Elliptic Dyson Models

Makoto Katori (Chuo Univ., Tokyo)

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Plan of my talk

1. Introduction

Martingales and 1-dim. Brownian Motion

Complex BM and Conformal Invariance

Entire Functions and Determinantal Martingale

Determinantal Measures and Processes

Problem

2. Results

Theorem 1, Theorem 2

3. Key Lemmas and Open Problems

1. Introduction

- **Stochastic analysis on interacting particle systems** is important to provide useful models describing **equilibrium and non-equilibrium phenomena studied in statistical physics**.

- **Determinantal process** is a stochastic system of interacting particles which is **integrable** in the sense that all spatio-temporal correlation functions are given by **determinants**. And all of them are controlled by a single function called the **spatio-temporal correlation kernel**.

- The **stochastic integrability of determinantal processes** is proved by showing that the Laplace transform of any multi-time joint probability density is expressed by the **spatio-temporal Fredholm determinant** associated with the correlation kernel.

[BR05] Borodin, A., Rains, E. M.: Eynard-Mehta theorem, Schur process, and their Pfaffian analog. J. Stat. Phys. 121, 291-317 (2005)

[KT07] Katori, M., Tanemura, H.: Noncolliding Brownian motion and determinantal processes. J. Stat. Phys. 129, 1233-1277 (2007)

- The purpose of this talk is to present new kinds of determinantal processes in which the interactions between particles are described by the **logarithmic derivatives of Jacobi's theta functions**.

- A classical example of determinantal processes is Dyson's Brownian motion model with parameter $\beta = 2$, which is a dynamical version of the eigenvalue statistics of random matrices in the Gaussian unitary ensemble (GUE), and we call it simply the **Dyson model**.

  [K16a] Katori, M.: "Bessel Processes, Schramm-Loewner Evolution, and the Dyson Model", SpringerBriefs in Mathematical Physics 11, Springer, Tokyo, (2016)

- We will extend the Dyson model to the **elliptic-function-level** in this talk.

- We use the notion of **martingales** in probability theory and the **elliptic determinantal evaluations of the Macdonald denominators** of the seven families of irreducible reduced affine root systems given by Rosengren and Schlosser (2006).

  [RS06] Rosengren, H., Schlosser, M.: Elliptic determinant evaluations and the Macdonald identities for affine root systems. Compositio Math. 142, 937-961 (2006)

- The present talk is based on the following three papers;

For Type A_{N-1}:

[K15] Katori, M.: Elliptic determinantal process of type A. Probab. Theory Relat. Fields 162, 637-677 (2015)

[K16b] Katori, M.: Elliptic Bessel processes and elliptic Dyson models realized as temporally inhomogeneous processes. J. Math. Phys. 57, 103302/1-32 (2016)

For Types $\mathrm{B}_N, \mathrm{B}_N^{\vee}, \mathrm{C}_N, \mathrm{C}_N^{\vee}, \mathrm{BC}_N, \mathrm{D}_N$:

[K17] Katori, M.: Elliptic determinantal processes and elliptic Dyson models. arXiv:math.PR/1703.03914

Martingales and 1-dim. Brownian Motion

- **Martingales** are the stochastic processes preserving their mean values and thus they represent **fluctuations**.

- A typical example of martingale is the **one-dim. standard Brownian motion**. Let $B(t), t \geq 0$ denote the position of the standard Brownian motion in $\mathbb{R}$ starting from the origin 0 at time $t = 0$.

- The **transition probability density** from $x \in \mathbb{R}$ to $y \in \mathbb{R}$ in time $t \geq 0$ is given by

$$p_{\mathrm{BM}}(t, y|x) = \begin{cases} \dfrac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}}, & t > 0, \\ \delta(x-y), & t = 0. \end{cases}$$

- For an arbitrary time sequence $0 \equiv t_0 < t_1 < \cdots < t_M < \infty$, $M \in \mathbb{N} \equiv \{1, 2, \dots\}$, and for any $A_m \in \mathcal{B}(\mathbb{R})$, $m = 1, 2, \dots, M$, $(x_0 \equiv 0)$,

$$\mathrm{P}\Big[B(t_m) \in A_m, m = 1, 2, \dots, M\Big] = \int_{A_1} dx^{(1)} \cdots \int_{A_M} dx^{(M)} \prod_{m=1}^{M} p_{\mathrm{BM}}(t_m - t_{m-1}, x^{(m)}|x^{(m-1)}).$$

- The collection of all paths is denoted by Ω and there is a subset $\widetilde{\Omega} \subset \Omega$ such that $\mathrm{P}[\widetilde{\Omega}] = 1$ and for any realization of path $\omega \in \widetilde{\Omega}$, $B(t) = B(t, \omega), t \geq 0$ is a real continuous function of t. In other words, $B(t), t \geq 0$ has a continuous path almost surely (a.s. for short).

- For each $t \in [0, \infty)$, we write the smallest σ-field (completely additive class of events) generated by the Brownian motion up to time t as $\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t)$.

- We have a nondecreasing family $\{\mathcal{F}_t : t \geq 0\}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for $0 \leq s < t < \infty$, which we call a **filtration**, and put $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t$.

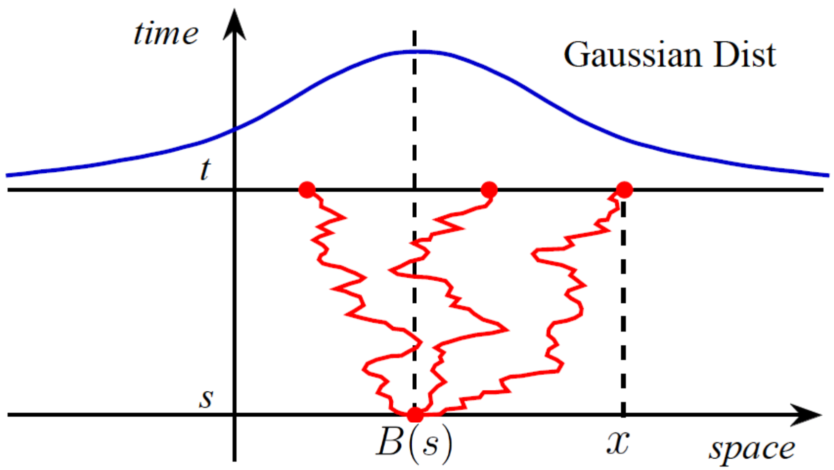
- The triplet $(\Omega, \mathcal{F}, \mathrm{P})$ is called the **probability space** for the one-dimensional standard Brownian motion.

- The **expectation** with respect to the probability law P is written as E.

- When we see $p_{\mathrm{BM}}(t, y|x)$ as a function of y, it is nothing but the probability density of the normal distribution with mean x and variance t, and hence it is easy to verify that

$$\mathrm{E}[B(t)|\mathcal{F}_s] = \int_{-\infty}^{\infty} x p_{\mathrm{BM}}(t-s, x|B(s))dx = B(s), \quad \text{a.s.} \quad 0 \leq s < t < \infty,$$

which means that $B(t), t \geq 0$ is a martingale.



- We see, however, $B(t)^n, t \geq 0, n \in \{2, 3, \dots\}$ are not martingales, since the generating function of $\mathrm{E}[B(t)^n|\mathcal{F}_s], n \in \mathbb{N}_0 \equiv \{0, 1, 2, \dots\}$, $0 < s \leq t < \infty$, with parameter $\alpha \in \mathbb{C}$ is calculated as

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \mathrm{E}[B(t)^n|\mathcal{F}_s] &= \mathrm{E}[e^{\alpha B(t)}|\mathcal{F}_s] = \int_{-\infty}^{\infty} e^{\alpha x} \frac{e^{-(x-B(s))^2/2(t-s)}}{\sqrt{2\pi(t-s)}} dx \\ &= e^{\alpha B(s)+(t-s)\alpha^2/2} \neq e^{\alpha B(s)}, \quad \text{a.s.}\end{aligned}$$

Complex BM and Conformal Invariance

- Now we assume that $\widetilde{B}(t), t \geq 0$ is a one-dimensional standard Brownian motion which is independent of $B(t), t \geq 0$, and its probability space is denoted by $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathrm{P}})$.
- Then we introduce the **complex Brownian motion** ($i \equiv \sqrt{-1}$),

$$Z(t) = B(t) + i\widetilde{B}(t), \quad t \geq 0.$$

- The probability space of $Z(t), t \geq 0$ is given by the product space $(\Omega, \mathcal{F}, \mathrm{P}) \otimes (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathrm{P}})$ and we write the expectation as $\mathbf{E} = \mathrm{E} \otimes \widetilde{\mathrm{E}}$.
- For the complex Brownian motion, by the independence of its real and imaginary parts,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \mathbf{E}[Z(t)^n | \mathcal{F}_s \otimes \widetilde{\mathcal{F}}_s] &= \mathbf{E}[e^{\alpha Z(t)} | \mathcal{F}_s \otimes \widetilde{\mathcal{F}}_s] \\
&= \mathrm{E}[e^{\alpha B(t)} | \mathcal{F}_s] \times \widetilde{\mathrm{E}}[e^{i\alpha \widetilde{B}(t)} | \widetilde{\mathcal{F}}_s] = e^{\alpha B(s) + (t-s)\alpha^2/2} \times e^{i\alpha \widetilde{B}(s) - (t-s)\alpha^2/2} \\
&= e^{\alpha Z(s)} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} Z(s)^n, \quad \text{a.s.} \quad 0 \leq s < t < \infty.
\end{aligned}$$

- The above implies that for any $n \in \mathbb{N}_0$, $Z(t)^n, t \geq 0$ is a **martingale**.

