod_{m=1}^{8} t_m = p^2 q^2$. (V.S., 2004)

More on such models $\Rightarrow$ Ruijsenaars, Razamat, ... talks.

BAILEY LEMMA

Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n-1)/2},$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n-3)/2}.$$

The second equalities follow from the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}x^n=(q;q)_\infty\theta(x;q).$$

Bailey (1949): sequences α_n, β_n form a Bailey pair (© G. Andrews), if

$$\beta_n=\sum_{k=0}^{n}\frac{\alpha_k}{(q;q)_{n-k}(aq;q)_{n+k}}.$$

Lemma. Given an above BP of sequences $\Rightarrow$

$$\sum_{n=0}^{\infty}a^n q^{n^2}\beta_n=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}a^n q^{n^2}\alpha_n.$$

Follows from the q-Gauss summation formula.

For $a=1$ and $\alpha_0=1$, $\alpha_n=(-1)^n q^{n(3n-1)/2}(1+q^n)$ one gets $\beta_n=1/(q;q)_n \Rightarrow$ the first RR-identity.

Further extension (G. Andrews, 1984; P. Paule, 1985): infinite chains of Bailey pairs.

Elliptic case: (V.S., 2001; $p=0$, Andrews, 2000)

$\alpha_n(a,k)$, $\beta_n(a,k)$ is an elliptic Bailey pair w.r.t. a and k, if

$$\beta_n(a,k)=\sum_{0\le m\le n} M_{nm}(a,k)\alpha_m(a,k), \text{ or } \beta(a,k)=M(a,k)\alpha(a,k)$$

with

$$M_{nm}(a,k)=\frac{\theta(k/a)_{n-m}\theta(k)_{n+m}}{\theta(q)_{n-m}\theta(aq)_{n+m}}\frac{\theta(aq^{2m};p)}{\theta(a;p)}a^{n-m},$$

where

$$\theta(a)_n=\theta(a;p)\theta(aq;p)\dots\theta(aq^{n-1};p)=\frac{\Gamma(q^n z;p,q)}{\Gamma(z;p,q)}.$$

Denote

$$D_{nm}(a;b,c)=D_m(a;b,c)\delta_{nm},$$
$$D_m(a;b,c)=\frac{\theta(b,c)_m}{\theta(aq/b,aq/c)_m}\left(\frac{aq}{bc}\right)^m.$$

Lemma. Given BPs $\alpha(a,t)$ and $\beta(a,t)$ w.r.t a and t,

$$\alpha'(a,k)=D(a;b,c)\alpha(a,t),$$
$$\beta'(a,k)=D(k;qt/b,qt/c)M(t,k)D(t;b,c)\beta(a,t),$$

where $qat=kbc$, are new BPs w.r.t. a and k.

From $\beta'(a,k) = M(a,k)\alpha'(a,k) \Rightarrow$ the matrix identity

$$M(a,k)D(a;b,c)M(t,a) = D(k;qt/b,qt/c)M(t,k)D(t;b,c)$$

$\Leftrightarrow$ the Frenkel–Turaev sum.

$M_{nm}(k,k) = D_{nm}(bc/q;b,c) = \delta_{nm} \Rightarrow M(a,k)M(k,a) = 1.$

ELLIPTIC FOURIER TRANSFORMATION

Definition of an integral transformation (V.S., 2003)

$$\beta(w,t) = M(t)_{wz}\alpha(z,t)$$

$$= \frac{(p;p)_\infty(q;q)_\infty}{4\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\Gamma(tw^{\pm1}z^{\pm1};p,q)}{\Gamma(t^2,z^{\pm2};p,q)}\alpha(z,t)\frac{dz}{z}.$$

Integral Bailey lemma:

$$\alpha'(w,st) = D(s;u,w)\alpha(w,t), \qquad D(s;u,w)D(s^{-1};u,w) = 1,$$

$$D(s;u,w) := \Gamma(\sqrt{pq}s^{-1}u^{\pm1}w^{\pm1};p,q),$$

$$\beta'(w,st) = D(t^{-1};u,w)M(s)_{wx}D(st;u,x)\beta(x,t).$$

From $\beta'(w,st) = M(st)_{wz}\alpha'(z,st) \Rightarrow$

$$M(s)_{wx}D(st;u,x)M(t)_{xz} = D(t;u,w)M(st)_{wz}D(s;u,z).$$

Braid ($MDM = DMD$) or operator star-triangle relation (STR)

Equivalent to the elliptic beta integral (V.S., 2000)

$$\frac{(p;p)_\infty(q;q)_\infty}{4\pi \mathrm{i}}\int_{\mathbb{T}}\frac{\prod_{j=1}^6\Gamma(t_jz^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{z}=\prod_{1\leq j<k\leq 6}\Gamma(t_jt_k;p,q),$$

where $\prod_{j=1}^6 t_j=pq$, $|t_j|<1$.

