# Quantum integrable systems of elliptic Calogero-Moser type

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# 1. Preamble

- Physical perspective: Systems of Calogero-Moser type are integrable one-dimensional $N$-particle systems that come in various versions: classical/quantum, nonrelativistic/relativistic, with special interactions given by rational/trigonometric/hyperbolic/elliptic functions.
- Harmonic analysis perspective: The quantum systems amount to commutative algebras of operators associated with root systems, with the differential/difference operator case corresponding to Lie groups/quantum groups; their symbols Poisson commute and amount to the classical versions.
- This talk focuses on the quantum elliptic systems associated with the root systems $A_{N-1}$ and $BC_N$.

# 2. The nr/PDO case

- The nonrelativistic/$A_{N-1}$ quantum Calogero-Moser (CM) Hamiltonian is given by

$$H_{\mathrm{nr}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\partial_{x_j}^2 + \frac{g(g-\hbar)}{m}\sum_{1\le j<k\le N} V(x_j - x_k),$$

with $\hbar > 0$ (Planck's constant), $m > 0$ (particle mass), $g \in \mathbb{R}$ (coupling constant), $V(x)$ pair potential of four types:

$$\begin{array}{rccl}
\text{I.} & 1/x^2 & & \text{(rational)} \\
\text{II.} & \pi^2/\alpha^2 \sinh^2(\pi x/\alpha), & \alpha > 0 & \text{(hyperbolic)} \\
\text{III.} & r^2/\sin^2(rx), & r > 0 & \text{(trigonometric)} \\
\text{IV.} & \wp(x;\pi/2r, i\alpha/2), & r,\alpha > 0 & \text{(elliptic)}
\end{array}$$

- Associated integrable system ($N$ commuting PDOs):

$$H_1 = -i\hbar \sum_{j=1}^{N} \partial_{x_j}, \quad H_2 = mH_{\mathrm{nr}},$$

$$H_k = \frac{(-i\hbar)^k}{k} \sum_{j=1}^{N} \partial_{x_j}^k + \mathrm{l.\ o.}, \quad k = 3, \ldots, N,$$

where l.o. = lower order in partials.

- Physical picture:

$$H_{\mathrm{nr}}, \quad P_{\mathrm{nr}} = H_1, \quad B = -m \sum_{j=1}^{N} x_j,$$

represent the Lie algebra of the Galilei group:

$$[H_{\mathrm{nr}}, P_{\mathrm{nr}}] = 0, [H_{\mathrm{nr}}, B] = i\hbar P_{\mathrm{nr}}, [P_{\mathrm{nr}}, B] = i\hbar Nm.$$

- The 'nonrelativistic'/$BC_N$ elliptic Hamiltonian is given by

$$H_{\mathrm{nr}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\partial_{x_j}^2 + \frac{g(g-\hbar)}{m}\sum_{\substack{1\le j<k\le N\\ \delta=+,-}}\wp(x_j-\delta x_k)$$

$$+\sum_{j=1}^{N}\sum_{t=0}^{3}\frac{g_t(g_t-\hbar)}{2m}\wp(x_j+\omega_t).$$

- It was introduced by Inozemtsev, who showed integrability of the classical version. On the quantum level there also exist $N-1$ additional pairwise commuting PDOs (Oshima/H. Sekiguchi) of orders $4,\ldots,2N$.
- The $N=1$ Schrödinger equation amounts to the Heun equation.

# 3. The rel/AΔO case

## 3A. Root system $A_{N-1}$

- The elliptic relativistic/$A_{N-1}$ systems are given by $N$ commuting AΔOs (analytic difference operators)

$$H_k(x) = \sum_{|I|=k} \prod_{\substack{m\in I\\ n\notin I}} f_-(x_m - x_n) \cdot \prod_{m\in I} e^{-i\hbar\beta\partial_{x_m}} \cdot \prod_{\substack{m\in I\\ n\notin I}} f_+(x_m - x_n),$$

  where $k = 1, \ldots, N$, $\beta > 0$, and

$$f_\pm(x)^2 = \sigma(x \pm i\beta g; \pi/2r, i\alpha/2)/\sigma(x; \pi/2r, i\alpha/2).$$