- This observation will be generalized as the following stronger statement;

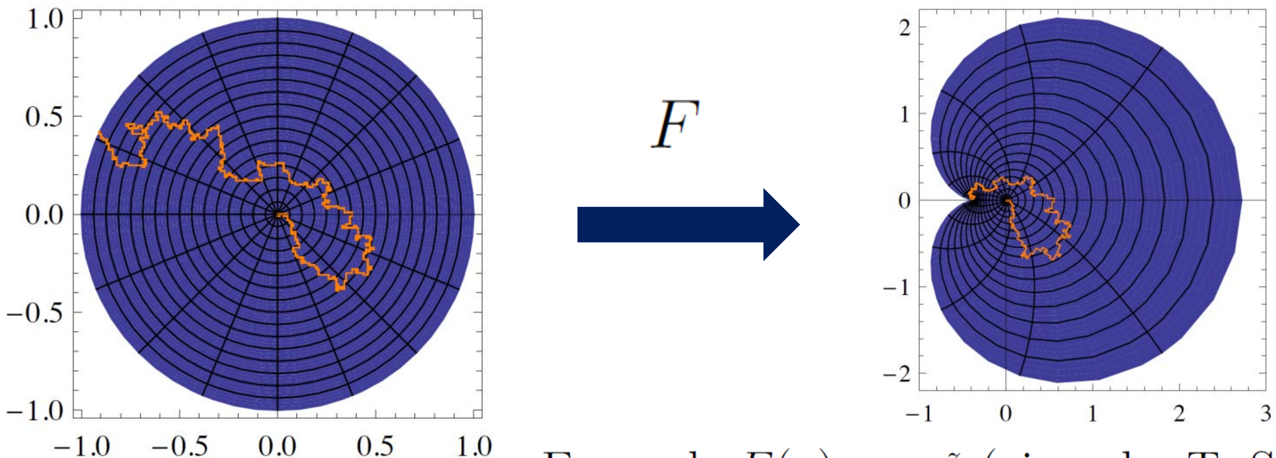
- If F is an **entire and non-constant function**, then $F(Z(t)), t \geq 0$ is a **time change of a complex Brownian motion**;

$$\Big(F(Z(t)) - F(Z(0))\Big)_{t \geq 0} \overset{\text{(law)}}{=} \Big(Z(T(t))\Big)_{t \geq 0},$$

$$\text{with a time change} \quad T(t) = \int_0^t |F'(Z(s))|^2 ds, \quad t \geq 0.$$

- This theorem is known as the **conformal invariance of the complex Brownian motion**, since $F(Z(t)), t \geq 0$ is a conformal map of $Z(t), t \geq 0$ (see, for instance, Section V.2 of [RY99]).

[RY99] Revuz, D., Yor, M.: "Continuous Martingales and Brownian Motion", 3rd edition, Springer, New York, (1999)



Example $F(z) = ze^z$ (given by T. Shirai)

- Hence $F(Z(t)), t \geq 0$ is a martingale;

$$\mathbf{E}[F(Z(t))|\mathcal{F}_s \otimes \widetilde{\mathcal{F}}_s] = F(Z(s)) \quad \text{a.s.} \quad 0 \leq s < t < \infty.$$

- If we take the expectation $\widetilde{\mathrm{E}}$ with respect to $\Im Z(\cdot) = \widetilde{B}(\cdot)$ of the both sides of the above equality, we have

$$\mathrm{E}\left[\widetilde{\mathrm{E}}[F(Z(t))]\middle|\mathcal{F}_s\right] = \widetilde{\mathrm{E}}[F(Z(s))] \quad \text{a.s.} \quad 0 \leq s < t < \infty.$$

- In this way we can obtain a martingale $\widehat{F}(t, B(t)) \equiv \widetilde{\mathrm{E}}[F(Z(t))], t \geq 0$ with respect to the one-dimensional Brownian motion.

- The present argument implies that if we have proper **entire functions**, then we will obtain **useful martingales** describing **intrinsic fluctuations involved in the interacting particle systems**.

Entire Functions and Det. Martingale

- Let $f_j, j \in \mathbb{Z}$, be an infinite series of linearly independent **entire functions**.
- For $N \in \{2, 3 \dots\}$, we define the Weyl chamber

$$\mathbb{W}_N = \{\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\},$$

and assume that $\boldsymbol{u} = (u_1, u_2, \dots, u_N) \in \mathbb{W}_N$.

- We then define a set of N distinct **entire and non-constant functions** of $z \in \mathbb{C}$ as

$$\Phi_{\boldsymbol{u}, u_j}(z) = \frac{\det_{1 \le \ell, m \le N} \Big[f_\ell(u_m + (z - u_m)\delta_{mj}) \Big]}{\det_{1 \le \ell, m \le N} \Big[f_\ell(u_m) \Big]}, \quad j = 1, 2, \dots, N.$$

- By this definition, $\det_{1 \le j,k \le N} \Big[\Phi_{\boldsymbol{u}, u_j}(z_k) \Big] = \frac{\det_{1 \le j,k \le N} \Big[f_j(z_k) \Big]}{\det_{1 \le j,k \le N} \Big[f_j(u_k) \Big]}, \ \boldsymbol{z} = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N.$

- We consider N pairs of **independent copies** $(B_k(t), \widetilde{B}_k(t)), t \geq 0, k = 1, 2, \dots, N$ of $(B(t), \widetilde{B}(t)), t \geq 0$ and define N independent complex Brownian motions

$$Z_k(t) = u_k + B_k(t) + i\widetilde{B}_k(t), \quad t \geq 0, \quad k = 1, 2, \dots, N,$$

each of which starts from $u_k \in \mathbb{R}$.

- The probability law and expectation of them are denoted by $\mathbf{P}_{\boldsymbol{u}} = \mathrm{P}_{\boldsymbol{u}} \otimes \widetilde{\mathrm{P}}$ and $\mathbf{E}_{\boldsymbol{u}} = \mathrm{E}_{\boldsymbol{u}} \otimes \widetilde{\mathrm{E}}$, respectively.

- Then for each complex Brownian motion $Z_k(t), t \geq 0$, $k = 1, 2, \dots, N$, we have N distinct **conformal maps** $\Phi_{\boldsymbol{u}, u_j}(Z_k(t)), t \geq 0$, $j = 1, 2, \dots, N$.

- By the conformal invariance of the complex Brownian motion mentioned above, they are N **distinct time-changes of complex Brownian motions** started from the real values, 0 or 1;

$$\Phi_{\boldsymbol{u},u_j}(Z_k(0)) = \Phi_{\boldsymbol{u},u_j}(u_k) = \delta_{jk}, \quad j,k = 1,2,\dots,N.$$

- Therefore, we can conclude that, if we **take expectation** $\widetilde{\mathrm{E}}$, we will obtain N **distinct martingales**,

$$\begin{aligned} M_{\boldsymbol{u},u_j}(t, B_k(t)) &\equiv \widetilde{\mathrm{E}}[\Phi_{\boldsymbol{u},u_j}(B_k(t) + i\widetilde{B}_k(t))] \\ &= \frac{\det_{1 \le \ell,m \le N}[\widehat{f}_\ell(t\delta_{mj}, u_m + (B_k(t) - u_m)\delta_{mj})]}{\det_{1 \le \ell,m \le N}[\widehat{f}_\ell(0, u_m)]}, \quad t \ge 0, \quad j = 1,2,\dots,N, \end{aligned}$$

for each one-dimensional Brownian motion $B_k(t), t \ge 0$, $k = 1,2,\dots,N$, where

$$\widehat{f}_\ell(t,x) = \widetilde{\mathrm{E}}[f_\ell(x + i\widetilde{B}(t))] = \int_{-\infty}^{\infty} f_\ell(x + i\widetilde{x}) p_{\mathrm{BM}}(t, \widetilde{x}|0) d\widetilde{x}, \quad \ell \in \mathbb{Z}.$$

- Then we define a function of $t \geq 0$ and $\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ by

$$D_{\boldsymbol{u}}(t, \boldsymbol{x}) = \det_{1 \leq j,k \leq N} [M_{\boldsymbol{u},u_j}(t, x_k)],$$

which is a martingale as a functional of $\boldsymbol{B}(t) = (B_1(t), B_2(t), \dots, B_N(t)), t \geq 0$;

$$\mathrm{E}_{\boldsymbol{u}}[D_{\boldsymbol{u}}(t, \boldsymbol{B}(t)) | \mathcal{F}_s] = D_{\boldsymbol{u}}(s, \boldsymbol{B}(s)), \quad 0 \leq s < t < \infty.$$

- By the multi-linearity of determinant, we see

$$D_{\boldsymbol{u}}(t, \boldsymbol{x}) \;=\; \widetilde{\mathrm{E}}\left[\det_{1 \leq j,k \leq N} [\Phi_{\boldsymbol{u},u_j}(x_k + i\widetilde{B}_k(t))] \right] = \frac{\det_{1 \leq j,k \leq N} [\widehat{f}_j(t, x_k)]}{\det_{1 \leq j,k \leq N} [\widehat{f}_j(0, u_k)]}.$$

- We call $D_{\boldsymbol{u}}(t, \boldsymbol{B}(t)), t \geq 0$, the **determinantal martingale**.

