q-Garnier system
and its autonomous limit

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Elliptic Hypergeometric Functions in Combinatorics, Integrable Systems and Physics

ESI, 24.Mar.2017

▶ Introduction

• **Garnier system** [Garnier (1912)] is a $2N$ variable extension of Painlevé VI equation ($N = 1$).

• **q-Garnier system** [Sakai (2005)] is its q-difference analog.

• In this talk, I will study the q-Garnier system and its **autonomous** limit from geometric/physical point of view.

Ref: [Nagao-Y, (arXiv.1601.01099)]

• **Autonomous limit of discrete Painlevé equations are integrable system (QRT system).**

• **Example** : **q-Painlevé VI** [Jimbo-Sakai (1996)]

$$T_t : \begin{pmatrix} a, b, c, d \\ r, s, t, u \end{pmatrix}, x, y \Bigg) \mapsto \Bigg(\begin{matrix} a, b, qc, qd \\ r, s, qt, qu \end{matrix}, \bar{x}, \bar{y} \end{pmatrix}, \qquad q = \frac{abtu}{cdrs},$$

$$\bar{x} = \frac{ab}{x}\frac{(\bar{y}+qt)(\bar{y}+qu)}{(\bar{y}+r)(\bar{y}+s)},$$

$$\bar{y} = \frac{rs}{y}\frac{(x+c)(x+d)}{(x+a)(x+b)}.$$

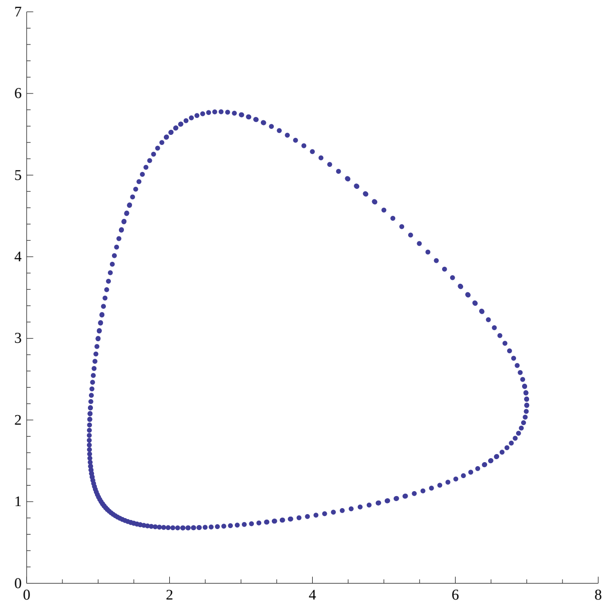
$\rightarrow$ a discrete dynamical system on (x, y) plane. (non-integrable in general)

- **When $q = 1$, q-P_{VI} is integrable.**

$$x \mapsto \frac{ab}{x}\frac{(y+t)(y+u)}{(y+r)(y+s)},$$

$$y \mapsto \frac{rs}{y}\frac{(x+c)(x+d)}{(x+a)(x+b)},$$

with a constraint $\dfrac{abtu}{cdrs} = 1$.



$\rightarrow$ conserved **elliptic** curve:

$$(x+a)(x+b)y^2+\{(r+s)x^2+ab(t+u)\}y$$
$$+rs(x+c)(x+d) = Hxy.$$

- We will generalize this to **hyper-elliptic** curves.

▸ Plan of the talk

1. Derive a **simple form** of Lax pair and evolution equations of **q-Garnier system**

2. Generalize **QRT system** to hyper-elliptic curve

3. Two **dual** formulations of q-Garnier system from **q-KP**

1. Lax pair and evolution equations of q-Garnier system

▶ Lax pair for q-Garnier (contiguous type)

$$L_2: \ F(x)\bar{y}(x) + G(x)y(x) - A(x)y(qx) = 0,$$

$$L_3: \ qx\bar{F}(x)y(qx) + G(x)\bar{y}(qx) - qtB(x)\bar{y}(x) = 0.$$

$$A(x) = \prod_{i=1}^{N+1}(x - a_i), \quad B(x) = \prod_{i=1}^{N+1}(x - b_i),$$

$$\begin{array}{ccc} \bar{y}(x) & - & \bar{y}(qx) \\ | & & | \\ y(x) & - & y(qx) \end{array}$$

$$F(x) = \sum_{i=0}^{N} f_i \ x^i, \quad G(x) = ct + \sum_{i=1}^{N} g_i \ x^i + x^{N+1}.$$

- Shift of parameters: $(\bar{a}_i, \bar{b}_i, \bar{c}, \bar{t}) = (a_i, b_i, c, qt)$.
- We have $2N(+1)$ **dynamical variables** f_i, g_i. (Overall normalization of $f_0, \ldots, f_N$ is a gauge factor)

- L_2 (or L_3) describing a deformation of

$$L_1: \quad A(x)F(\frac{x}{q})y(qx) - R(x)y(x) + tB(\frac{x}{q})F(x)y(\frac{x}{q}) = 0,$$

where $R(x)$ is a polynomial of degree $2N+1$.