Inversion relation $t\to t^{-1}$: (V.S., Warnaar, 2004)

$$M(t^{-1})_{wz}M(t)_{zx}f(x)=f(w).$$

Inversion = sign change (like in the Fourier transformation)

Ell. beta int. $\Rightarrow$ an explicit example of BP $\beta(w,t),\alpha(z,t)$. From $\beta'=M\alpha'\Rightarrow W(E_7)$-symmetry for $V(\underline{t};p,q)$. (V.S., 2003)

Take $\mathbf{t}=(t_1,t_2,t_3,t_4)$ and consider $\mathcal{S}_4$-group generators:

$s_1(\mathbf{t})=(t_2,t_1,t_3,t_4),\ s_2(\mathbf{t})=(t_1,t_3,t_2,t_4),\ s_3(\mathbf{t})=(t_1,t_2,t_4,t_3).$

Define

$$[\mathrm{S}_1(\mathbf{t})f](z_1,z_2):=M(t_1/t_2)_{z_1z}f(z,z_2),$$
$$[\mathrm{S}_2(\mathbf{t})f](z_1,z_2):=D(t_2/t_3;z_1,z_2)f(z_1,z_2),$$
$$[\mathrm{S}_3(\mathbf{t})f](z_1,z_2):=M(t_3/t_4)_{z_2z}f(z_1,z).$$

Take the twisted multiplication rule $\mathrm{S}_j\mathrm{S}_k:=\mathrm{S}_j(s_k(\mathbf{t}))\mathrm{S}_k(\mathbf{t})$

$\Rightarrow$ The Coxeter relations

$$\mathrm{S}_j^2=1,\quad \mathrm{S}_i\mathrm{S}_j=\mathrm{S}_j\mathrm{S}_i\ \text{ for }\ |i-j|>1,\quad \mathrm{S}_j\mathrm{S}_{j+1}\mathrm{S}_j=\mathrm{S}_{j+1}\mathrm{S}_j\mathrm{S}_{j+1}.$$

Ell. beta integral = star-triangle rel. = Coxeter rel.

BAILEY LEMMA ON ROOT SYSTEMS

The A_n-operator ($SU(n+1)$ group) (V.S., Warnaar, 2004)

$$M(t)_{wz}f(z) := \frac{(p;p)_\infty^n (q;q)_\infty^n}{(2\pi\mathrm{i})^n (n+1)!} \times \int_{\mathbb{T}^n} \frac{\prod_{j,k=1}^{n+1}\Gamma(tw_jz_k^{-1})f(z)}{\Gamma(t^{n+1})\prod_{1\leq j<k\leq n+1}\Gamma(z_jz_k^{-1},z_j^{-1}z_k)}\prod_{k=1}^n\frac{dz_k}{z_k},$$

where $\prod_{k=1}^{n+1} z_k = 1$.

Bailey lemma. Given BPs $\alpha(z,t), \beta(w,t)$ related by

$$\beta(w,t) = M(t)_{wz}\alpha(z,t),$$

the functions

$$\alpha'(w,st) = D(s, t^{-\frac{n-1}{2}}u)_w\alpha(w,t),$$

$$D(t,u)_z := \prod_{j=1}^{n+1}\Gamma(\sqrt{pq}t^{-\frac{n+1}{2}}\frac{u}{z_j}, \sqrt{pq}t^{-\frac{n+1}{2}}\frac{z_j}{u}), \quad D(t^{-1}) = D(t)^{-1},$$

$$\beta'(w,st) = D(t^{-1}, s^{\frac{n-1}{2}}u)_wM(s)_{wz}D(ts,u)_z\beta(z,t)$$

form a new BP

$$\beta'(w,st) = M(st)_{wz}\alpha'(z,st).$$

The proof is based on the operator identity

$$M(s)_{wz}D(st,u)_zM(t)_{zx} = D(t, s^{\frac{n-1}{2}}u)_wM(st)_{wx}D(s,t^{-\frac{n-1}{2}}u)_x,$$

equivalent to the A_n-elliptic beta integral (V.S., 2003)

$$\frac{(p;p)_\infty^n(q;q)_\infty^n}{(2\pi \mathrm{i})^n(n+1)!}\int_{\mathbb{T}^n}\frac{\prod_{j=1}^{n+1}\prod_{k=1}^{n+2}\Gamma(s_kz_j,t_kz_j^{-1};p,q)}{\prod_{1\leq j<k\leq n+1}\Gamma(z_jz_k^{-1},z_j^{-1}z_k;p,q)}\prod_{j=1}^n\frac{dz_j}{z_j}$$

$$=\prod_{k=1}^{n+2}\Gamma\Big(\frac{S}{s_k},\frac{T}{t_k};p,q\Big)\prod_{k,l=1}^{n+2}\Gamma(s_kt_l;p,q),$$

$$S=\prod_{j=1}^{n+2}s_j,\ T=\prod_{j=1}^{n+2}t_j,\ ST=pq,\ |t_i|,|s_i|<1.$$

Complete proofs: Rains, 2003, V.S., 2004

$\Rightarrow$ Applications to quiver gauge theories: Brünner, V.S., 2016

SOLUTION OF THE YANG-BAXTER EQUATION

The Yang-Baxter equation

$$\mathbb{R}_{12}(u-v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{23}(v) = \mathbb{R}_{23}(v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{12}(u-v)$$

$\mathbb{R}_{jk}$ acts in $\mathbb{V}_j \otimes \mathbb{V}_k \subset \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3 \subset \Phi(z_1, z_2, z_3),\ z_j \in \mathbb{C}$.