  Thus,

$$f_+(x)^2 f_-(x)^2 = \sigma(i\beta g)^2(\wp(i\beta g) - \wp(x)).$$

- Physical picture: $\beta = 1/mc$ and $c$ = light speed;

$$H_{\rm rel} = mc^2[H_1(x) + H_1(-x)], \quad P_{\rm rel} = mc[H_1(x) - H_1(-x)],$$

  and $B$ yield the Lie algebra of the Poincaré group:

$$[H_{\rm rel}, P_{\rm rel}] = 0, [H_{\rm rel}, B] = i\hbar P_{\rm rel}, [P_{\rm rel}, B] = i\hbar c^{-2} H_{\rm rel}.$$

- The nonrelativistic limit $c \to \infty$ gives

$$P_{\mathrm{rel}} \to P_{\mathrm{nr}}, \quad H_{\mathrm{rel}} - Nmc^2 \to H_{\mathrm{nr}}.$$

- The hyperbolic and elliptic regimes have two length scales, namely

$$a_+ \equiv \alpha, \quad (\text{imaginary period/interaction length}),$$

  and

$$a_- \equiv \hbar/mc, \quad (\text{shift step size/Compton wave length}).$$

- The above family of AΔOs $H_k$ with $a_+$ and $a_-$ interchanged yields a second family commuting with the first one. Hence, eigenfunctions of one family that are symmetric under interchange of $a_+$ and $a_-$ (modular-invariant) are joint eigenfunctions of both families. (In the hyperbolic case this can be tied in with the modular quantum groups introduced by Faddeev.)

- To bring out modular symmetry and another $\mathbb{Z}_2$ symmetry, it is crucial to reparametrize the commuting AΔOs $H_1, \ldots, H_N$. To this end (and also for later purposes) we need the elliptic gamma function $G(r, a_+, a_-; z)$ and allied functions. We have

$$G(z) := \prod_{m,n=0}^{\infty} \frac{1 - q_+^{2m+1} q_-^{2n+1} e^{-2irz}}{1 - q_+^{2m+1} q_-^{2n+1} e^{2irz}},$$

  where $q_\pm := \exp(-ra_\pm)$. It corresponds to two elliptic curves with real period $\pi/r$ and imaginary periods $ia_+, ia_-$.
- We also need the RHS functions in the AΔEs to which $G$ is the minimal solution:

$$\frac{G(z + ia_\delta/2)}{G(z - ia_\delta/2)} = R_{-\delta}(z), \quad \delta = +, -,$$

$$R_\delta(z) = R(r, a_\delta; z) = \prod_{l=0}^{\infty} (1 - q_\delta^{2l+1} e^{2irz})(z \to -z).$$

  (Thus $R_\delta$ is even and $\pi/r$-periodic.)

- Crucial $G$-features are

$$G(r, a_+, a_-; z) = G(r, a_-, a_+; z), \quad \text{(modular invariance)}$$

$$G(\lambda^{-1} r, \lambda a_+, \lambda a_-; \lambda z) = G(r, a_+, a_-; z), \quad \text{(scale invariance)}$$

$$G(-z) = 1/G(z), \quad \text{(reflection equation)}$$

$$G(z) = \exp\left(i \sum_{n=1}^{\infty} \frac{\sin(2nrz)}{2n \sinh(nra_+) \sinh(nra_-)}\right), \quad |\Im z| < (a_+ + a_-)/2.$$

$$\lim_{a_- \downarrow 0} \frac{G(r, a_+, a_-; z - ia_-\kappa)}{G(r, a_+, a_-; z - ia_-\lambda)} = \exp((\lambda - \kappa) \ln R(r, a_+; z)).$$

- Relation to conventions for elliptic hypergeometric functions: Put $p = q_+^2$, $q = q_-^2$, $x = \exp(2irz)$, to get

$$\theta_p(x) = R(r, a_+; z + ia_+/2), \ \theta_q(x) = R(r, a_-; z + ia_-/2),$$

$$\Gamma_{p,q}(x) = G(r, a_+, a_-; z - i(a_+ + a_-)/2).$$

- Returning to the AΔOs, we need a Harish-Chandra function

$$c(z) := G(z + ia - ib)/G(z + ia), \quad a := (a_+ + a_-)/2,$$

weight function $w(z) := 1/c(z)c(-z)$ and scattering function

$$u(z) := c(z)/c(-z).$$

Their multi-variate versions are

$$F(x) := \prod_{1\leq j<k\leq N} f(x_j - x_k), \quad f = c, w, u.$$

- Setting

$$\rho_{\delta,\pm}(z) := R_\delta(z \pm (ia_\delta/2 - ib))/R_\delta(z \pm ia_\delta/2),$$