- Compatibility of L_2, L_3 (or L_1) $\to$ q-**Garnier system**:

$$xF(x)\bar{F}(x) = tA(x)B(x) \quad \text{for} \quad G(x) = 0,$$

$$G(x)\underline{G}(x) = tA(x)B(x) \quad \text{for} \quad F(x) = 0.$$

- Rewrite the L_1 equation as

$$\Big[A(x)F(\frac{x}{q})T - R(x) + tB(\frac{x}{q})F(x)T^{-1}\Big]y(x) = 0,$$

where $Tx = qxT$.

- In $q \to 1$, the L_1 equation is divisible by $F(x)$ and becomes

$$\boxed{A(x)T - U(x) + \frac{tB(x)}{T} = 0.}$$

This is an algebraic equation in commuting variables (x, T) of bi-degree $(N+1, 2)$. $\to$ a hyper-elliptic curve of genus $g = N$. (Seiberg-Witten curve)

- We will see that autonomous q-Garnier system $=$ generalized QRT system on hyper-elliptic curve.

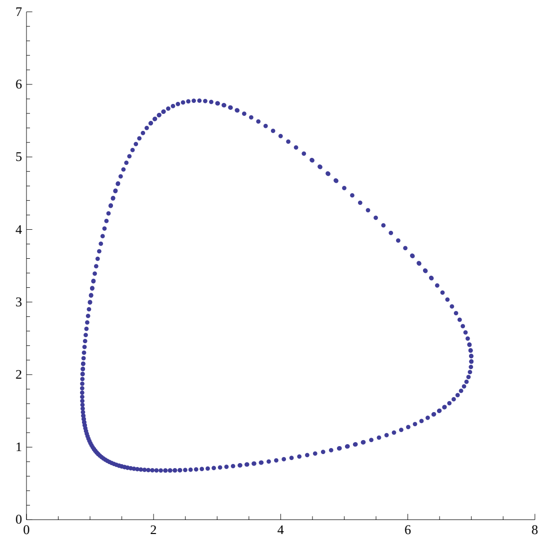
2. QRT system and its hyper-elliptic generalization

▸ QRT system [Quispel-Roberts-Tompson (1989)] [Tsuda (2004)]

• It is an **integrable mapping** on plane preserving a family of curves of bi-degree $(2,2)$:

$$C_u \; : \; \varphi_1(x,y) + u\,\varphi_2(x,y) = 0.$$

• We have two **flips** r_x, r_y



$$r_x : (x,y) \mapsto (\bar{x}, y), \qquad r_y : (x,y) \mapsto (x, \bar{y}).$$

Iterations of r_x, r_y = QRT system.

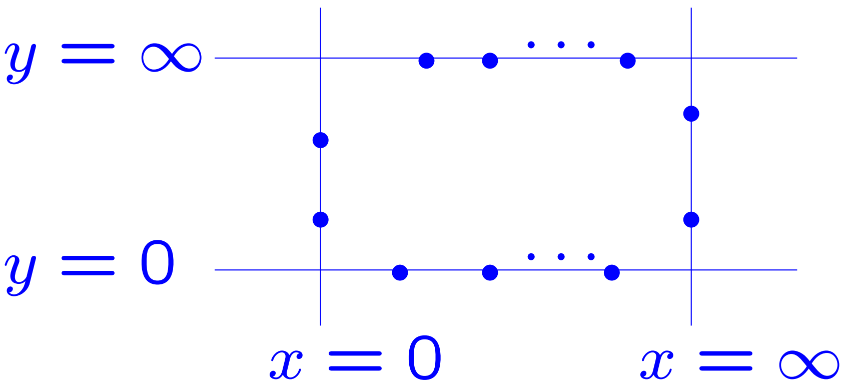
$\rightarrow$ The curve C_u is conserved.

• We generalize this to **hyper-elliptic curves**

• Consider a family of curves of bi-degree $(N+1, 2)$

$$C_u : A(x)y - U(x) + \frac{tB(x)}{y} = 0,$$

passing through the following $2N+6$ points:



(with one constraint)

• Coefficients of $U(x) = \sum_{i=0}^{N+1} u_i x^i$ are free (except for u_0, u_{N+1})

$\rightarrow$ N-**dimensional family** of hyper-elliptic curves of $g = N$.