Determinantal Measures and Processes

- Now we introduce a measure $\widehat{\mathbb{P}}_{\boldsymbol{u}}$, which is complex-valued in general, but is absolutely continuous to $\mathbf{P}_{\boldsymbol{u}}$ as

$$\widehat{\mathbb{P}}_{\boldsymbol{u}}\Big|_{\mathcal{F}_t} = D_{\boldsymbol{u}}(t, \boldsymbol{B}(t))\mathbf{P}_{\boldsymbol{u}}\Big|_{\mathcal{F}_t}, \quad t \geq 0.$$

- This measure defines N particle systems on $\mathbb{R}$,

$$\widehat{\boldsymbol{X}}(t) = (\widehat{X}_1(t), \dots, \widehat{X}_N(t)), \quad t \geq 0,$$

starting from $\boldsymbol{u} \in \mathbb{W}_N$ and each particle of which has a continuous path a.s.

- We consider the **unlabeled configuration (a measure-valued process)** of $\widehat{\boldsymbol{X}}(t)$ as

$$\widehat{\Xi}(t, \cdot) = \sum_{j=1}^{N} \delta_{\widehat{X}_j(t)}(\cdot), \quad t \geq 0,$$

where, for $y \in \mathbb{R}$, $\delta_y(\cdot)$ denotes the **delta measure** such that $\delta_y(\{x\}) = 1$ if $x = y$ and $\delta_y(\{x\}) = 0$ otherwise.

- Consider an arbitrary number $M \in \mathbb{N}$ and an arbitrary set of strictly increasing times $\boldsymbol{t} = \{t_1, t_2, \dots, t_M\}$, $0 \equiv t_0 < t_1 < \cdots < t_M < \infty$. Let $\mathrm{C_c}(\mathbb{R})$ be the set of all continuous real-valued functions with compact supports on $\mathbb{R}$. For $\boldsymbol{g} = (g_{t_1}, g_{t_1} \dots, g_{t_M}) \in \mathrm{C_c}(\mathbb{R})^M$, we consider the following functional of $\boldsymbol{g}$,

$$\widehat{\mathbb{L}}_{\boldsymbol{u}}[\boldsymbol{t}, \boldsymbol{g}] = \widehat{\mathbb{E}}_{\boldsymbol{u}}\left[\exp\left\{\sum_{m=1}^{M} \int_{\mathbb{R}} g_{t_m}(x) \widehat{\Xi}(t_m, dx)\right\}\right],$$

which is the **Laplace transform of the multi-time distribution function** $\widehat{\mathbb{P}}_{\boldsymbol{u}}$ on a set of times $\boldsymbol{t}$ with the functions $\boldsymbol{g}$.

- If we put $\chi_t = e^{g_t} - 1$, then the above can be written as

$$\widehat{\mathbb{L}}_{\boldsymbol{u}}[\boldsymbol{t}, \boldsymbol{g}] = \widehat{\mathbb{E}}_{\boldsymbol{u}}\left[\prod_{m=1}^{M} \prod_{j=1}^{N} \{1 + \chi_{t_m}(\widehat{X}_j(t_m))\}\right].$$

- Explicit expression is given by the following **multiple integrals**,

$$\begin{aligned} \widehat{\mathbb{L}}_{\boldsymbol{u}}[\boldsymbol{t}, \boldsymbol{g}] &= \int_{\mathbb{R}^N} d\boldsymbol{x}^{(1)} \cdots \int_{\mathbb{R}^N} d\boldsymbol{x}^{(M)} D_{\boldsymbol{u}}(t_M, \boldsymbol{x}^{(M)}) \\ &\times \prod_{m=1}^{M} \prod_{j=1}^{N} \left[p_{\mathrm{BM}}(t_m - t_{m-1}, x_j^{(m)} | x_j^{(m-1)}) \{1 + \chi_{t_m}(x_j^{(m)})\}\right], \end{aligned}$$

where $x_j^{(0)} = u_j, j = 1, 2, \dots, N$, $\boldsymbol{x}^{(m)} = (x_1^{(m)}, \dots, x_N^{(m)})$, and $d\boldsymbol{x}^{(m)} = \prod_{j=1}^{N} dx_j^{(m)}$, $m = 1, 2, \dots, M$.

- Let $\mathbf{1}(\omega)$ be the indicator function of ω; $\mathbf{1}(\omega) = 1$ if ω is satisfied, and $\mathbf{1} = 0$ otherwise.
- The following was proved as Theorem 1.3 in [K14].

[K14] Katori, M.: Determinantal martingales and noncolliding diffusion processes. Stochastic Process. Appl. 124, 3724-3768 (2014)

Proposition 1

Put $\widehat{\mathbb{K}}_{\boldsymbol{u}}(s,x;t,y) = \sum_{j=1}^{N} p_{\rm BM}(s,x|u_j) M_{\boldsymbol{u},u_j}(t,y) - \mathbf{1}(s>t) p_{\rm BM}(s-t,x|y)$,
$s,t>0$, $x,y \in \mathbb{R}$. For an arbitrary number $M \in \mathbb{N}$, an arbitrary set of strictly increasing times $\boldsymbol{t} = \{t_1, t_2, \dots, t_M\}$, $0 \equiv t_0 < t_1 < \cdots < t_M < \infty$, and $\boldsymbol{g} = (g_{t_1}, g_{t_1} \dots, g_{t_M}) \in \mathrm{C}_{\rm c}(\mathbb{R})^M$,

$$\widehat{\mathbb{L}}_{\boldsymbol{u}}[\boldsymbol{t}, \boldsymbol{g}] = \mathop{\mathrm{Det}}_{\substack{s,t\in\boldsymbol{t},\\ x,y\in\mathbb{R}}} \Big[\delta_{st}\delta(x-y) + \widehat{\mathbb{K}}_{\boldsymbol{u}}(s,x;t,y)\chi_t(y) \Big],$$

where the RHS denotes the spatio-temporal Fredholm determinant with $\widehat{\mathbb{K}}_{\boldsymbol{u}}$,

$$\mathop{\mathrm{Det}}_{\substack{s,t\in\boldsymbol{t},\\ x,y\in\mathbb{R}}} \Big[\delta_{st}\delta(x-y) + \widehat{\mathbb{K}}_{\boldsymbol{u}}(s,x;t,y)\chi_t(y) \Big]$$

$$\equiv \sum_{\substack{0\le N_m \le N,\\ 1\le m\le M}} \int_{\prod_{m=1}^{M} \mathbb{W}_{N_m}} \prod_{m=1}^{M} d\boldsymbol{x}_{N_m}^{(m)} \prod_{j=1}^{N_m} \chi_{t_m}(x_j^{(m)}) \mathop{\det}_{\substack{1\le j\le N_m, 1\le k\le N_n,\\ 1\le m,n\le M}} [\widehat{\mathbb{K}}_{\boldsymbol{u}}(t_m, x_j^{(m)}; t_n, x_k^{(n)})],$$

where $d\boldsymbol{x}_{N_m}^{(m)} = \prod_{j=1}^{N_m} dx_j^{(m)}$, $m = 1, 2, \dots, M$, and the term with $N_m = 0, 1 \le \forall m \le M$ in the RHS should be interpreted as 1.

Problem

- This proposition is general and it proves that the N-**particle system** $\widehat{\boldsymbol{X}}(t) = (\widehat{X}_1(t), \dots, \widehat{X}_N(t))$, $t \geq 0$ is **determinantal**.

- The measure
$$\widehat{\mathbb{P}}_{\boldsymbol{u}}\Big|_{\mathcal{F}_t} = D_{\boldsymbol{u}}(t, \boldsymbol{B}(t))\mathbf{P}_{\boldsymbol{u}}\Big|_{\mathcal{F}_t}, \quad t \geq 0.$$
which governs this particle system is, however, **complex-valued in general**, and hence **the system is unphysical**.

- The **problem** is to clarify the proper conditions to construct a non-negative-definite real measure, *i.e.* **the probability measure**, which defines a **physical system of interacting particles**.

- As a matter of course, this problem depends on the **choice of an infinite set of linearly independent entire functions** $f_j, j \in \mathbb{Z}$.