Each $\mathbb{V} \Leftrightarrow u$ (spectral parameter) and g (spin, $\propto 2\ell+1$ for rank 1 algebras). Define:

$$u_{1,2} = \frac{u \pm g_1}{2}, \quad v_{1,2} = \frac{v \pm g_2}{2}, \quad w_{1,2} = \frac{w \pm g_3}{2}.$$

and

$$\mathbb{R}_{12} := \mathbb{P}_{12}\,\mathrm{R}_{12}(u_1, u_2|v_1, v_2), \quad \mathbb{P}_{12}\,\Phi(z_1, z_2) = \Phi(z_2, z_1), \quad \text{etc.}$$

Then YBE takes the form:

$$\mathrm{R}_{23}(u_1, u_2|v_1, v_2)\,\mathrm{R}_{12}(u_1, u_2|w_1, w_2)\,\mathrm{R}_{23}(v_1, v_2|w_1, w_2)$$
$$= \mathrm{R}_{12}(v_1, v_2|w_1, w_2)\,\mathrm{R}_{23}(u_1, u_2|w_1, w_2)\,\mathrm{R}_{12}(u_1, u_2|v_1, v_2).$$

Theorem: (Derkachov, V.S., 2012)

$$\mathrm{R}_{12}(\mathbf{u}) := \mathrm{S}_2\mathrm{S}_1\mathrm{S}_3\mathrm{S}_2 = \mathrm{S}_2(s_1 s_3 s_2 \mathbf{u})\,\mathrm{S}_1(s_3 s_2 \mathbf{u})\,\mathrm{S}_3(s_2 \mathbf{u})\,\mathrm{S}_2(\mathbf{u}),$$

where S_j are the Bailey lemma entries:

$$\mathrm{S}_j(\mathbf{u}) \Leftrightarrow \mathrm{S}_j(\mathbf{t}) \text{ with } t_{1,2} = e^{2\pi i u_{1,2}},\ t_{3,4} = e^{2\pi i v_{1,2}},\ z \to e^{2\pi i z}.$$

Analogously, $\mathrm{R}_{23}(\mathbf{u}) = \mathrm{S}_4\mathrm{S}_3\mathrm{S}_5\mathrm{S}_4$. **Proof**:

$$\mathrm{S}_4\mathrm{S}_3\mathrm{S}_5\mathrm{S}_4 \cdot \mathrm{S}_2\mathrm{S}_1\mathrm{S}_3\mathrm{S}_2 \cdot \mathrm{S}_4\mathrm{S}_3\mathrm{S}_5\mathrm{S}_4 = \mathrm{S}_2\mathrm{S}_3\mathrm{S}_1\mathrm{S}_2 \cdot \mathrm{S}_4\mathrm{S}_3\mathrm{S}_5\mathrm{S}_4 \cdot \mathrm{S}_2\mathrm{S}_1\mathrm{S}_3\mathrm{S}_2,$$

i.e. YBE is a word identity in $\mathcal{S}_6$.

The formalism: Derkachov, 2006; Derkachov, Manashov, 2006 ...

This R-operator can be reduced to:

- Baxter's (1972) 8-vertex model R-matrix: $\mathbb{V}_j = \mathbb{C}^2$,

$$\mathbb{R}_{12}(u) = \sum_{a=0}^{3} w_a(u)\, \sigma_a \otimes \sigma_a, \qquad w_a(u) = \frac{\theta_{a+1}(u+\eta|\tau)}{\theta_{a+1}(\eta|\tau)}$$

- Sklyanin's (1982) L-operator: $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{C}^2, \dim \mathbb{V}_3 = \infty$,

$$\begin{aligned}
&\mathrm{L}(u) := \sum_{a=0}^{3} w_a(u)\, \sigma_a \otimes \mathbf{S}^a, \\
&\mathbf{S}^\alpha\, \mathbf{S}^\beta - \mathbf{S}^\beta\, \mathbf{S}^\alpha = i \left(\mathbf{S}^0\, \mathbf{S}^\gamma + \mathbf{S}^\gamma\, \mathbf{S}^0\right) , \\
&\mathbf{S}^0\, \mathbf{S}^\alpha - \mathbf{S}^\alpha\, \mathbf{S}^0 = i\, \mathbf{J}_{\beta\gamma} \left(\mathbf{S}^\beta\, \mathbf{S}^\gamma + \mathbf{S}^\gamma\, \mathbf{S}^\beta\right) ,
\end{aligned}$$

$(\alpha, \beta, \gamma) =$ a cycle of $(1, 2, 3)$, $\mathbf{J}_{12} = \frac{\theta_1^2(\eta)\theta_4^2(\eta)}{\theta_2^2(\eta)\theta_3^2(\eta)}$, etc.
Sklyanin: $\mathbf{S}^a = \mathbf{S}^a(g; \eta, \tau) =$ 2nd order difference operators.