• To fix the parameters u_i, we need **N points** $Q_i = (x_i, y_i)$.

$$\mathcal{D} = \{Q_1, \cdots, Q_N\} = \text{dynamical variables.}$$

However, the evolution equation (addition formula) on hyper-elliptic curve is **not bi-rational** in (x_i, y_i)-coordinates.

• Following Mumford, we represent the divisor $\mathcal{D}$ by a pair of polynomials $F(x), G(x)$ (degree $N, N+1$) such that

$$\boxed{F(x_i) = 0, \quad y_i = \frac{G(x_i)}{A(x_i)}.}$$

$\rightarrow$ one can define two bi-rational flips

$$r_x : (F, G) \mapsto (\bar{F}, G), \quad r_y : (F, G) \mapsto (F, \bar{G}).$$

• y-flip:

C_u is deg $= 2$ in y $\quad\rightarrow\quad$ hyper-elliptic involution

$$y_i \rightarrow \bar{y}_i, \qquad y_i \bar{y}_i = \frac{tB(x_i)}{A(x_i)}.$$

Since $y_i = \dfrac{G(x_i)}{A(x_i)}$ for $F(x_i) = 0$, we have

$$\boxed{G(x)\bar{G}(x) = tA(x)B(x) \quad \text{for } F(x) = 0.}$$

• This corresponds to the equation for $G(x)$ of q-Garnier system.

• x-flip:

Note that

$$G(x)\left[A(x)y - U(x) + \frac{tB(x)}{y}\right] \Big/ . \; y = \frac{G(x)}{A(x)}$$
$$= G(x)^2 - U(x)G(x) + tA(x)B(x) =: P(x).$$

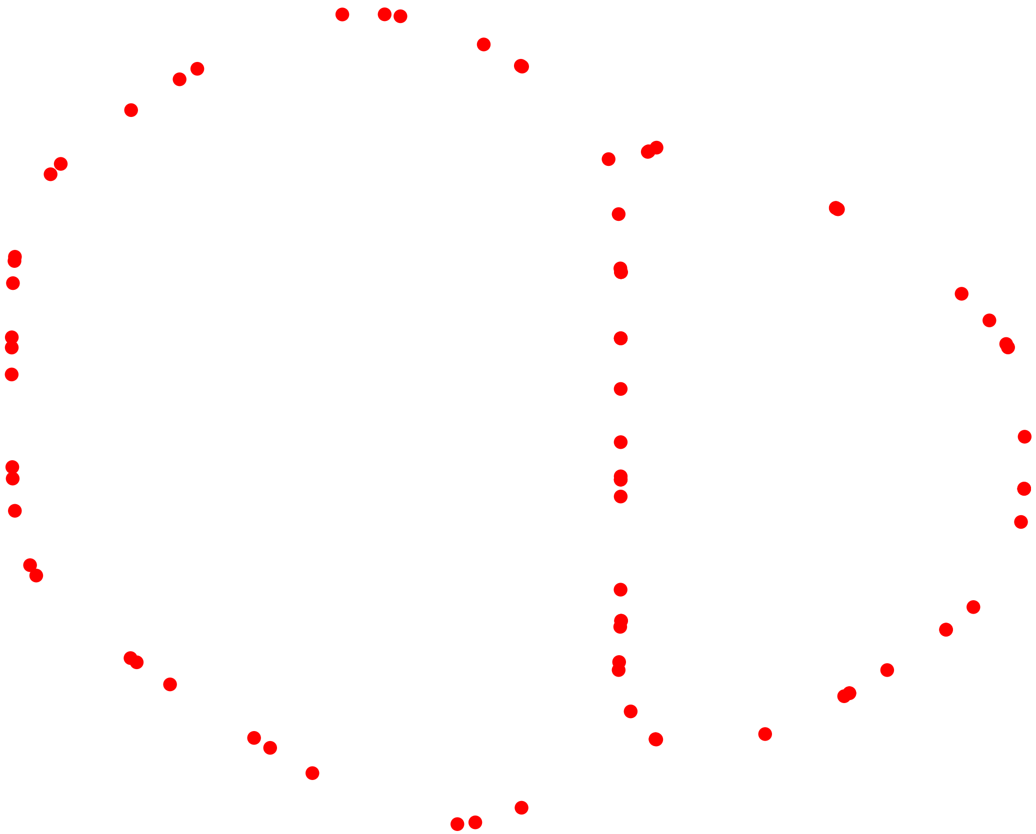
From $Q_i \in C_u$ (and some additional conditions), one can define $\bar{F}(x)$ by $P(x) = xF(x)\bar{F}(x)$, i.e.