?

$$\Phi_{\boldsymbol{u},u_j} \Rightarrow M_{\boldsymbol{u},u_j}(t,x) \Rightarrow D_{\boldsymbol{u}}(t,\boldsymbol{x}) \Rightarrow \widehat{\mathbb{P}}_{\boldsymbol{u}} \Rightarrow \mathbb{P}_{\boldsymbol{u}}$$

2. Results

In this talk we report the results when we choose $f_j, j \in \mathbb{Z}$ as

$$\begin{aligned}
f_j^{\mathrm{A}_{N-1}}(z;\tau) &= e^{iJ^{\mathrm{A}_{N-1}}(j)z/r}\vartheta_1\left(J^{\mathrm{A}_{N-1}}(j)\tau + \frac{\mathcal{N}^{\mathrm{A}_{N-1}}z}{2\pi r} + \frac{1-(-1)^N}{4}; \mathcal{N}^{\mathrm{A}_{N-1}}\tau\right), \\
f_j^{\mathrm{R}}(z;\tau) &= e^{iJ^{\mathrm{R}}(j)z/r}\vartheta_1\left(J^{\mathrm{R}}(j)\tau + \frac{\mathcal{N}^{\mathrm{R}}z}{2\pi r}; \mathcal{N}^{\mathrm{R}}\tau\right) \\
&\quad - e^{-iJ^{\mathrm{R}}(j)z/r}\vartheta_1\left(J^{\mathrm{R}}(j)\tau - \frac{\mathcal{N}^{\mathrm{R}}z}{2\pi r}; \mathcal{N}^{\mathrm{R}}\tau\right) \quad \text{for } \mathrm{R} = \mathrm{B}_N, \mathrm{B}_N^\vee, \\
f_j^{\mathrm{R}}(z;\tau) &= e^{iJ^{\mathrm{R}}(j)z/r}\vartheta_1\left(J^{\mathrm{R}}(j)\tau + \frac{\mathcal{N}^{\mathrm{R}}z}{2\pi r} + \frac{1}{2}; \mathcal{N}^{\mathrm{R}}\tau\right) \\
&\quad - e^{-iJ^{\mathrm{R}}(j)z/r}\vartheta_1\left(J^{\mathrm{R}}(j)\tau - \frac{\mathcal{N}^{\mathrm{R}}z}{2\pi r} + \frac{1}{2}; \mathcal{N}^{\mathrm{R}}\tau\right) \quad \text{for } \mathrm{R} = \mathrm{C}_N, \mathrm{C}_N^\vee, \mathrm{BC}_N, \\
f_j^{\mathrm{D}_N}(z;\tau) &= e^{iJ^{\mathrm{D}_N}(j)z/r}\vartheta_1\left(J^{\mathrm{D}_N}(j)\tau + \frac{\mathcal{N}^{\mathrm{D}_N}z}{2\pi r} + \frac{1}{2}; \mathcal{N}^{\mathrm{D}_N}\tau\right) \\
&\quad + e^{-iJ^{\mathrm{D}_N}(j)z/r}\vartheta_1\left(J^{\mathrm{D}_N}(j)\tau - \frac{\mathcal{N}^{\mathrm{D}_N}z}{2\pi r} + \frac{1}{2}; \mathcal{N}^{\mathrm{D}_N}\tau\right),
\end{aligned}$$

$z \in \mathbb{C}, j \in \mathbb{Z}$, with $N \in \mathbb{N}$, $0 < r < \infty$, and $\tau \in \mathbb{C}, 0 < \Im\tau < \infty$, where

$$J^{\mathrm{R}}(j) = \begin{cases} j-1, & \mathrm{R} = \mathrm{A}_{N-1}, \mathrm{B}_N, \mathrm{B}_N^\vee, \mathrm{D}_N, \\ j, & \mathrm{R} = \mathrm{C}_N, \mathrm{BC}_N, \\ j-1/2, & \mathrm{R} = \mathrm{C}_N^\vee, \end{cases} \qquad \mathcal{N}^{\mathrm{R}} = \begin{cases} N, & \mathrm{R} = \mathrm{A}_{N-1}, \\ 2N-1, & \mathrm{R} = \mathrm{B}_N, \\ 2N, & \mathrm{R} = \mathrm{B}_N^\vee, \mathrm{C}_N^\vee, \\ 2(N+1), & \mathrm{R} = \mathrm{C}_N, \\ 2N+1, & \mathrm{R} = \mathrm{BC}_N, \\ 2(N-1), & \mathrm{R} = \mathrm{D}_N, \end{cases}$$

and ϑ_1 is one of Jacobi's theta functions defined below.

Let

$$z = e^{v\pi i}, \quad q = e^{\tau\pi i},$$

where $v \in \mathbb{C}$ and $\Im\tau > 0$. Here the Jacobi theta functions are denoted as follows,

$$\begin{aligned}
\vartheta_0(v;\tau) &= \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} z^{2n} = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{\tau\pi i n^2}\cos(2n\pi v),\\
\vartheta_1(v;\tau) &= i\sum_{n\in\mathbb{Z}}(-1)^n q^{(n-1/2)^2} z^{2n-1} = 2\sum_{n=1}^{\infty}(-1)^{n-1} e^{\tau\pi i(n-1/2)^2}\sin\{(2n-1)\pi v\},\\
\vartheta_2(v;\tau) &= \sum_{n\in\mathbb{Z}} q^{(n-1/2)^2} z^{2n-1} = 2\sum_{n=1}^{\infty} e^{\tau\pi i(n-1/2)^2}\cos\{(2n-1)\pi v\},\\
\vartheta_3(v;\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2} z^{2n} = 1 + 2\sum_{n=1}^{\infty} e^{\tau\pi i n^2}\cos(2n\pi v).
\end{aligned}$$

- These functions $f_j^{\mathrm{R}}(z;\tau), j \in \mathbb{Z}$ were used to express the **determinant evaluations by Rosengren and Schlosser** [RS06] for the **Macdonald denominators** $W_{\mathrm{R}}(x)$ for the seven families of **irreducible reduced affine root systems** $\mathrm{R} = \mathrm{A}_{N-1}$, B_N, $\mathrm{B}_N^\vee$, C_N, $\mathrm{C}_N^\vee$, BC_N, D_N.

[RS06] Rosengren, H., Schlosser, M.: Elliptic determinant evaluations and the Macdonald identities for affine root systems. Compositio Math. 142, 937-961 (2006)

- Let $0 < r < \infty$, $0 < t_* < \infty$, and $N \in \{2, 3, \dots\}$.
- Assume that

$$\boldsymbol{u} \in \mathbb{W}_N^{(0,2\pi r)} \equiv \{\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N : 0 < x_1 < \cdots < x_N < 2\pi r\} \quad \text{for } \mathrm{R} = \mathrm{A}_{N-1},$$
$$\boldsymbol{u} \in \mathbb{W}_N^{(0,\pi r)} \equiv \{\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N : 0 < x_1 < \cdots < x_N < \pi r\} \quad \text{for } \mathrm{R} \neq \mathrm{A}_{N-1}.$$

- Assume $t \in [0, t_*)$ and let

$$\tau^{\mathrm{R}}(t) = \frac{i\mathcal{N}^{\mathrm{R}}(t_* - t)}{2\pi r^2} \quad \text{with} \quad \mathcal{N}^{\mathrm{R}} = \begin{cases} N, & \mathrm{R} = \mathrm{A}_{N-1}, \\ 2N-1, & \mathrm{R} = \mathrm{B}_N, \\ 2N, & \mathrm{R} = \mathrm{B}_N^{\vee}, \mathrm{C}_N^{\vee}, \\ 2(N+1), & \mathrm{R} = \mathrm{C}_N, \\ 2N+1, & \mathrm{R} = \mathrm{BC}_N, \\ 2(N-1), & \mathrm{R} = \mathrm{D}_N. \end{cases}$$

$$\Phi_{\boldsymbol{u},u_j} \Rightarrow M_{\boldsymbol{u},u_j}(t,x) \Rightarrow D_{\boldsymbol{u}}(t,\boldsymbol{x}) \Rightarrow \widehat{\mathbb{P}}_{\boldsymbol{u}} \Rightarrow \mathbb{P}_{\boldsymbol{u}}$$

- For $\mathrm{R} = \mathrm{A}_{N-1}$, we obtain

$$D_{\boldsymbol{u}}^{\mathrm{A}_{N-1}}(t,\boldsymbol{x}) = \frac{c_0^{\mathrm{A}_{N-1}}(\tau^{\mathrm{A}_{N-1}}(t))}{c_0^{\mathrm{A}_{N-1}}(\tau^{\mathrm{A}_{N-1}}(0))} \frac{\vartheta_1((\sum_{\ell=1}^N x_\ell - \kappa_N)/2\pi r; \tau^{\mathrm{A}_{N-1}}(t))}{\vartheta_1((\sum_{\ell=1}^N u_\ell - \kappa_N)/2\pi r; \tau^{\mathrm{A}_{N-1}}(0))} \times \prod_{1\le j<k\le N} \frac{\vartheta_1((x_k - x_j)/2\pi r; \tau^{\mathrm{A}_{N-1}}(t))}{\vartheta_1((u_k - u_j)/2\pi r; \tau^{\mathrm{A}_{N-1}}(0))},$$

where

$$\kappa_N = \begin{cases} \pi r(N-1), & N \text{ is even}, \\ \pi r(N-2), & N \text{ is odd}, \end{cases}$$

$$c_0^{\mathrm{A}_{N-1}}(\tau) = \eta(\tau)^{-(N-1)(N-2)/2}.$$

- Here $\eta(\tau)$ denotes the **Dedekind modular function**

$$\eta(\tau) = e^{\tau\pi i/12} \prod_{n=1}^{\infty} (1 - e^{2n\tau\pi i}), \quad \Im\tau > 0.$$

$$\Phi_{\boldsymbol{u},u_j} \Rightarrow M_{\boldsymbol{u},u_j}(t,x) \Rightarrow D_{\boldsymbol{u}}(t,\boldsymbol{x}) \Rightarrow \widehat{\mathbb{P}}_{\boldsymbol{u}} \Rightarrow \mathbb{P}_{\boldsymbol{u}}$$