The detailed reduction analysis: $\Rightarrow$ Chicherin's talk.

Intertwining relations. For $p = e^{2\pi i\tau}, \quad q = e^{4\pi i\eta}, \quad t = e^{-2\pi i g}$,

$$\mathrm{M}(t)\,\mathbf{S}^a(g) = \mathbf{S}^a(-g)\,\mathrm{M}(t).$$

Equivalently, $\ell \to -1-\ell$ or $u_1 \to u_2$.

Uniqueness of $M(t)$ (from this rel.) ? Iff: analyticity in $e^{2\pi i z_j}$ and $\exists$ **the elliptic modular double** (V.S., 2008)

$\mathrm{M}(t)$ is p, q symmetric ($\tau \leftrightarrow 2\eta$) $\Rightarrow$ a second Sklyanin algebra: $\tilde{\mathbf{S}}^a(g;\eta,\tau) := \mathbf{S}^a(g;\tau/2,2\eta)$ and

$$\mathrm{M}(t)\,\tilde{\mathbf{S}}^a(g) = \tilde{\mathbf{S}}^a(-g)\,\mathrm{M}(t).$$

The cross-commutation relations

$$\mathbf{S}^a\tilde{\mathbf{S}}^b = \tilde{\mathbf{S}}^b\mathbf{S}^a, \quad a,b \in \{0,3\} \text{ or } a,b \in \{1,2\},$$
$$\mathbf{S}^a\tilde{\mathbf{S}}^b = -\tilde{\mathbf{S}}^b\mathbf{S}^a, \quad a \in \{0,3\},\ b \in \{1,2\} \text{ or } a \in \{1,2\},\ b \in \{0,3\}.$$

Faddeev's (1999) modular double: $U_q(sl(2)) \otimes U_{\tilde{q}}(sl(2))$

The explicit R-operator

$$[\mathbb{R}_{12}(\mathbf{u})f](x_1,x_2) = \Gamma(\sqrt{pq}x_1^{\pm1}x_2^{\pm1}e^{2\pi\mathrm{i}(v_2-u_1)};p,q)$$
$$\times \int_{\mathbb{T}^2} \frac{\Gamma(e^{2\pi\mathrm{i}(v_1-u_1)}x_2^{\pm1}x^{\pm1}, e^{2\pi\mathrm{i}(v_2-u_2)}x_1^{\pm1}y^{\pm1};p,q)}{\Gamma(e^{4\pi\mathrm{i}(v_1-u_1)}, e^{4\pi\mathrm{i}(v_2-u_2)}, x^{\pm2}, y^{\pm2};p,q)}$$
$$\times \Gamma(\sqrt{pq}e^{2\pi\mathrm{i}(v_1-u_2)}x^{\pm1}y^{\pm1};p,q)f(x,y)\frac{dx}{x}\frac{dy}{y}.$$

Related $2d$ lattice model

Act by the operator STR on a δ-function (a localized spin) $\Rightarrow$

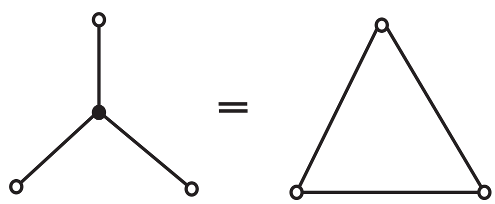
$$\int_0^{2\pi} \rho(u) W(\xi-\alpha;x,u)W(\alpha+\gamma;y,u)W(\xi-\gamma;w,u)du$$
$$= \chi W(\alpha;y,w)W(\xi-\alpha-\gamma;x,w)W(\gamma;x,y),$$

where

$$W(\alpha;x,y) = \Gamma(e^{-\alpha}e^{\mathrm{i}(\pm x \pm y)};p,q)$$
$$\rho(u) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi}\theta(e^{2\mathrm{i}u};p)\theta(e^{-2\mathrm{i}u};q),$$
$$\chi = \Gamma(e^{-\alpha},e^{-\gamma},e^{\alpha+\gamma-\xi};p,q), \qquad e^{-\xi}=\sqrt{pq}.$$