$$xF(x)\bar{F}(x) = tA(x)B(x) \quad \text{for } G(x) = 0.$$

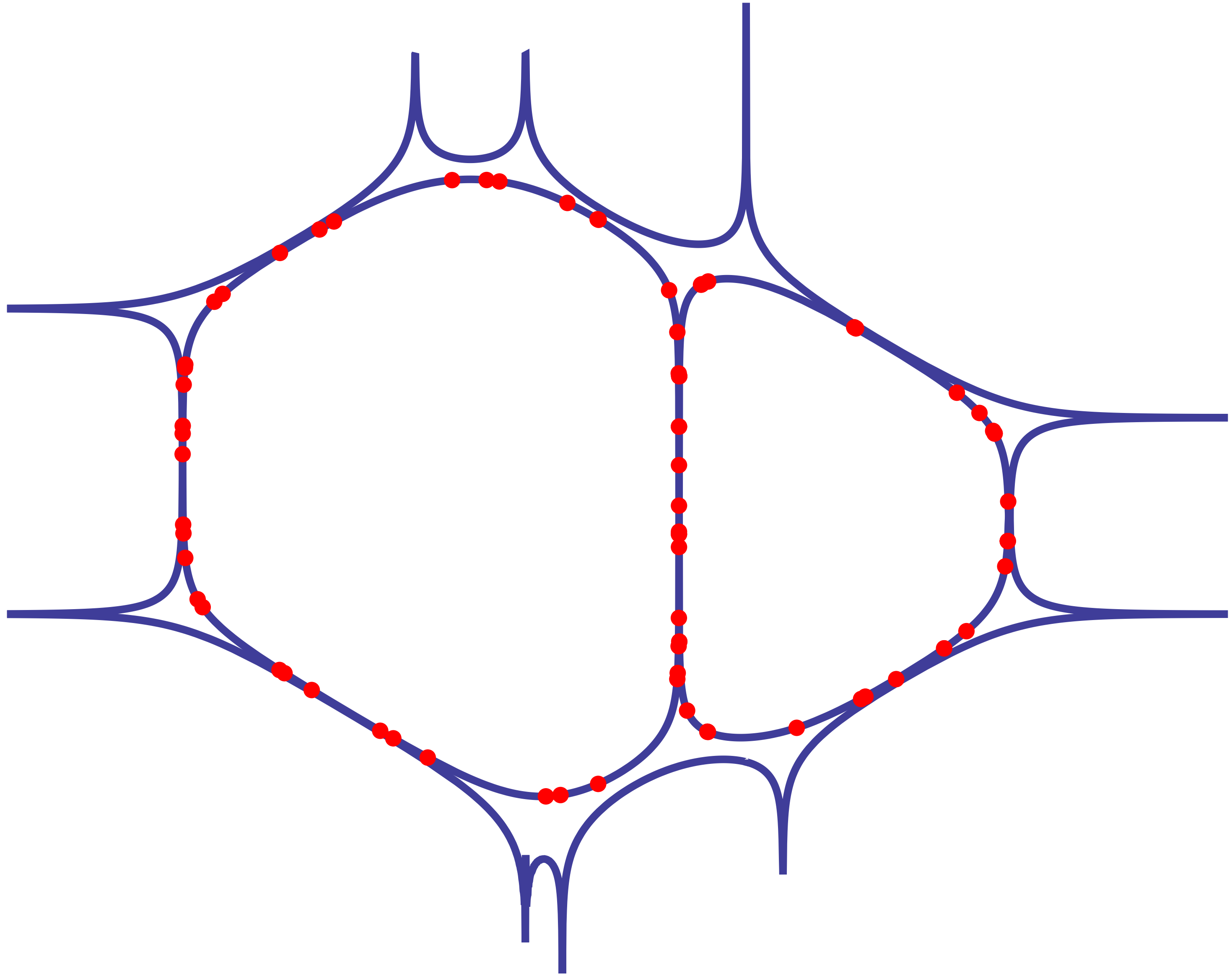
• This corresponds to the equation for $F(x)$ of q-Garnier system. Hence, **autonomous q-Garnier = hyper-elliptic QRT**.