we introduce $2N$ commuting Hamiltonians

$$H_{k,\delta}(x) := \sum_{|I|=k} \prod_{\substack{m\in I \\ n\notin I}} \Big(\rho_{\delta,+}(x_m - x_n)\rho_{\delta,-}(x_m - x_n - ia_{-\delta})\Big)^{1/2}$$

$$\times \prod_{m\in I} e^{-ia_{-\delta}\partial_{x_m}}, \quad k = 1, \ldots, N, \quad \delta = +, -.$$

- Now $H_{k,+}$ amounts to the previous $H_k$ up to a multiplicative constant. The present normalization entails invariance under $b \mapsto 2a - b$.
- We also need $2N$ AΔOs

$$A_{k,\delta}(x) := W(x)^{-1/2} H_{k,\delta}(x) W(x)^{1/2}.$$

Using the $G$-AΔEs they can be written as

$$A_{k,\delta}(x) = \sum_{|I|=k} \prod_{\substack{m\in I \\ n\notin I}} \rho_{\delta,+}(x_m - x_n) \cdot \prod_{m\in I} e^{-ia_{-\delta}\partial_{x_m}}.$$

They are not invariant under $b \mapsto 2a - b$, since $W(x)$ is not. But since $U(x)$ is invariant, the AΔOs

$$\mathcal{A}_{k,\delta} := U(x)^{-1/2} H_{k,\delta} U(x)^{1/2} = C(x)^{-1} A_{k,\delta} C(x),$$

are invariant. Each of these three AΔO-families is crucial for further developments.

# 3. The rel/AΔO case

## 3B. Root system $BC_N$

- A 'relativistic' Hamiltonian $H_{vD}$ for the $BC_N$ case is due to van Diejen; the associated $N-1$ commuting Hamiltonians were shown to exist by Hikami/Komori, and will not be considered here. As in the $A_{N-1}$ case, we need AΔOs $H_{\pm}$, $A_{\pm}$ and $\mathcal{A}_{\pm}$, with $H_+$ of the form
$$H_+ = C_1 H_{vD} + C_2, \quad C_1, C_2 \in \mathbb{C}^*.$$
As before, these choices reveal non-manifest symmetries.
- In order to detail the $N=1$ AΔOs, we again need a Harish-Chandra function
$$c_e(z) := \frac{1}{G(2z+ia)} \prod_{\mu=0}^{7} G(z - i\gamma_\mu), \quad \gamma_0, \ldots, \gamma_7 \in \mathbb{C},$$
weight function $w_e(z) := 1/c_e(z)c_e(-z)$ and scattering function $u_e(z) := c_e(z)/c_e(-z)$.

- Once again, we have the relations

$$A_\delta(z) = w_e(z)^{-1/2} H_\delta(z) w_e(z)^{1/2},$$

$$\mathcal{A}_\delta(z) = u_e(z)^{-1/2} H_\delta(z) u_e(z)^{1/2} = c_e(z)^{-1} A_\delta(z) c_e(z).$$

Here, $A_\delta$ is of the form

$$A_\delta = V_\delta(z) \exp(-ia_{-\delta}\partial_z) + (z \to -z) + V_{b,\delta}(z),$$

with

$$V_\delta(z) := c_e(z)/c_e(z - ia_{-\delta}).$$

- Letting

$$V_{a,\delta}(z) := V_\delta(-z) V_\delta(z + ia_{-\delta}),$$

it follows that we have

$$H_\delta = V_{a,\delta}(z)^{1/2} \exp(ia_{-\delta}\partial_z) + (z \to -z) + V_{b,\delta}(z),$$

$$\mathcal{A}_\delta = \exp(-ia_{-\delta}\partial_z) + V_{a,\delta}(z) \exp(ia_{-\delta}\partial_z) + V_{b,\delta}(z).$$