- For $\mathrm{R} = \mathrm{B}_N, \mathrm{B}_N^\vee, \mathrm{C}_N, \mathrm{C}_N^\vee, \mathrm{BC}_N, \mathrm{D}_N$, we obtain

$$\begin{aligned}
D_{\boldsymbol{u}}^{\mathrm{R}}(t,\boldsymbol{x}) &= \frac{c_0^{\mathrm{R}}(\tau^{\mathrm{R}}(t))}{c_0^{\mathrm{R}}(\tau^{\mathrm{R}}(0))} \prod_{\ell=1}^{N} \frac{\vartheta_1(c_1^{\mathrm{R}} x_\ell/2\pi r; c_2^{\mathrm{R}}\tau^{\mathrm{R}}(t))}{\vartheta_1(c_1^{\mathrm{R}} u_\ell/2\pi r; c_2^{\mathrm{R}}\tau^{\mathrm{R}}(0))} \\
&\times \prod_{1\le j<k\le N} \frac{\vartheta_1((x_k-x_j)/2\pi r;\tau^{\mathrm{R}}(t))}{\vartheta_1((u_k-u_j)/2\pi r;\tau^{\mathrm{R}}(0))} \frac{\vartheta_1((x_k+x_j)/2\pi r;\tau^{\mathrm{R}}(t))}{\vartheta_1((u_k+u_j)/2\pi r;\tau^{\mathrm{R}}(0))} \quad \text{for } \mathrm{R} = \mathrm{B}_N, \mathrm{B}_N^\vee, \mathrm{C}_N, \mathrm{C}_N^\vee, \\
D_{\boldsymbol{u}}^{\mathrm{BC}_N}(t,\boldsymbol{x}) &= \frac{c_0^{\mathrm{BC}_N}(\tau^{\mathrm{BC}_N}(t))}{c_0^{\mathrm{BC}_N}(\tau^{\mathrm{BC}_N}(0))} \prod_{\ell=1}^{N} \frac{\vartheta_1(x_\ell/2\pi r;\tau^{\mathrm{BC}_N}(t))}{\vartheta_1(u_\ell/2\pi r;\tau^{\mathrm{BC}_N}(0))} \frac{\vartheta_0(x_\ell/\pi r; 2\tau^{\mathrm{BC}_N}(t))}{\vartheta_0(u_\ell/\pi r; 2\tau^{\mathrm{BC}_N}(0))} \\
&\times \prod_{1\le j<k\le N} \frac{\vartheta_1((x_k-x_j)/2\pi r;\tau^{\mathrm{BC}_N}(t))}{\vartheta_1((u_k-u_j)/2\pi r;\tau^{\mathrm{BC}_N}(0))} \frac{\vartheta_1((x_k+x_j)/2\pi r;\tau^{\mathrm{BC}_N}(t))}{\vartheta_1((u_k+u_j)/2\pi r;\tau^{\mathrm{BC}_N}(0))}, \\
D_{\boldsymbol{u}}^{\mathrm{D}_N}(t,\boldsymbol{x}) &= \frac{c_0^{\mathrm{D}_N}(\tau^{\mathrm{D}_N}(t))}{c_0^{\mathrm{D}_N}(\tau^{\mathrm{D}_N}(0))} \prod_{1\le j<k\le N} \frac{\vartheta_1((x_k-x_j)/2\pi r;\tau^{\mathrm{D}_N}(t))}{\vartheta_1((u_k-u_j)/2\pi r;\tau^{\mathrm{D}_N}(0))} \frac{\vartheta_1((x_k+x_j)/2\pi r;\tau^{\mathrm{D}_N}(t))}{\vartheta_1((u_k+u_j)/2\pi r;\tau^{\mathrm{D}_N}(0))},
\end{aligned}$$

where

$$\begin{aligned}
c_0^{\mathrm{R}}(\tau) &= \eta(\tau)^{-N(N-1)} \quad \text{for } \mathrm{R} = \mathrm{B}_N, \mathrm{C}_N, \\
c_0^{\mathrm{B}_N^\vee}(\tau) &= \eta(\tau)^{-(N-1)^2}\eta(2\tau)^{-(N-1)}, \\
c_0^{\mathrm{C}_N^\vee}(\tau) &= \eta(\tau)^{-(N-1)^2}\eta(\tau/2)^{-(N-1)}, \\
c_0^{\mathrm{BC}_N}(\tau) &= \eta(\tau)^{-N(N-1)}\eta(2\tau)^{-N}, \\
c_0^{\mathrm{D}_N}(\tau) &= \eta(\tau)^{-N(N-2)},
\end{aligned}$$

and

$$\begin{aligned}
&c_1^{\mathrm{B}_N} = c_1^{\mathrm{C}_N^\vee} = 1, \quad c_1^{\mathrm{B}_N^\vee} = c_1^{\mathrm{C}_N} = 2, \\
&c_2^{\mathrm{B}_N} = c_2^{\mathrm{C}_N} = 1, \quad c_2^{\mathrm{B}_N^\vee} = 2, \quad c_2^{\mathrm{C}_N^\vee} = 1/2.
\end{aligned}$$

$$\Phi_{\boldsymbol{u},u_j} \Rightarrow M_{\boldsymbol{u},u_j}(t,x) \Rightarrow D_{\boldsymbol{u}}(t,\boldsymbol{x}) \Rightarrow \widehat{\mathbb{P}}_{\boldsymbol{u}} \Rightarrow \mathbb{P}_{\boldsymbol{u}}$$

For $\mathrm{R} = \mathrm{A}_{N-1}$:

- On a circumference of circle $[0, 2\pi r)$, consider the N-particle system of one-dimensional standard BMs, $\boldsymbol{B}(t) = (B_1(t), \dots, B_N(t))$, $t \geq 0$ started at $\boldsymbol{u} = (u_1, \dots, u_N) \in \mathbb{W}_N^{(0,2\pi r)}$ with the following 'wrapped' transition probability densities.

- When N is even,

$$\begin{aligned} p_{\mathrm{even}N}^{[0,2\pi r]}(t,y|x) &= \sum_{k=-\infty}^{\infty} p_{\mathrm{BM}}(t, y+2\pi r k|x) \\ &= p_{\mathrm{BM}}(t,y|x)\vartheta_3\left(\frac{i(y-x)r}{t}; \frac{2\pi i r^2}{t}\right) = \frac{1}{2\pi r}\vartheta_3\left(\frac{y-x}{2\pi r}; \frac{it}{2\pi r^2}\right), \end{aligned}$$

and when N is odd,

$$\begin{aligned} p_{\mathrm{odd}N}^{[0,2\pi r]}(t,y|x) &= \sum_{k=-\infty}^{\infty} (-1)^k p_{\mathrm{BM}}(t, y+2\pi r k|x) \\ &= p_{\mathrm{BM}}(t,y|x)\vartheta_0\left(\frac{i(y-x)r}{t}; \frac{2\pi i r^2}{t}\right) = \frac{1}{2\pi r}\vartheta_2\left(\frac{y-x}{2\pi r}; \frac{it}{2\pi r^2}\right), \end{aligned}$$

- We write the probability law of such a system of **wrapped Brownian motions** on a circumference of circle $[0, 2\pi r)$ as $\mathbf{P}_{\boldsymbol{u}}^{[0,2\pi r)}$.

For $\mathrm{R} = \mathrm{B}_N, \mathrm{B}_N^\vee, \mathrm{C}_N, \mathrm{C}_N^\vee, \mathrm{BC}_N, \mathrm{D}_N$:

- In the interval $[0, \pi r]$, consider the N-particle system of one-dimensional standard BMs $\boldsymbol{B}(t) = (B_1(t), \dots, B_N(t))$, $t \geq 0$ started at $\boldsymbol{u} = (u_1, \dots, u_N) \in \mathbb{W}_N^{(0,\pi r)}$ with **either an absorbing or reflecting boundary conditions at** 0 **and** πr.