▶ **Numerical example**. $N = 2$ case

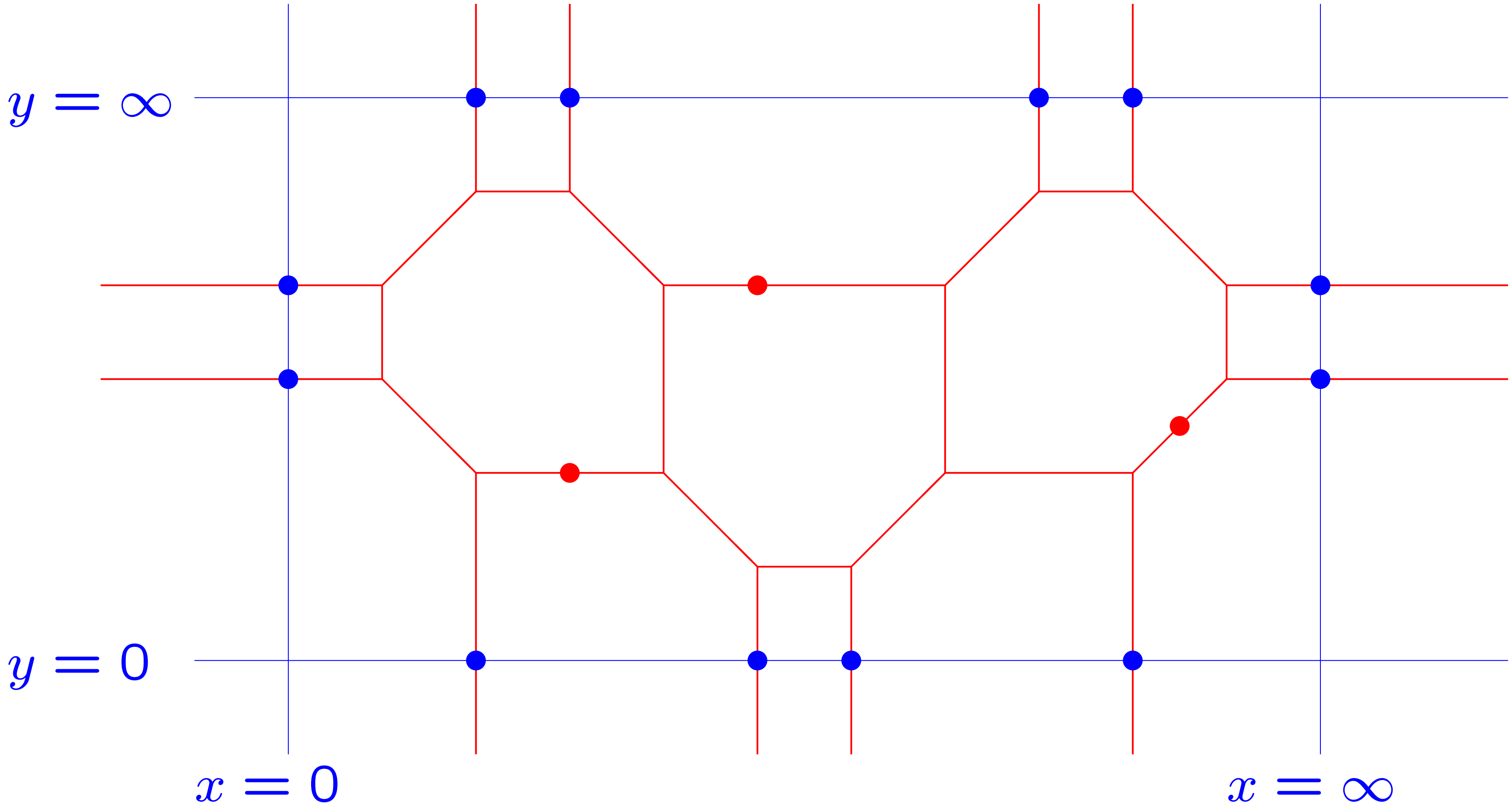
log-log plot of $(x_1, y_1), (x_2, y_2)$



- Conserved curve (amoeba)

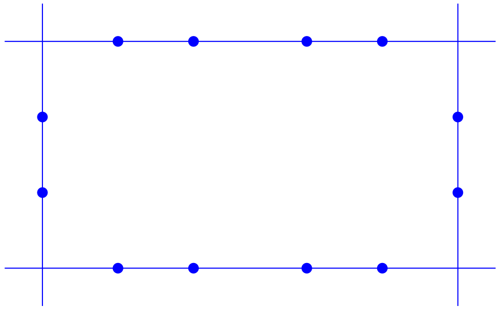


- Tropical limit ($N = 3$)



▶ Variations.

• So far, the $2N+6$ points are on **four lines**:



$\rightarrow$ can be generalized to any **bi-degree (2,2) curve**.