$\Rightarrow$ ell. beta integral = functional star-triangle relation

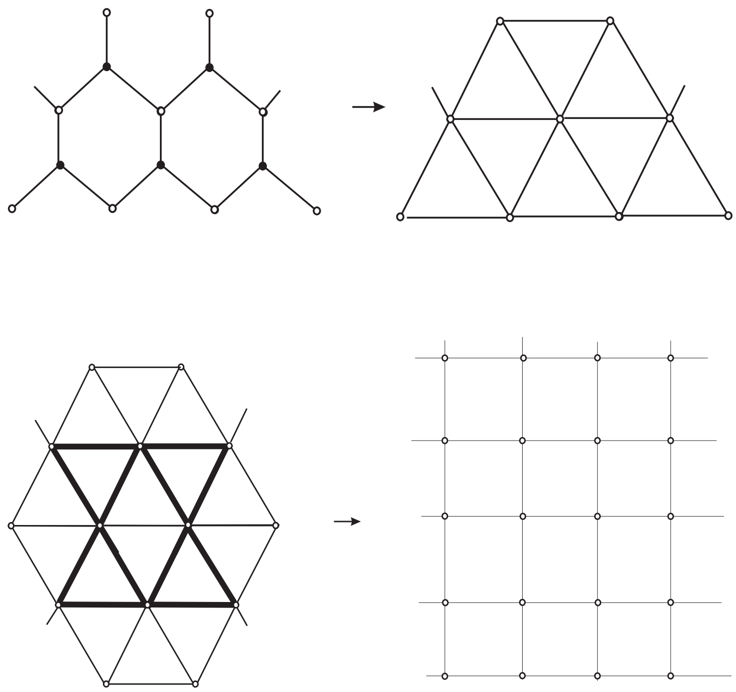
(Bazhanov, Sergeev, 2010)



Ising type models: circles carry spins $z, w, \ldots$, edges carry Boltzmann weights W, black circle = integration (summation) over z-spin values.

$W \propto$ kernel of $M(t)$ (even for A_n-case, Bazhanov-Sergeev, 2011).

Honeycomb, triangluar, and square lattices:



Asymptotics of the Ising partition function ⇒ a Mahler measure

A more detailed consideration ⇒ Gahramanov, Kels, Yagi talks

SUPERCONFORMAL INDEX

Four-dimensional $\mathcal{N}=1$ SUSY gauge field theory:

$G_{full} = SU(2,2|1) \times G \times F$

$J_i, \overline{J}_i$ ($SU(2)$ subgroup generators, or Lorentz rotations),
$P_\mu, Q_\alpha, \overline{Q}_{\dot\alpha}$ (supertranslations), $\alpha, \dot\alpha = 1, 2$
$K_\mu, S_\alpha, \overline{S}_{\dot\alpha}$ (special superconformal transformations),
H (dilations) and R ($U(1)_R$-rotations).
Internal symmetries: gauge group G^a, flavor symmetry F_k.

Superconformal algebra $su(2,2|1)$:

$$\mathcal{M}_{\mathcal{A}}^{\ \mathcal{B}} = \begin{pmatrix} M_\alpha^{\ \beta} + \frac{1}{2}\delta_\alpha^\beta H & \frac{1}{2}P_{\alpha\dot\beta} \\ \frac{1}{2}K^{\dot\alpha\beta} & \overline{M}^{\dot\alpha}_{\ \dot\beta} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} H \end{pmatrix},$$

$$\mathcal{Q}_{\mathcal{A}} = \begin{pmatrix} Q_\alpha \\ \overline{S}^{\dot\alpha} \end{pmatrix}, \ \overline{\mathcal{Q}}^{\mathcal{B}} = \begin{pmatrix} S^\beta, & \overline{Q}_{\dot\beta} \end{pmatrix}$$

$$[\mathcal{M}_{\mathcal{A}}^{\ \mathcal{B}}, \mathcal{M}_{\mathcal{C}}^{\ \mathcal{D}}] = \delta_{\mathcal{C}}^{\mathcal{B}} \mathcal{M}_{\mathcal{A}}^{\ \mathcal{D}} - \delta_{\mathcal{A}}^{\mathcal{D}} \mathcal{M}_{\mathcal{C}}^{\mathcal{B}},$$

$$[\mathcal{M}_{\mathcal{A}}^{\ \mathcal{B}}, \mathcal{Q}_{\mathcal{C}}] = \delta_{\mathcal{C}}^{\mathcal{B}} \mathcal{Q}_{\mathcal{A}} - \frac{1}{4}\delta_{\mathcal{A}}^{\mathcal{B}} \mathcal{Q}_{\mathcal{C}}, \qquad [\mathcal{M}_{\mathcal{A}}^{\ \mathcal{B}}, \overline{\mathcal{Q}}^{\mathcal{C}}] = -\delta_{\mathcal{A}}^{\mathcal{C}} \overline{\mathcal{Q}}^{\mathcal{B}} + \frac{1}{4}\delta_{\mathcal{A}}^{\mathcal{B}} \overline{\mathcal{Q}}^{\mathcal{C}},$$

$$[R, \mathcal{Q}_{\mathcal{A}}] = -\mathcal{Q}_A, \qquad [R, \overline{\mathcal{Q}}^{\mathcal{B}}] = \overline{\mathcal{Q}}^{\mathcal{B}},$$

$$\{\mathcal{Q}_{\mathcal{A}}, \overline{\mathcal{Q}}^{\mathcal{B}}\} = 4\mathcal{M}_{\mathcal{A}}^{\mathcal{B}} + 3\delta_{\mathcal{A}}^{\mathcal{B}} R, \qquad \{\mathcal{Q}_{\mathcal{A}}, \mathcal{Q}_{\mathcal{B}}\} = 0, \qquad \{\overline{\mathcal{Q}}^{\mathcal{A}}, \overline{\mathcal{Q}}^{\mathcal{B}}\} = 0,$$

$$\delta_{\mathcal{A}}^{\mathcal{B}} = \begin{pmatrix} \delta_\alpha^\beta & 0 \\ 0 & \delta^{\dot\alpha}_{\dot\beta} \end{pmatrix}.$$