- The transition probability density of each particle is denoted by $p^{[0,\pi r]}$. By the reflection principle of Brownian motion, if **both boundaries are absorbing**, it is given by

$$\begin{aligned}
p_{\mathrm{abs}}^{[0,\pi r]}(t, y|x) &= \sum_{k=-\infty}^{\infty} \{p_{\mathrm{BM}}(t, y + 2\pi r k|x) - p_{\mathrm{BM}}(t, y + 2\pi r k| - x)\} \\
&= p_{\mathrm{BM}}(t, y|x)\vartheta_3\left(\frac{i(y-x)r}{t}; \frac{2\pi i r^2}{t}\right) - p_{\mathrm{BM}}(t, y|-x)\vartheta_3\left(\frac{i(y+x)r}{t}; \frac{2\pi i r^2}{t}\right) \\
&= \frac{1}{2\pi r}\left\{\vartheta_3\left(\frac{y-x}{2\pi r}; \frac{it}{2\pi r^2}\right) - \vartheta_3\left(\frac{y+x}{2\pi r}; \frac{it}{2\pi r^2}\right)\right\}, \quad x, y \in (0, \pi r), t \geq 0,
\end{aligned}$$

and if **both are reflecting**, it is given by

$$\begin{aligned}
p_{\mathrm{ref}}^{[0,\pi r]}(t, y|x) &= \sum_{k=-\infty}^{\infty} \{p_{\mathrm{BM}}(t, y + 2\pi r k|x) + p_{\mathrm{BM}}(t, y + 2\pi r k| - x)\} \\
&= p_{\mathrm{BM}}(t, y|x)\vartheta_3\left(\frac{i(y-x)r}{t}; \frac{2\pi i r^2}{t}\right) + p_{\mathrm{BM}}(t, y|-x)\vartheta_3\left(\frac{i(y+x)r}{t}; \frac{2\pi i r^2}{t}\right) \\
&= \frac{1}{2\pi r}\left\{\vartheta_3\left(\frac{y-x}{2\pi r}; \frac{it}{2\pi r^2}\right) + \vartheta_3\left(\frac{y+x}{2\pi r}; \frac{it}{2\pi r^2}\right)\right\}, \quad x, y \in [0, \pi r], t \geq 0.
\end{aligned}$$

- We write the probability law of such a system of boundary-conditioned Brownian motions in $[0, \pi r]$ as $\mathbf{P}_{\boldsymbol{u}}^{[0,\pi r]}$.

$$\Phi_{\boldsymbol{u},u_j} \Rightarrow M_{\boldsymbol{u},u_j}(t,x) \Rightarrow D_{\boldsymbol{u}}(t,\boldsymbol{x}) \Rightarrow \widehat{\mathbb{P}}_{\boldsymbol{u}} \Rightarrow \mathbb{P}_{\boldsymbol{u}}$$

- In $\mathbf{P}_{\boldsymbol{u}}^{[0,2\pi r)}$ for $\mathrm{R} = \mathrm{A}_{N-1}$ and in $\mathbf{P}_{\boldsymbol{u}}^{[0,\pi r]}$ for $\mathrm{R} \neq \mathrm{A}_{N-1}$, put

$$T_{\text{collision}} = \inf\{t > 0 : B_j(t) = B_k(t) \quad \text{for any } j \neq k\},$$

i.e., the **first collision-time** of the N-particle system of Brownian motions.

- Then we define

$$\left.\mathbb{P}_{\boldsymbol{u}}^{\mathrm{A}_{N-1}}\right|_{\mathcal{F}_t} = \mathbf{1}(T_{\text{collision}} > t) D_{\boldsymbol{u}}^{\mathrm{A}_{N-1}}(t, \boldsymbol{B}(t)) \left.\mathbf{P}_{\boldsymbol{u}}^{[0,2\pi r)}\right|_{\mathcal{F}_t}, \quad t \in [0, t_*),$$

$$\left.\mathbb{P}_{\boldsymbol{u}}^{\mathrm{R}}\right|_{\mathcal{F}_t} = \mathbf{1}(T_{\text{collision}} > t) D_{\boldsymbol{u}}^{\mathrm{R}}(t, \boldsymbol{B}(t)) \left.\mathbf{P}_{\boldsymbol{u}}^{[0,\pi r]}\right|_{\mathcal{F}_t}, \quad t \in [0, t_*), \quad \text{for } \mathrm{R} \neq \mathrm{A}_{N-1}.$$

Theorem 1

(i) The above defined $\mathbb{P}^{\mathrm{R}}_{\boldsymbol{u}}$ give **probability measures** and define measure-valued stochastic processes

$$\Xi^{\mathrm{R}}(t,\cdot)=\sum_{j=1}^{N}\delta_{X_j^{\mathrm{R}}(t)}(\cdot),\quad t\in[0,t_*).$$

(ii) The processs $((\Xi^{\mathrm{R}}(t))_{t\in[0,t_*)},\mathbb{P}^{\mathrm{R}}_{\boldsymbol{u}})$ are **determinantal with the spatio-temporal correlation kernels**

$$\mathbb{K}^{\mathrm{A}_{N-1}}_{\boldsymbol{u}}(s,x;t,y) = \sum_{j=1}^{N} p^{[0,2\pi r]}(s,x|u_j)M^{\mathrm{R}}_{\boldsymbol{u},u_j}(t,y)-\mathbf{1}(s>t)p^{[0,2\pi r)}(s-t,x|y),$$

$$\mathbb{K}^{\mathrm{R}}_{\boldsymbol{u}}(s,x;t,y) = \sum_{j=1}^{N} p^{[0,\pi r]}(s,x|u_j)M^{\mathrm{R}}_{\boldsymbol{u},u_j}(t,y)-\mathbf{1}(s>t)p^{[0,\pi r]}(s-t,x|y)\quad \text{for } \mathrm{R}\neq \mathrm{A}_{N-1},$$

where

$$M^{\mathrm{R}}_{\boldsymbol{u},u_j}(t,x)=\widetilde{\mathrm{E}}[\Phi^{\mathrm{R}}_{\boldsymbol{u},u_j}(x+i\widetilde{B}(t))]=\int_{-\infty}^{\infty}\Phi^{\mathrm{R}}_{\boldsymbol{u},u_j}(x+i\widetilde{x})p_{\mathrm{BM}}(t,\widetilde{x}|0)d\widetilde{x}.$$

$$\Phi_{\boldsymbol{u},u_j} \Rightarrow M_{\boldsymbol{u},u_j}(t,x) \Rightarrow D_{\boldsymbol{u}}(t,\boldsymbol{x}) \Rightarrow \widehat{\mathbb{P}}_{\boldsymbol{u}} \Rightarrow \mathbb{P}_{\boldsymbol{u}}$$

Here

$$\Phi^{\mathrm{A}_{N-1}}_{\boldsymbol{u},u_j}(z) = \frac{\vartheta_1((\sum_{\ell=1}^N u_\ell + z - u_j)/2\pi r; \tau^{\mathrm{A}_{N-1}}(0))}{\vartheta_1(\sum_{\ell=1}^N u_\ell/2\pi r; \tau^{\mathrm{A}_{N-1}}(0))} \prod_{\substack{1\le\ell\le N,\\ \ell\ne j}} \frac{\vartheta_1((z-u_\ell)/2\pi r; \tau^{\mathrm{R}}(0))}{\vartheta_1((u_j-u_\ell)/2\pi r; \tau^{\mathrm{R}}(0)},$$

$$\Phi^{\mathrm{R}}_{\boldsymbol{u},u_j}(z) = \frac{\vartheta_1(c_1^{\mathrm{R}} z/2\pi r; c_2^{\mathrm{R}}\tau^{\mathrm{R}}(0))}{\vartheta_1(c_1^{\mathrm{R}} u_j/2\pi r; c_2^{\mathrm{R}}\tau^{\mathrm{R}}(0))}$$
$$\times \prod_{\substack{1\le\ell\le N,\\ \ell\ne j}} \frac{\vartheta_1((z-u_\ell)/2\pi r; \tau^{\mathrm{R}}(0))}{\vartheta_1((u_j-u_\ell)/2\pi r; \tau^{\mathrm{R}}(0)} \frac{\vartheta_1((z+u_\ell)/2\pi r; \tau^{\mathrm{R}}(0))}{\vartheta_1((u_j+u_\ell)/2\pi r; \tau^{\mathrm{R}}(0))} \quad \textit{for } \mathrm{R} = \mathrm{B}_N, \mathrm{B}_N^\vee, \mathrm{C}_N, \mathrm{C}_N^\vee,$$

$$\Phi^{\mathrm{BC}_N}_{\boldsymbol{u},u_j}(z) = \frac{\vartheta_1(z/2\pi r; \tau^{\mathrm{BC}_N}(0))}{\vartheta_1(u_j/2\pi r; \tau^{\mathrm{BC}_N}(0))} \frac{\vartheta_0(z/\pi r; 2\tau^{\mathrm{BC}_N}(0))}{\vartheta_0(u_j/\pi r; 2\tau^{\mathrm{BC}_N}(0))}$$
$$\times \prod_{\substack{1\le\ell\le N,\\ \ell\ne j}} \frac{\vartheta_1((z-u_\ell)/2\pi r; \tau^{\mathrm{BC}_N}(0))}{\vartheta_1((u_j-u_\ell)/2\pi r; \tau^{\mathrm{BC}_N}(0)} \frac{\vartheta_1((z+u_\ell)/2\pi r; \tau^{\mathrm{BC}_N}(0))}{\vartheta_1((u_j+u_\ell)/2\pi r; \tau^{\mathrm{BC}_N}(0))},$$

$$\Phi^{\mathrm{D}_N}_{\boldsymbol{u},u_j}(z) = \prod_{\substack{1\le\ell\le N,\\ \ell\ne j}} \frac{\vartheta_1((z-u_\ell)/2\pi r; \tau^{\mathrm{D}_N}(0))}{\vartheta_1((u_j-u_\ell)/2\pi r; \tau^{\mathrm{D}_N}(0)} \frac{\vartheta_1((z+u_\ell)/2\pi r; \tau^{\mathrm{D}_N}(0))}{\vartheta_1((u_j+u_\ell)/2\pi r; \tau^{\mathrm{D}_N}(0))}.$$