• In the most generic case, we obtain **hyper-elliptic QRT system of elliptic difference type**.
($5d \rightarrow 6d$ lift in gauge theory)

• Its non-autonomous deformation gives an **elliptic Garnier system**. Equivalent (?) to the one constructed by Rains-Ormerod [Rains-Ormerod(2016)].

- **An elliptic Garnier system.** $[x] = (x, p/x; p)_\infty$

$$L_2 \ : \ F(x)[\frac{k}{x^2}]\overline{y}(\frac{x}{q}) - G(x)A(\frac{k}{x})y(\frac{x}{q}) + G(\frac{k}{x})A(x)y(x),$$

$$L_3 \ : \ F(x)[\frac{k}{x^2}]\underline{y}(x) - \underline{G}(x)B(\frac{k}{x})y(x) + \underline{G}(\frac{k}{x})B(x)y(\frac{x}{q}),$$

$$A(x) = \prod_{i=1}^{N+3}[\frac{x}{a_i}], \ B(x) = \prod_{i=1}^{N+3}[\frac{x}{b_i}], \ F(x) = wx\prod_{i=1}^{N}[\frac{x}{\lambda_i}][\frac{k}{\lambda_i x}],$$

$$G(x) = x\prod_{i=1}^{N+1}[\frac{x}{\mu_i}], \ \prod_{i=1}^{N+1}\mu_i = \ell, \ \overline{k} = k/q, \ \overline{\ell} = q\ell.$$

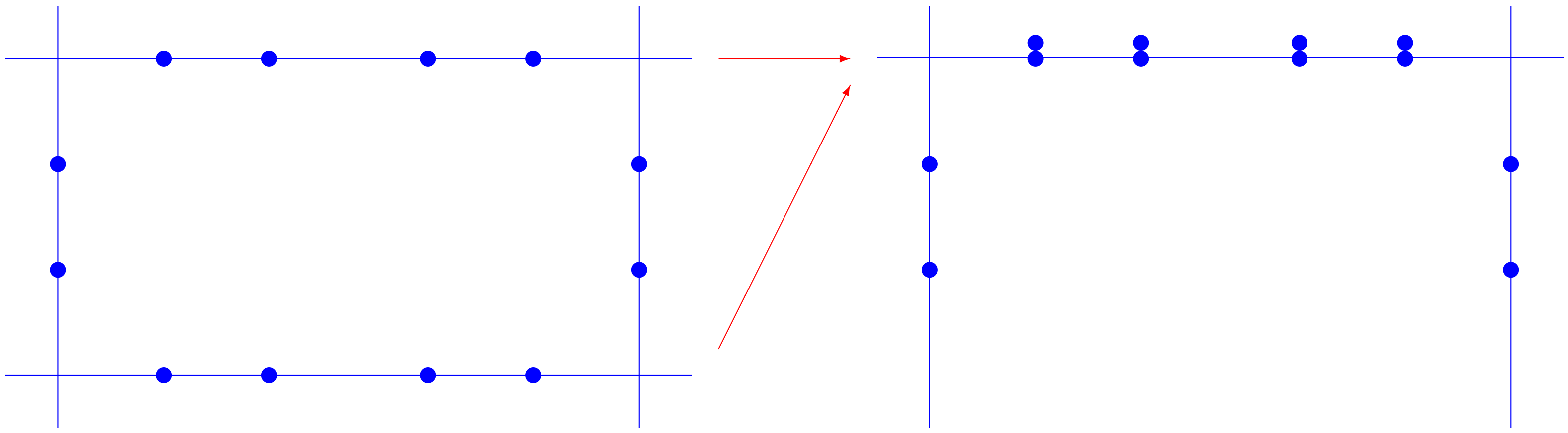
- It has special solutions given in terms of **elliptic HG** :

$$\tau = \det[\ {}_{2N+10}V_{2N+9}\ ].$$

$(N = 1 \to$ ell-$E_8^{(1)}$. [Noumi-Tsujimoto-Y.(2013)])

• One can also consider various **degenerate configurations** of $2N+6$ points.

Example. Differential Garnier system:



($5d \to 4d$ reduction in gauge theory)

3. Two dual formulations from q-KP

• (m,n)**-reduced Lax operator for** q**-KP**.

$$\boldsymbol{\Psi}(qz) = \boldsymbol{\mathcal{A}}(z)\boldsymbol{\Psi}(z), \quad \boldsymbol{\mathcal{A}}(z) = d(t)X_m(z)\cdots X_1(z),$$

$$X_i(z) = \begin{bmatrix} x_{i,1} & 1 & & & \\ & x_{i,2} & 1 & & \\ & & \ddots & \ddots & \\ & & & x_{i,n-1} & 1 \\ r_i z & & & & x_{i,n} \end{bmatrix},$$

$$d(t) = \mathrm{diag}(t_1, \cdots, t_n).$$

• $W(A^{(1)}_{m-1}) \times W(A^{(1)}_{n-1})$ symmetry [Kajiwara-Noumi-Y (2002)]
($\leftarrow$ Tropical R-map, Yang-Baxter map.)