For $Q=\overline{Q}_1$ and $Q^\dagger=-\overline{S}_1$, one has $Q^2=(Q^\dagger)^2=0$ and

$$\{Q,Q^\dagger\}=2\mathcal{H},\qquad \mathcal{H}=H-2\overline{J}_3-3R/2$$

The superconformal index: (KMMR, Romelsberger, 2005)

$$I(y;p,q)=\mathrm{Tr}\Big((-1)^{\mathrm{F}}p^{\mathcal{R}/2+J_3}q^{\mathcal{R}/2-J_3}\prod_k y_k^{F_k}\,e^{-\beta\mathcal{H}}\Big),$$

$\mathcal{R}=H-R/2$ and F is the fermion number,
$p,q,y_k,e^{-\beta}$ are group parameters (fugacities).

It counts BPS states $\mathcal{H}|\psi\rangle=0$ or cohomology of Q, $Q^\dagger$ operators (hence, no β-dependence).

"Physical" (not rigorous) computation yields a matrix integral:

$$I(y;p,q)\;=\;\int_G d\mu(z)\,\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{ind}\big(p^n,q^n,z^n,y^n\big)\right)$$

with the Haar measure $d\mu(z)$ and single particle states index

$$\mathrm{ind}(p,q,z,y)=\frac{2pq-p-q}{(1-p)(1-q)}\chi_{adj_G}(z)$$
$$+\sum_j\frac{(pq)^{R_j/2}\chi_{r_F,j}(y)\chi_{r_G,j}(z)-(pq)^{1-R_j/2}\chi_{\bar{r}_F,j}(y)\chi_{\bar{r}_G,j}(z)}{(1-p)(1-q)}.$$

$\chi_{R_F,j}(y)$ and $\chi_{R_G,j}(z)$ are characters of the respective representations, and R_j are R-charges of fields.

A single chiral field contribution with $F=U(1)$:

$$I=\exp\Big(\sum_{n=1}^{\infty}\frac{y^n-(pqy^{-1})^n}{n(1-p^n)(1-q^n)}\Big)=\Gamma(y;p,q).$$

For the unitary group $SU(N)$, $z=(z_1,\ldots,z_N)$, $\prod_{a=1}^N z_a=1$,

$$\int_{SU(N)}d\mu(z) \;=\; \frac{1}{N!}\int_{\mathbb{T}^{N-1}}\Delta(z)\Delta(z^{-1})\prod_{a=1}^{N-1}\frac{dz_a}{2\pi\mathrm{i}z_a},$$

$$\Delta(z) \;=\; \prod_{1\leq a<b\leq N}(z_a-z_b), \qquad \text{the Vandermonde determinant.}$$

Where is the elliptic beta integral ?

The left-hand side: $G=SU(2)$, $F=SU(6)$, representations

1) vector superfield: $(adj,1)$,

$\chi_{SU(2),adj}(z)=z^2+z^{-2}+1$,

2) chiral superfield: (f,f),

$\chi_{SU(2),f}(z)=z+z^{-1}, \qquad R_f=1/3$,

$$\chi_{SU(6),f}(y)=\sum_{k=1}^6 y_k,\quad \chi_{SU(6),\bar f}(y)=\sum_{k=1}^6 y_k^{-1},\quad \prod_{k=1}^6 y_k=1,$$

and $t_k=(pq)^{1/6}y_k$, $k=1,\ldots,6$. Balancing = $SU(6)$-unitarity.

The right-hand side: $G = 1$, $F = SU(6)$ with the single chiral superfield $T_A: \quad \Phi_{ij} = -\Phi_{ji}$,

$$\chi_{SU(6),T_A}(y) = \sum_{1\leq i<j\leq 6} y_i y_j, \qquad R_{T_A} = 2/3.$$

A Wess-Zumino type theory for the confined colored particles.

The elliptic beta integral describes the confinement phenomenon in the simplest $4d$ supersymmetric quantum chromodynamics.

Seiberg, 1994; Dolan-Osborn, 2008.

The process of integrals' evaluation = transition from UV (weak coupling) to IR (strong coupling) physics.

EHIs = new matrix models

EHIs = new computable path integrals in $4d$ QFT

Symmetries of EHIs = general Seiberg dualities.