Note that $\tau^{\mathrm{R}}(t)$ *and* c_1^{R}, c_2^{R} *for* $\mathrm{R} = \mathrm{B}_N$, $\mathrm{B}_N^\vee$, C_N, $\mathrm{C}_N^\vee$ *are given before.*

- Assume $t_* > 0, r > 0, \mathcal{N} > 0$, and let

$$A^{2\pi r}_{\mathcal{N}}(t_* - t, x) = \left[\frac{1}{2\pi r}\frac{d}{dv}\log\vartheta_1(v;\tau)\right]_{v=x/2\pi r, \tau=i\mathcal{N}(t_*-t)/2\pi r^2} = \frac{1}{2\pi r}\frac{\vartheta_1'(x/2\pi r; i\mathcal{N}(t_*-t)/2\pi r^2)}{\vartheta_1(x/2\pi r; i\mathcal{N}(t_*-t)/2\pi r^2)},$$

where $\vartheta_1'(v;\tau) = d\vartheta_1(v;\tau)/dv$. As a function of $x \in \mathbb{R}$, $A^{2\pi r}_{\mathcal{N}}(t_* - t, x)$ is odd,

$$A^{2\pi r}_{\mathcal{N}}(t_* - t, -x) = -A^{2\pi r}_{\mathcal{N}}(t_* - t, x),$$

and periodic with period $2\pi r$

$$A^{2\pi r}_{\mathcal{N}}(t_* - t, x + 2m\pi r) = A^{2\pi r}_{\mathcal{N}}(t_* - t, x), \quad m \in \mathbb{Z}.$$

It has only simple poles at $x = 2m\pi r$, and simple zeroes at $x = (2m+1)\pi r, m \in \mathbb{Z}$.

Theorem 2

Put an absorbing (resp. reflecting) boundary condition at both endpoints in $[0, \pi r]$ for $\mathrm{R} = \mathrm{B}_N$ (resp. $\mathrm{R} = \mathrm{D}_N$). Then the **determinantal processes** with $\mathrm{R} = \mathrm{A}_{N-1}, \mathrm{B}_N$ and D_N solve the following **systems of SDEs**, respectively, for $t \in [0, t_*)$,

$$(\mathrm{A}_{N-1}) \quad X_j^{\mathrm{A}_{N-1}}(t) = u_j + W_j(t) + \int_0^t A_N^{2\pi r}\Big(t_* - s, \sum_{\ell=1}^N X_\ell^{\mathrm{A}_{N-1}}(s) - \kappa_N\Big) ds$$

$$+ \sum_{\substack{1 \le k \le N, \\ k \neq j}} \int_0^t A_N^{2\pi r}(t_* - s, X_j^{\mathrm{A}_{N-1}}(s) - X_k^{\mathrm{A}_{N-1}}(s)) ds, \quad j = 1, \dots, N, \text{ in } \mathbb{R},$$

$$(\mathrm{B}_N) \quad X_j^{\mathrm{B}_N}(t) = u_j + W_j(t) + \int_0^t A_{2N-1}^{2\pi r}(t_* - s, X_j^{\mathrm{B}_N}(s)) ds$$

$$+ \sum_{\substack{1 \le k \le N, \\ k \neq j}} \int_0^t (A_{2N-1}^{2\pi r}(t_* - s, X_j^{\mathrm{B}_N}(s) - X_k^{\mathrm{B}_N}(s)) + A_{2N-1}^{2\pi r}(t_* - s, X_j^{\mathrm{B}_N}(s) + X_k^{\mathrm{B}_N}(s))) ds,$$

$j = 1, 2, \dots, N$, in the interval $[0, \pi r]$ with an absorbing boundary condition at 0 and πr,

$$(\mathrm{D}_N) \quad X_j^{\mathrm{D}_N}(t) = u_j + W_j(t)$$

$$+ \sum_{\substack{1 \le k \le N, \\ k \neq j}} \int_0^t (A_{2(N-1)}^{2\pi r}(t_* - s, X_j^{\mathrm{D}_N}(s) - X_k^{\mathrm{D}_N}(s)) + A_{2(N-1)}^{2\pi r}(t_* - s, X_j^{\mathrm{D}_N}(s) + X_k^{\mathrm{D}_N}(s))) ds,$$

$j = 1, 2, \dots, N$, in the interval $[0, \pi r]$ with a reflecting boundary condition at 0 and πr.

- We have $\lim_{t_* \to \infty} A_N^{2\pi r}(t_* - t, x) = \frac{1}{2r} \cot\left(\frac{x}{2r}\right)$.
- Hence in the limit $t_* \to \infty$, the above systems become the following **temporally homogeneous systems of SDEs** for $t \in [0, \infty)$;

$$(\mathrm{A}_{N-1}) \qquad X_j^{\mathrm{A}_{N-1}}(t) = u_j + W_j(t) - \frac{1}{2r}\int_0^t \tan\left(\frac{1}{2r}\sum_{\ell=1}^N X_\ell^{\mathrm{A}_{N-1}}(s)\right) ds$$
$$+\frac{1}{2r}\sum_{\substack{1\le k\le N,\\ k\neq j}}\int_0^t \cot\left(\frac{X_j^{\mathrm{A}_{N-1}}(s) - X_k^{\mathrm{A}_{N-1}}(s)}{2r}\right) ds, \quad j = 1, 2, \dots, N, \text{ in } \mathbb{R},$$

$$(\mathrm{B}_N) \qquad X_j^{\mathrm{B}_N}(t) = u_j + W_j(t) + \frac{1}{2r}\int_0^t \cot\left(\frac{X_j^{\mathrm{B}_N}(s)}{2r}\right) ds$$
$$+\frac{1}{2r}\sum_{\substack{1\le k\le N,\\ k\neq j}}\int_0^t \left\{\cot\left(\frac{X_j^{\mathrm{B}_N}(s) - X_k^{\mathrm{B}_N}(s)}{2r}\right) + \cot\left(\frac{X_j^{\mathrm{B}_N}(s) + X_k^{\mathrm{B}_N}(s)}{2r}\right)\right\} ds,$$

$j = 1, 2, \dots, N$, in $[0, \pi r]$ with an absorbing boundary condition at 0 and πr,

$$(\mathrm{D}_N) \qquad X_j^{\mathrm{D}_N}(t) = u_j + W_j(t)$$
$$+\frac{1}{2r}\sum_{\substack{1\le k\le N,\\ k\neq j}}\int_0^t \left\{\cot\left(\frac{X_j^{\mathrm{D}_N}(s) - X_k^{\mathrm{D}_N}(s)}{2r}\right) + \cot\left(\frac{X_j^{\mathrm{D}_N}(s) + X_k^{\mathrm{D}_N}(s)}{2r}\right)\right\} ds,$$

$j = 1, 2, \dots, N$, in $[0, \pi r]$ with a reflecting boundary condition at 0 and πr.

- Moreover, we have $\lim_{r\to\infty} \frac{1}{2r} \cot\left(\frac{x}{2r}\right) = \frac{1}{x}, \quad \lim_{r\to\infty} \frac{1}{2r} \tan\left(\frac{x}{2r}\right) = 0.$
- Then in the $r \to \infty$ limit, these systems are reduced to be the follows for $t \in [0, \infty)$,

$$
\begin{aligned}
(\mathrm{A}_{N-1}) \quad X_j^{\mathrm{A}_{N-1}}(t) &= u_j + W_j(t) + \sum_{\substack{1\le k\le N,\\ k\neq j}} \int_0^t \frac{1}{X_j^{\mathrm{A}_{N-1}}(s) - X_k^{\mathrm{A}_{N-1}}(s)} ds, \\
& \quad j = 1, 2, \dots, N, \text{ in } \mathbb{R}, \\
(\mathrm{B}_N) \quad X_j^{\mathrm{B}_N}(t) &= u_j + W_j(t) + \int_0^t \frac{1}{X_j^{\mathrm{B}_N}(s)} ds \\
&+ \sum_{\substack{1\le k\le N,\\ k\neq j}} \int_0^t \left\{ \frac{1}{X_j^{\mathrm{B}_N}(s) - X_k^{\mathrm{B}_N}(s)} + \frac{1}{X_j^{\mathrm{B}_N}(s) + X_k^{\mathrm{B}_N}(s)} \right\} ds, \\
& \quad j = 1, 2, \dots, N, \text{ in } (0, \infty), \\
(\mathrm{D}_N) \quad X_j^{\mathrm{D}_N}(t) &= u_j + W_j(t) \\
&+ \sum_{\substack{1\le k\le N,\\ k\neq j}} \int_0^t \left\{ \frac{1}{X_j^{\mathrm{D}_N}(s) - X_k^{\mathrm{D}_N}(s)} + \frac{1}{X_j^{\mathrm{D}_N}(s) + X_k^{\mathrm{D}_N}(s)} \right\} ds, \\
& \quad j = 1, 2, \dots, N, \text{ in } [0, \infty) \text{ with a reflecting condition at the origin.}
\end{aligned}
$$

- They are the original Dyson model (Dyson's Brownian motion model with parameter $\beta = 2$), noncolliding absorbing Brownian motions, and noncolliding reflecting Brownian motions, respectively.