• Interpretation [Inoue-Lam-Pylyavskyy (2016)] as cluster integrable system.

▶ A duality : m $\leftrightarrow$ n.

- This duality is known as
  Spectral duality [Milonov-Morozov-Runov-Zenkevich-Zotov (2012)], $\cdots$
  Fiber-base duality [Mitev-Pomoni-Taki-Yagi (2014)], $\cdots$.

$$\mathbf{\Psi}(qz) = \boldsymbol{\mathcal{A}}(z)\mathbf{\Psi}(z) = d(t)X_m(z)\cdots X_2(z)X_1(z)\mathbf{\Psi}(z).$$

We put $\mathbf{\Psi}_1 = \mathbf{\Psi}$, $\mathbf{\Psi}_{i+1} = X_i\mathbf{\Psi}_i$ $(1 \le i \le m)$, and $\psi_{i,j} = (\mathbf{\Psi_i})_j$, then

$$\psi_{i+1,j} = x_{i,j}\psi_{i,j} + \psi_{i,j+1},$$
$$\psi_{m+1,j} = t_j^{-1}T_z\psi_{1,j}, \qquad \psi_{i,n+1} = r_i z\psi_{i,1}.$$

These relations are symmetric under the exchange:

$m \leftrightarrow n$, $\psi_{i,j} \leftrightarrow \psi_{j,i}$, $x_{i,j} \leftrightarrow -x_{j,i}$, $r_k \leftrightarrow t_k^{-1}$, $z \leftrightarrow T_z$. □

▶ Two Lax form of q-Garnier system

$$\Psi(qz) = \mathcal{A}(z)\Psi(z), \quad \bar{\Psi}(z) = \mathcal{B}(z)\Psi(z).$$

(i) $(m,n) = (2, 2N+2)$ case: [Suzuki (2015)]

$$\mathcal{A}(z) = \begin{bmatrix} * & * & * & & & \\ & * & * & * & & \\ & & \vdots & \vdots & & \\ & & & * & * & \\ & & & & * & \end{bmatrix} + \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ * & & & & & \\ * & * & & & & \end{bmatrix} z.$$

(ii) $(m,n) = (2N+2, 2)$ case:(= [Sakai (2005)], up to gauge)

$$\mathcal{A}(z) = \begin{bmatrix} * & * \\ & * \end{bmatrix} + \begin{bmatrix} * & * \\ * & * \end{bmatrix} z + \cdots + \begin{bmatrix} * & * \\ * & * \end{bmatrix} z^N + \begin{bmatrix} * & \\ * & * \end{bmatrix} z^{N+1}.$$

- Case (i): We consider the deformation

$$\boldsymbol{\mathcal{A}}(z) = X_2(z)X_1(z), \quad \boldsymbol{\mathcal{B}}(z) = X_1(z),$$

$d(t) = 1$. The time evolution equation ($\leftarrow$ compatibility condition $\bar{\boldsymbol{\mathcal{A}}}(z)\boldsymbol{\mathcal{B}}(z) = \boldsymbol{\mathcal{B}}(qz)\boldsymbol{\mathcal{A}}(z)$) is explicitly written as follows: (the Yang-Baxter-map)

$$\bar{x}_j = x_j \frac{P_{j+1}}{P_j}, \quad \bar{y}_j = y_j \frac{P_j}{P_{j+1}},$$

$$P_1 = \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} x_j \right) \left(\prod_{j=k+1}^{n} y_j \right), \quad P_{j+1} = \mu\pi(P_j),$$

where $\pi(x_j) = \mu^{-\delta_{j,n}} x_{j+1}$ (similar for y_j) , $\mu = \dfrac{r_2}{qr_1}$, $\bar{r}_1 = r_2$, $\bar{r}_2 = qr_1$.