Seiberg duality:

"Electric" theory:

	$SU(N_c)$	$SU(N_f)_l$	$SU(N_f)_r$	$U(1)_B$	$U(1)_R$
Q	f	f	1	1	$\widetilde{N}_c/N_f$
$\widetilde{Q}$	$\overline{f}$	1	$\overline{f}$	-1	$\widetilde{N}_c/N_f$
V	adj	1	1	0	1

"Magnetic" theory:

	$SU(\widetilde{N}_c)$	$SU(N_f)_l$	$SU(N_f)_r$	$U(1)_B$	$U(1)_R$
q	f	$\overline{f}$	1	$N_c/\widetilde{N}_c$	N_c/N_f
$\widetilde{q}$	$\overline{f}$	1	f	$-N_c/\widetilde{N}_c$	N_c/N_f
M	1	f	$\overline{f}$	0	$2\widetilde{N}_c/N_f$
$\widetilde{V}$	adj	1	1	0	1

$\widetilde{N}_c = N_f - N_c$

Seiberg conjecture: these two $\mathcal{N} = 1$ SYM theories have the same physics at their IR fixed points

The electric theory index:

$$I_E = \kappa_{N_c} \int_{\mathbb{T}^{N_c-1}} \frac{\prod_{i=1}^{N_f} \prod_{j=1}^{N_c} \Gamma(s_i z_j, t_i^{-1} z_j^{-1})}{\prod_{1\leq i<j\leq N_c} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j)} \prod_{j=1}^{N_c-1} \frac{dz_j}{z_j},$$

$$\prod_{j=1}^{N_c} z_j = 1, \qquad \kappa_N = \frac{(p;p)_\infty^{N-1} (q;q)_\infty^{N-1}}{N!(2\pi \mathrm{i})^{N-1}}.$$

The magnetic theory: $\qquad I_M = \kappa_{\tilde{N}_c} \prod_{i,j=1}^{N_f} \Gamma(s_i t_j^{-1}) \times$

$$\times \int_{\mathbb{T}^{\tilde{N}_c-1}} \frac{\prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}_c} \Gamma(S^{\frac{1}{\tilde{N}_c}} s_i^{-1} x_j, T^{-\frac{1}{\tilde{N}_c}} t_i x_j^{-1})}{\prod_{1\leq i<j\leq \tilde{N}_c} \Gamma(x_i x_j^{-1}, x_i^{-1} x_j)} \prod_{j=1}^{\tilde{N}_c-1} \frac{dx_j}{x_j},$$

where $\prod_{j=1}^{\tilde{N}_c} x_j = 1, \quad \tilde{N}_c = N_f - N_c,$

$S = \prod_{i=1}^{N_f} s_i, \quad T = \prod_{i=1}^{N_f} t_i, \quad ST^{-1} = (pq)^{N_f - N_c}.$

Theorem: $I_E = I_M$

For $N_c = 2$, $N_f = 3, 4$ and $N_f = N_c + 1$ (V.S., 2000-2003), for general N_c, N_f (Rains, 2003)

Partial summary of the results.

Joint work with G.S. Vartanov (2008–2014):

• Systematic construction of SCIs for known dualities $\Rightarrow$ about 50 conjectural identities for EHIs

• Systematic physical interpretation of mathematical EHI identities $\Rightarrow$ about 20 new Seiberg dualities

• “Vanishing” (delta function behavior) of SCIs/EHIs $\longleftrightarrow$ chiral symmetry breaking

• $SL(3,\mathbb{Z})$-modularity of EHIs $\longleftrightarrow$ 't Hooft anomaly matching conditions

Gadde, Pomoni, Rastelli, Razamat, Yan, Gaiotto, 2009-..:

• $4d$ $\mathcal{N}=2$ SCIs and $2d$ topological field theories, $Y^{p,q}$-quiver gauge theories

• Relations to Macdonald polynomials, insertion of surface defects, Ruijsenaars type integrable systems, ... $\Rightarrow$ Razamat's talk

Dolan, Vartanov, V.S., Gadde, Yan, Imamura, 2011:

• $4d$ SCIs $\to$ $3d$ squashed sphere partition functions (EHIs $\to$ hyperbolic integrals)

V.S., 2010:

• $4d$ SCIs for quiver theories describe partition functions of $2d$ lattice spin systems. Seiberg duality $\simeq$ Kramers-Wannier type duality. Similarly, $3d$ quiver gauge theories $\Rightarrow$ Faddeev-Volkov type models

Kim, Imamura, Yokoyama, KSV, Kapustin, 2011; Gahramanov, Rosengren, Kels,...

- $3d$ SCIs $\simeq$ bilateral sums of q-hypergeometric integrals

STRONG DEPENDENCE ON THE TOPOLOGY OF CURVED SPACES

Flat space-time $\mathbb{R}^4 \to \mathbb{S}^3 \times \mathbb{R} \to \mathbb{S}^3 \times \mathbb{S}^1 \simeq \mathcal{M}_{p,q} = \mathbb{C}^2/\{0,0\}$: $(z_1, z_2) = (pz_1, qz_2)$, the Hopf surface.

Further deformations:

Benini, Nishioka, Yamazaki, Razamat, Willet, Kels, V.S., ...:

- SCIs for lens space theories $(\mathbb{S}^3/\mathbb{Z}_r) \times \mathbb{S}^1$, relation to $3d$ SCIs, finite sums of EHIs, new EHI identities

Kallen, Qui, Zabzine, Winding, Imamura, Kim^2-Lee, ...