- Using the $G$-A$\Delta$Es, the functions $V_\delta(z)$ and $V_{a,\delta}(z)$ can be expressed solely in terms of $R_\delta(z)$. In particular,

$$V_{a,\delta}(z) = D_\delta(z)^{-1} \prod_{\mu=0}^{7} \prod_{\tau=+,-} R_\delta(z + \tau i\gamma_\mu + ia_{-\delta}/2),$$

  with the denominator $D_\delta(z)$ a product of $\gamma$-independent $R_\delta$-functions. As a result, $V_{a,\delta}(z)$ is elliptic in $z$ and has $B_8$-symmetry in $\gamma$. (I. e., invariance under $S_8$ and sign flips.)
- The additive potential $V_{b,\delta}(z)$ is also elliptic and can be characterized in terms of its residues at 4 simple poles in a period cell. It admits an explicit formula from which $D_8$-symmetry in $\gamma$ can be read off. (I. e., $S_8$ and even sign flips.)
- As a consequence, the A$\Delta$Os $H_\pm$ and $\mathcal{A}_\pm$ are $D_8$-invariant. (But $w_e(z)$ is not, so $A_\pm$ are not.)

- The generators $S_0, S_1, S_2, S_3$ of the Sklyanin algebra have representations (labeled by $\nu \in \mathbb{C}^*$) as AΔOs acting on even meromorphic functions. In these representations the quadratic part of the algebra is 9-dimensional. It can be viewed as the linear combinations of the van Diejen AΔOs $A_+(z)$ (with $\sum_\mu \gamma_\mu$ fixed), plus the constants. In fact, the generators themselves are represented by AΔOs that can be regarded as special van Diejen AΔOs. (See E. Rains/S. R., CMP 2013 for these results and other ones.)
- The 4-coupling Heun operator can be tied in with Painlevé VI (via the so-called Painlevé-Calogero correspondence). The conjecture (S. R., Bonn EIS Workshop 2008) that the 8-coupling 'relativistic' Heun (i. e., van Diejen) operator has a similar relation to the Sakai elliptic difference Painlevé equation is still open, but Takemura has recently shown that this relation holds true at lower levels in the Sakai hierarchy.

- Turning finally to 'relativistic' $BC_N$ with $N > 1$, the commuting modular pair $H_\pm$ of defining Hamiltonians is of the form

$$\sum_{j=1}^{N} \Big( \mathcal{V}_{j,\pm}(x)^{1/2} e^{-ia_\mp \partial_{x_j}} \mathcal{V}_{j,\pm}(-x)^{1/2} + (x \to -x) \Big) + \mathcal{V}_\pm(x).$$

Here, we have

$$\mathcal{V}_{j,\delta}(x) := V_\delta(x_j) \prod_{\substack{k \neq j \\ \tau=+,-}} \frac{R_\delta(x_j - \tau x_k - ib + ia_\delta/2)}{R_\delta(x_j - \tau x_k + ia_\delta/2)},$$

with $V_\delta(z)$ the previous $BC_1$ coefficient, and with $\mathcal{V}_\delta(x)$ an elliptic function whose definition we skip.

- Next, we introduce the Harish-Chandra function

$$C(x) := \prod_{j=1}^{N} c_e(x_j) \cdot \prod_{\substack{1 \leq j < k \leq N \\ \tau=+,-}} \frac{G(x_j - \tau x_k - ib + ia)}{G(x_j - \tau x_k + ia)},$$

weight function $W(x) := 1/C(x)C(-x)$ and scattering function $U(x) := C(x)/C(-x)$.

- Then we get again the two $H_\delta$-avatars

$$A_\delta(x) := W(x)^{-1/2} H_\delta(x) W(x)^{1/2},$$

and

$$\mathcal{A}_\delta(x) := U(x)^{-1/2} H_\delta(x) U(x)^{1/2} = C(x)^{-1} A_\delta(x) C(x).$$

- The AΔOs $A_\pm$ and $H_\pm$ are $BC_N$-invariant, whereas $\mathcal{A}_\pm$ are not invariant under sign changes of $x_j$ (since $C(x)$ is not). The AΔOs $\mathcal{A}_\pm$ and $H_\pm$ are $D_8$-invariant, whereas $A_\pm$ are not invariant under even sign changes of $\gamma_\mu$ (since $C(x)$ is not).
- This 9-coupling family admits a great many degenerations and limits. In particular, the trigonometric specialization of $A_+$ is the 5-coupling Koornwinder AΔO, which has Koornwinder-Macdonald polynomials as eigenfunctions, and the 'nonrelativistic' limit of $H_+$ yields the previous 5-coupling Inozemtsev PDO.