- Case (ii)

$$\boldsymbol{\mathcal{A}}(z) = \begin{bmatrix} r_1^{-1} & 0 \\ 0 & r_2^{-1} \end{bmatrix} X_{2N+2} \cdots X_1, \quad \boldsymbol{\mathcal{B}}(z) = \begin{bmatrix} 0 & 1 \\ z & y_1 - \bar{x}_1 \end{bmatrix}.$$

Then, for $F(z) = \boldsymbol{\mathcal{A}}(z)_{12}$, $G(z) = \boldsymbol{\mathcal{A}}(z)_{11}$, we have

$$zF(z)\bar{F}(z) = -\det \boldsymbol{\mathcal{A}}(z) \qquad \text{for} \quad G(z) = 0,$$
$$G(z)\bar{G}(z) = \det \boldsymbol{\mathcal{A}}(z) \qquad \text{for} \quad \bar{F}(z) = 0.$$

▶ Summary

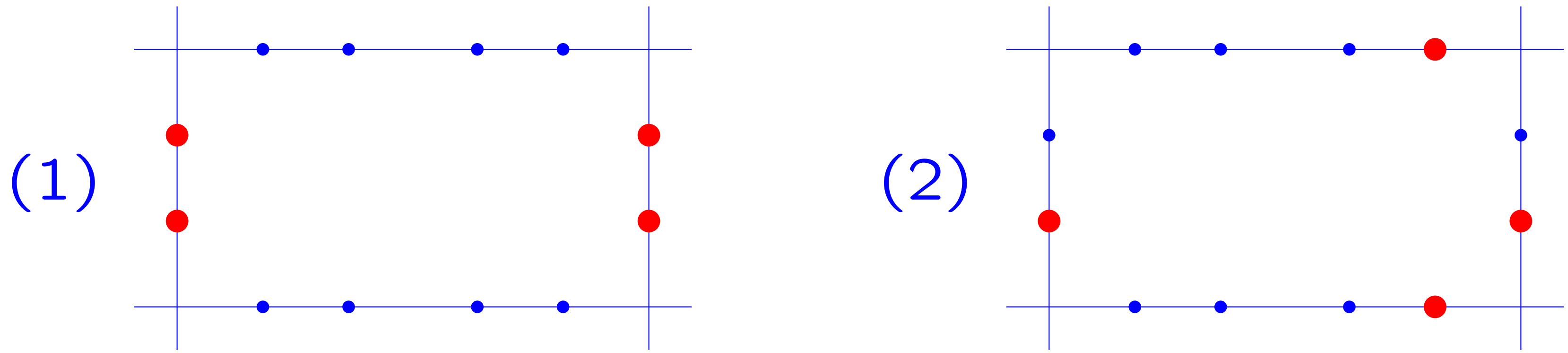
We studied the q-**Garnier system** (and elliptic Garnier system) from a geometric points of view.

- Simple form of the Lax pair and evolution equations are given.
- Autonomous limit is interpreted as generalized QRT system.
- As reduction of q-KP, two dual Lax formulations are obtained.

Thank you!

▶ Special solutions. [Nagao-Y (2016)]

Consider the configurations where the points ● are in special position (ratio $\in q^{\mathbb{Z}}$).



Then the q-Garnier system has terminating HG solutions such that the τ-**functions** are given by determinants of
(1) **q-Appell-Lauricella functions** φ_D of N-variables,
(2) **generalized q-hypergeometric functions** ${}_{N+1}\varphi_N$.

• **The idea of the derivation** : **Padé method**.

(1) Padé approximation (differential grid):

$$\psi(x) = \prod_{i=1}^{N+1} \frac{(a_i x; q)_\infty}{(b_i x; q)_\infty} = \frac{P_m(x)}{Q_n(x)} + O(x^{m+n+1}), \quad x \to 0.$$

(2) Padé interpolation (on q-grid):

$$\psi(x) = c^{\log_q x} \prod_{i=1}^{N} \frac{(a_i x; q)_\infty}{(b_i x : q)_\infty} = \frac{P_m(x)}{Q_n(x)}, \quad (x = 1, q, \ldots, q^{m+n}).$$

• **Key fact**. The functions $y(x) = P_m(x), \psi(x)Q_n(x)$ give the solutions of the Lax linear equations L_2, L_3 (and L_1) .

• **Schur function representation of τ-function**

q-Garnier:

$$\tau_{m,n} = \det\left[p_{m-i+j}\right]_{i,j=1}^{n}, \qquad \exp\left(\sum_{k=1}^{\infty} t_k x^k\right) = \sum_{k=0}^{\infty} p_k x^k,$$

$$t_k = \sum_{i=1}^{N+1} \frac{b_i^k - a_i^k}{k(1-q^k)}.$$

differential Garnier:

$$a_i = s_i q^{\alpha_i},\ b_i = s_i,\ q \to 1, \text{ hence } t_k = \sum_{i=1}^{N+1} \frac{\alpha_i s_i^k}{k}.$$