3. Key Lemmas and Open Problems

- Please see the following for proofs.

For Type A_{N-1}**:**

[K15] Katori, M.: Elliptic determinantal process of type A. Probab. Theory Relat. Fields 162, 637-677 (2015)

[K16b] Katori, M.: Elliptic Bessel processes and elliptic Dyson models realized as temporally inhomogeneous processes. J. Math. Phys. 57, 103302/1-32 (2016)

For Types $\mathrm{B}_N, \mathrm{B}_N^\vee, \mathrm{C}_N, \mathrm{C}_N^\vee, \mathrm{BC}_N, \mathrm{D}_N$**:**

[K17] Katori, M.: Elliptic determinantal processes and elliptic Dyson models. arXiv:math.PR/1703.03914

- Here I only remark the key lemmas for Theorems.

A Key Lemma for Theorem 1

Lemma 1

For $\mathrm{R} = \mathrm{A}_{N-1}, \mathrm{B}_N, \mathrm{B}_N^\vee, \mathrm{C}_N, \mathrm{C}_N^\vee, \mathrm{BC}_N, \mathrm{D}_N$,

$$\begin{aligned} \widehat{f}_j^{\mathrm{R}}(t, x; \tau^{\mathrm{R}}(0)) &\equiv \widetilde{\mathrm{E}}[f_j^{\mathrm{R}}(x + i\widetilde{B}(t); \tau^{\mathrm{R}}(0)] \\ &= e^{J^{\mathrm{R}}(j)^2 t/2r^2} f_j^{\mathrm{R}}(x; \tau^{\mathrm{R}}(t)), \quad t \in [0, t_*), \quad j \in \mathbb{Z}, \end{aligned}$$

where

$$\tau^{\mathrm{R}}(t) = \frac{i\mathcal{N}^{\mathrm{R}}(t_* - t)}{2\pi r^2}.$$

taking expectation $\widetilde{\mathrm{E}}$ of the imaginary parts $\Longrightarrow$ making martingales

$\Longrightarrow$ time shift $\tau^{\mathrm{R}}(0) \to \tau^{\mathrm{R}}(t)$ and factors $e^{J^{\mathrm{R}}(j)^2 t/2r^2}$

$\Longrightarrow$ factors expressed by the Dedekind modular function $\eta(\tau)$

e.g., $c_0^{\mathrm{A}_{N-1}}(\tau) = \eta(\tau)^{-(N-1)(N-2)/2}$

A Key Lemma for Theorem 2

Theorem 2 is proved by solving the backward **Kolmogorov equations** in the form; for example for $\mathrm{R} = \mathrm{B}_N$,

$$-\frac{\partial p^{\mathrm{B}_N}(t, \boldsymbol{y}|s, \boldsymbol{x})}{\partial s} = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2 p^{\mathrm{B}_N}(t, \boldsymbol{y}|s, \boldsymbol{x})}{\partial x_j^2} + \sum_{j=1}^{N} A_{2N-1}^{2\pi r}(t_* - s, x_j) \frac{\partial p^{\mathrm{B}_N}(t, \boldsymbol{y}|s, \boldsymbol{x})}{\partial x_j}$$
$$+ \sum_{\substack{1 \leq j,k \leq N, \\ j \neq k}} (A_{2N-1}^{2\pi r}(t_* - s, x_j - x_k) + A_{2N-1}^{2\pi r}(t_* - s, x_j + x_k)) \frac{\partial p^{\mathrm{B}_N}(t, \boldsymbol{y}|s, \boldsymbol{x})}{\partial x_j}$$

under the condition $\lim_{s \uparrow t} p^{\mathrm{B}_N}(t, \boldsymbol{y}|s, \boldsymbol{x}) = \prod_{j=1}^{N} \delta(x_j - y_j)$.

A Key Lemma for Theorem 2

Lemma 2

The backward Kolmogorov equations for $\mathrm{R} = \mathrm{A}_{N-1}, \mathrm{B}_N, \mathrm{D}_N$ can be reduced to the following simple equations which determine the factors $c_0^{\mathrm{R}}(\tau^{\mathrm{R}}(s))$,

$$\frac{d \log c_0^{\mathrm{A}_{N-1}}(s)}{ds} = -N(N-1)(N-2)\frac{1}{4\pi r}\eta_N^1(t_* - s),$$
$$\frac{d \log c_0^{\mathrm{B}_N}(s)}{ds} = -N(N-1)(2N-1)\frac{1}{2\pi r}\eta_{2N-1}^1(t_* - s),$$
$$\frac{d \log c_0^{\mathrm{D}_N}(s)}{ds} = -N(N-1)(N-2)\frac{1}{\pi r}\eta_{2(N-1)}^1(t_* - s),$$

where

$$\eta_N^1(t_* - s) = \frac{\pi^2}{\omega_1}\left(\frac{1}{12} - 2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right)\Bigg|_{\omega_1 = \pi r, q = e^{-N(t_* - s)/2r^2}}.$$

Open Problem 1

- We have identified the systems of SDEs which are solved by the three families of determinantal processes of types A_{N-1}, B_N, and D_N.

- The systems of SDEs for other cases $R = B_N^{\vee}, C_N, C_N^{\vee}, BC_N$ are not yet clarified.

- We have used the **functional equation**

$$\begin{aligned}(\zeta_{\mathcal{N}}(t_* - s, z+u) - \zeta_{\mathcal{N}}(t_* - s, z) - \zeta_{\mathcal{N}}(t_* - s, u))^2 \\ = \wp_{\mathcal{N}}(t_* - s, z+u) + \wp_{\mathcal{N}}(t_* - s, z) + \wp_{\mathcal{N}}(t_* - s, u),\end{aligned}$$

with

$$\zeta_{\mathcal{N}}(t_* - s, x) \equiv \zeta(x|2\omega_1, 2\omega_3)\Big|_{\omega_1=\pi r, \omega_3 = i\mathcal{N}(t_*-s)/2r},$$

$$\wp_{\mathcal{N}}(t_* - s, x) \equiv \wp(x|2\omega_1, 2\omega_3)\Big|_{\omega_1=\pi r, \omega_3 = i\mathcal{N}(t_*-s)/2r},$$

where the Weierstrass $\wp$ function and zeta function ζ are defined by

$$\wp(z|2\omega_1, 2\omega_3) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left[\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{{\Omega_{m,n}}^2}\right],$$

$$\zeta(z|2\omega_1, 2\omega_3) = \frac{1}{z} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left[\frac{1}{z-\Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{{\Omega_{m,n}}^2}\right], \quad \Omega_{m,n} = 2m\omega_1 + 2n\omega_3,$$

- It seems to be that we need the similar equations for $\zeta, \wp$ with different periods ω_1, ω_3.

Open Problem 2

- Connections to other elliptic-function-level models ?
  - random matrix theory
  - probabilistic discrete models with elliptic weights

    [Sch07] Schlosser, M.: Elliptic enumeration of nonintersecting lattice paths. J. Combin. Theory Ser. A 114, 505-521 (2007)

    [BGR10] Borodin, A., Gorin, V., Rains, E. M.: q-distributions on boxed plane partitions. Sel. Math. (N. S.) 16, 731-789 (2010)

    [Betea11] Betea, D.: Elliptically distributed lozenge tilings of a hexagon. arXiv:math-ph/1110.4176
  - stochastic Komatu-Loewner evolution in doubly connected domains

    [CFR15] Chen, Z.-Q., Fukushima, M., Rhode, S.: Chordal Komatu–Loewner equation and Brownian motion with darning in multiply connected domain. Trans. Amer. Math. Soc. 368, 4065-4114 (2016)

- The present talk was based on the following three papers;

<u>**For Type A_{N-1}:**</u>

[K15] Katori, M.: Elliptic determinantal process of type A. Probab. Theory Relat. Fields 162, 637-677 (2015)

[K16b] Katori, M.: Elliptic Bessel processes and elliptic Dyson models realized as temporally inhomogeneous processes. J. Math. Phys. 57, 103302/1-32 (2016)

<u>**For Types $\mathrm{B}_N, \mathrm{B}_N^{\vee}, \mathrm{C}_N, \mathrm{C}_N^{\vee}, \mathrm{BC}_N, \mathrm{D}_N$:**</u>

[K17] Katori, M.: Elliptic determinantal processes and elliptic Dyson models. arXiv:math.PR/1703.03914

Thank you very much for your attention.

- I have shown that the complex structures and conformal invariance are hidden in the determinantal solutions for nonequilibrium-statistical-physics models.
- The present argument for the solvability of the models using **conformal invariance and martingale processes** can be discusses in a unified way including **classical diffusion processes**, the **Schramm-Loewner evolution (SLE)**, and the interacting particle systems in the **KPZ universality class**.

A Lecture Note entitled
`Bessel Processes, Schramm-Loewner Evolution, and the Dyson Model'
was published (2016) as
SpringerBriefs in Mathematical Physics 11