# 4. Eigenfunctions

- Given a set of commuting operators, the obvious first problem is to show or rule out the existence of joint eigenfunctions. In case joint eigenfunctions exist, the next problem is to obtain explicit information about them. Finally, with sufficient information available, the problem of finding a Hilbert space reinterpretation of the commuting operators can be addressed.
- For the Hilbert space joint eigenfunction problem, the spectral theorem is of little use, since it assumes the existence of commuting self-adjoint operators. The PDOs/AΔOs are only formally self-adjoint, however.
- Especially in the AΔO case, there are hardly any 'useful' existence results available. In fact, already for the 1-variable case there are simple examples of commuting AΔOs without joint eigenfunctions.

- To explain this in more detail, we first look at

$$A = \exp(-ia\partial_z), \quad B = \exp(-ib\partial_z), \quad a, b > 0, \quad a/b \notin \mathbb{Q}.$$

The AΔOs $A$ and $B$ commute, but the only solutions to the joint eigenvalue equation $AF = F, \ BF = F$, are the constant functions.

- Now consider the AΔO pair

$$C = (1 + \exp(2\pi z/b))A, \quad D = (1 + \exp(2\pi z/a))B.$$

Clearly, $C$ and $D$ still commute. Even so, no joint solutions to

$$CF = \lambda F, \quad DF = \mu F,$$

exist for any $\lambda, \mu \in \mathbb{C}$. (This can be proved by first solving each equation via the hyperbolic gamma function, and then requiring equality to arrive at a contradiction.)

- Abundant results on eigenfunctions exist for the Lamé/Heun cases (equivalently, the nonrelativistic $A_1/BC_1$ cases). Far less is known about their relativistic counterparts (more on this in my Thursday seminar).
- For $A_{N-1}$ with $N > 2$ there are results of 'Bethe Ansatz' type. They are restricted to certain discrete couplings and to the defining Hamiltonian (Felder/Varchenko for the PDO case, Billey for the AΔO case); likewise, under these restrictions finite-dimensional invariant subspaces have been shown to exist (Hasegawa, Hikami/Komori).
- Results by Komori/Takemura on the $A_{N-1}$ nr/PDO case yield existence of joint Hilbert space eigenfunctions reducing to (basically) the Jack-Sutherland polynomials in the trigonometric limit. Since perturbation theory is used, restrictions on the imaginary period and the coupling are present.

# 5. Kernel functions

- Given a pair of operators $H_1(x)$ and $H_2(y)$, a kernel function is a function $K(x, y)$ satisfying

$$H_1(x)K(x, y) = H_2(y)K(x, y).$$

  Here, $x$ and $y$ may vary over spaces of different dimension. Reinterpreting $K(x, y)$ as the kernel of an integral operator $\mathcal{I}$, the operator $\mathcal{I}$ can be used (in "favorable" cases, as explained later) to connect eigenfunctions of $H_2$ to those of $H_1$.
- For the above elliptic $N$-variable Hamiltonians, kernel functions with both $x$ and $y$ varying over $\mathbb{C}^N$ are known, imposing one coupling constraint for the $BC_N$ case with $N > 1$. Probably the earliest result (with $H_1, H_2$ Lamé operators) is due to Whittaker (1915).

- The first multi-variate kernel function result was obtained by Langmann (2000). It pertains to the defining $A_{N-1}$ PDO. Specifically, $H_1$ and $H_2$ equal (with $m = \hbar = 1$)

$$H_{nr} = -\frac{1}{2}\sum_{j=1}^{N} \partial_{x_j}^2 + g(g-1)\sum_{1\le j<k\le N} \wp(x_j - x_k),$$

and his kernel function amounts to

$$W_{nr}(x)^{1/2} W_{nr}(y)^{1/2} \prod_{j,k=1}^{N} R(x_j - y_k + \xi)^{-g},$$

$$W_{nr}(x) := \Big(\prod_{1\le j<k\le N} R(x_j - x_k + i\alpha/2) R(x_j - x_k - i\alpha/2)\Big)^g.$$

He has used this as a starting point to derive perturbative formulas for $H_{nr}$-eigenfunctions.