- SCIs for $5d$ and $6d$ theories $\Rightarrow$ higher order hyperbolic and elliptic gamma functions.

Assel, Cassani, Martelli, Di Pietro, Komargodski, Lorenzen, Sparks, 2014,...

- computation of the partition function for the Hopf surface

$$Z(\mathcal{M}_{p,q}) = \int [d\phi] e^{-S[\phi]} = e^{-\beta E_{Cas}} I(t; p, q) \simeq \text{SCI}.$$

E_{Cas} = the Casimir energy, $t \simeq$ background gauge fields.

A completely different interpretation of EHI parameters !

+ Many other results and names (sorry for not mentioning)

The modified elliptic gamma function (V.S., 2003)

Take $\omega_1, \omega_2, \omega_3 \in \mathbb{C}$ and define

$$p = e^{2\pi \mathrm{i}\omega_3/\omega_2}, \quad q = e^{2\pi \mathrm{i}\omega_1/\omega_2}, \quad r = e^{2\pi \mathrm{i}\omega_3/\omega_1}$$

$\tau \to -1/\tau$ modular transformations

$$\widetilde{p} = e^{-2\pi \mathrm{i}\omega_2/\omega_3}, \quad \widetilde{q} = e^{-2\pi \mathrm{i}\omega_2/\omega_1}, \quad \widetilde{r} = e^{-2\pi \mathrm{i}\omega_1/\omega_3}$$

Elliptic gamma functions: special solutions of the finite difference equation

$$f(u+\omega_1) = \theta(e^{2\pi \mathrm{i} u/\omega_2}; p) f(u). \qquad (*)$$

A particular solution for $|q| < 1$: $f(u) = \Gamma(e^{2\pi \mathrm{i} u/\omega_2}; p, q)$

This $f(u)$ satisfies two more equations

$$f(u+\omega_2) = f(u), \qquad f(u+\omega_3) = \theta(e^{2\pi \mathrm{i} u/\omega_2}; q) f(u)$$

defining it uniquely (for $\sum_{i=1}^3 n_i\omega_i \neq 0,\ n_i \in \mathbb{Z}$) up to a multiplication by constant.

Another elliptic gamma function (**well defined for** $|q| = 1$):

$$\mathcal{G}(u;\omega) := \Gamma(e^{2\pi \mathrm{i} u/\omega_2}; p, q)\Gamma(r e^{-2\pi \mathrm{i} u/\omega_1}; \widetilde{q}, r).$$

Satisfies (*) and two other equations

$$\mathcal{G}(u+\omega_2) = \theta(e^{2\pi i u/\omega_1}; r)\mathcal{G}(u), \quad \mathcal{G}(u+\omega_3) = e^{-\pi i B_{2,2}(u,\omega_1,\omega_2)}\mathcal{G}(u),$$

$B_{2,2}$ is a 2nd order Bernoulli polynomial

$$B_{2,2}(u,\omega_1,\omega_2) = \frac{u^2}{\omega_1\omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}.$$

A different representation (following from an $SL(3,\mathbb{Z})$ modular transformation, Felder, Varchenko, 1999):

$$\mathcal{G}(u;\omega) \ = \ e^{-\frac{\pi i}{3}B_{3,3}(u;\omega)}\Gamma(e^{-2\pi i u/w_3};\widetilde{r},\widetilde{p}),$$

where $B_{3,3}$ is a 3rd order Bernoulli polynomial

$$B_{3,3}(u;\omega) = \frac{1}{\omega_1\omega_2\omega_3}\Big(u - \frac{1}{2}\sum_{k=1}^{3}\omega_k\Big)\Big((u - \frac{1}{2}\sum_{k=1}^{3}\omega_k)^2 - \frac{1}{4}\sum_{k=1}^{3}\omega_k^2\Big)$$

Multiple Bernoulli polynomials:

$$\frac{x^m e^{xu}}{\prod_{k=1}^{m}(e^{\omega_k x}-1)} = \sum_{n=0}^{\infty} B_{m,n}(u;\omega_1,\ldots,\omega_m)\frac{x^n}{n!}$$

Modified EHIs: $I^{mod} := I(t;p,q)\,[\text{after}\,\Gamma(z;p,q) \to \mathcal{G}(u;\omega)]$.

It turns out: $I^{mod} = e^{\varphi}I(t;p,q)\,[\text{after}\,\omega_2,\omega_3 \to -\omega_3,\omega_2]$

(van Diejen, V.S., 2003)

"Duality": $I_E^{mod} = I_M^{mod} \Longleftrightarrow \varphi_E = \varphi_M$

$\Rightarrow$ 't Hooft anomaly matching (Vartanov, V.S., 2012)

Equivalently, a set of 7 Diophantine equations of different forms

$\Rightarrow$ relation to the Casimir energy $\varphi \simeq -\beta E_{Cas}$ for $\beta \to \infty$: Brünner, Regalado, V.S., 2016

CONCLUSION: interesting special functions emerge from applications and live in applications.