- In later work (partly joint with Takemura), he obtains so-called source identities. They can be specialized to obtain various kernel identities for elliptic PDOs with more than one mass.

- Kernel functions for the $2N$ commuting $A_{N-1}$ AΔOs were first presented at the Kyoto EIS Workshop (S. R., 2004). For $A_{k,\delta}$ one can take in particular

$$K_\xi(x,y) = \prod_{j,k=1}^{N} \frac{G(x_j - y_k - ib/2 + \xi)}{G(x_j - y_k + ib/2 + \xi)}, \quad \xi \in \mathbb{C}.$$

- Taking the nonrelativistic limit of the $H_{k,\delta}$-kernel function

$$W(x)^{1/2} W(y)^{1/2} K_\xi(x,y),$$

  we get Langmann's kernel function, together with the kernel function property for the higher-order commuting PDOs.
- At the 2004 Kyoto EIS Workshop I also introduced similar kernel functions for the defining $BC_N$ AΔO and PDO. (For $N > 1$ one balancing condition occurs.)

- The concept of a kernel function is still unfamiliar to many colleagues. Once it is understood, a natural question is: What are kernel functions good for?
- Indeed, given an operator $H(x)$ with eigenfunctions

$$H(x)\psi_m(x) = E_m\psi_m(x), \quad m = 0, 1, 2, ..., M \leq \infty,$$

  any function $K(x, y)$ of the form

$$K(x, y) = \sum_{m=0}^{M} \lambda_m\psi_m(x)\psi_m(y),$$

  satisfies the kernel identity

$$H(x)K(x, y) = H(y)K(x, y)$$

  (formally in case $M = \infty$). As a consequence, kernel functions exist in profusion.
- Key point: Once one has found such a kernel identity for a given Hamiltonian $H$, one can use $K(x, y)$ in "favorable" cases as the kernel of an integral operator $\mathcal{I}$ whose eigenfunctions are also $H$-eigenfunctions.

- To explain why the qualifier "favorable" is needed, consider e. g. a finite-rank kernel of the form

$$K(x,y)=\sum_{m=0}^{M}\lambda_m\psi_m(x)\psi_m(y),\quad 0<\lambda_0<\cdots<\lambda_M,$$

with $\psi_m(x)$ real-valued smooth functions such that

$$\int_0^1\psi_m(x)\psi_n(x)dx=\delta_{nm}.$$

Thus $\mathcal{I}$ is a self-adjoint operator on $L^2((0,1),dx)$ with eigenfunctions $\psi_0,\ldots,\psi_M$ and infinite-dimensional null space.
- Snag: A kernel identity $(H(x)-H(y))K(x,y)$ does not imply that $H(x)$ has eigenfunctions $\psi_m(x)$. For instance, take $M=1$ and define $H$ to be zero on $\{\psi_0,\psi_1\}^{\perp}$, and

$$(H\psi_0)(x):=E_0\psi_0(x)+c\lambda_1\psi_1(x),\ (H\psi_1)(x):=E_1\psi_1(x)+c\lambda_0\psi_0(x),$$

with $E_0,E_1,c>0$ (say). Then the kernel identity easily follows, yet it is plain that $\psi_0$ and $\psi_1$ are not eigenfunctions of $H$.

- Worse yet, for the above elliptic commuting Hamiltonians, it is not at all clear that the explicit kernel functions just surveyed have a bearing on eigenfunctions. Indeed, to begin with, the existence and features of joint eigenfunctions are unknown in most cases.
- Crux: It can be shown that the (very special) kernel functions at hand do give rise to "favorable" cases, provided suitable Hilbert spaces are chosen and the couplings are restricted to suitable polytopes.
- More specifically, for the relativistic cases the kernel functions furnish the only tool (to date) to solve the long-standing problem of promoting the commuting AΔOs to bona fide self-adjoint commuting Hilbert space operators, with an orthonormal base of eigenfunctions arising from the integral operators associated to the kernel functions.
- In my Thursday talk I shall explain this in more detail.