An elliptic Painlevé equation from next-nearest-neighbor translation on the $E_8^{(1)}$ lattice

Nobutaka Nakazono (The University of Sydney)

Joint work with

Prof. Nalini Joshi (The University of Sydney)

Supported by the Australian Research Council

FL120100094 and DP130100967

Workshop on Elliptic Hypergeometric Functions in Combinatorics, Integrable Systems and Physics

ESI, Boltzmann Lecture Hall, Vienna, Austria, 21 Mar. 2017

Abstract

Elliptic Painlevé equations head the list of the differential and discrete Painlevé equations. The well known elliptic Painlevé equation is given by a nearest neighbor vector on the $E_8^{(1)}$ weight lattice.

In this poster, we present an elliptic Painlevé equation, which is obtained by a next-nearest-neighbor vector. We also show that its projectively-reduced equation is the elliptic difference equation found by Ramani, Carstea and Grammaticos in 2009 from the reduction of the discrete analogue of the Krichever-Novikov equation.

Painlevé equations

In the early 20th-century, in order to find a new class of special functions, Painlevé and Gambier classified all rational ordinary differential equations of second order of the form $y'' = F(y', y, t)$, where $y = y(t)$ and $' = d/dt$, with the Painlevé property (solutions do not have movable branch points). As a result, they obtained six new equations.
According to the classification by the rational surfaces (space of initial values) by Okamoto [6], the Painlevé equations can be classified into 8 types. From the view point of this classification, $\mathrm{P_{III}}$ can be divided into three types by the values of parameters. Therefore, we have the following diagram of degeneration:

$$\begin{array}{ccccccccc}
\underset{(D_4^{(1)})}{\mathrm{P_{VI}}} & \rightarrow & \underset{(D_5^{(1)})}{\mathrm{P_{V}}} & \rightarrow & \underset{(D_6^{(1)})}{\mathrm{P_{III}^{D_6^{(1)}}}} & \rightarrow & \underset{(D_7^{(1)})}{\mathrm{P_{III}^{D_7^{(1)}}}} & \rightarrow & \underset{(D_8^{(1)})}{\mathrm{P_{III}^{D_8^{(1)}}}} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \underset{(E_6^{(1)})}{\mathrm{P_{IV}}} & \rightarrow & \underset{(E_7^{(1)})}{\mathrm{P_{II}}} & \rightarrow & \underset{(E_8^{(1)})}{\mathrm{P_{I}}} & &
\end{array}$$

Discrete Painlevé equations

Discrete Painlevé equations are nonlinear ordinary difference equations of second order, which can be reduced to the Painlevé equations through appropriate limiting processes. It is well known that there are three difference types (elliptic-, multiplicative- and additive-type).

Originally, discrete Painlevé equations appeared in the model of 2D quantum gravity and the theory of orthogonal polynomials [3, 5]. In 1991, Grammaticos *et al.* introduced the singularity confinement criterion as the discrete version of the Painlevé property [4]. Since then, many kinds of discrete Painlevé systems were found.

In 2001, Sakai [8] showed the classification of discrete Painlevé equations by space of initial values. Discrete Painlevé equations are characterized by their space of initial values constructed by the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at eight base points (i.e. points where the system is ill defined because it approaches 0/0). They are classified into 19 types according to the configuration of the base points.

Discrete type	Type of surface
Elliptic	$A_0^{(1)}$
Multiplicative	$A_0^{(1)*}$, $A_1^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$, $\ldots$, $A_7^{(1)}$, $A_7^{(1)'}$
Additive	$A_0^{(1)**}$, $A_1^{(1)*}$, $A_2^{(1)*}$, $D_4^{(1)}$, $\ldots$, $D_7^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$

RCG equation

Ramani-Carstea-Grammaticos [7] obtained the following ordinary difference equation (RCG equation) from the partial difference equation called Lattice Krichever-Novikov (KN) system [1].

$$\tilde{y} = \frac{(1-k^2\mathrm{sz}^4)\mathrm{c}_e\mathrm{d}_e\,xy - (\mathrm{c}_e^2-\mathrm{cz}^2)\mathrm{cz}\,\mathrm{dz} - (1-k^2\mathrm{s}_e^2\mathrm{sz}^2)\mathrm{cz}\,\mathrm{dz}\,x^2}{k^2(\mathrm{c}_e^2-\mathrm{cz}^2)\mathrm{cz}\,\mathrm{dz}\,x^2y - (1-k^2\mathrm{sz}^4)\mathrm{c}_e\mathrm{d}_e\,x + (1-k^2\mathrm{s}_e^2\mathrm{sz}^2)\mathrm{cz}\,\mathrm{dz}\,y}$$

$$\tilde{x} = \frac{(1-k^2\widehat{\mathrm{sz}}^4)\mathrm{c}_o\mathrm{d}_o\,\tilde{y}x - (\mathrm{c}_o^2-\widehat{\mathrm{cz}}^2)\widehat{\mathrm{cz}}\,\widehat{\mathrm{dz}} - (1-k^2\mathrm{s}_o^2\widehat{\mathrm{sz}}^2)\widehat{\mathrm{cz}}\,\widehat{\mathrm{dz}}\,\tilde{y}^2}{k^2(\mathrm{c}_o^2-\widehat{\mathrm{cz}}^2)\widehat{\mathrm{cz}}\,\widehat{\mathrm{dz}}\,\tilde{y}^2x - (1-k^2\widehat{\mathrm{sz}}^4)\mathrm{c}_o\mathrm{d}_o\,\tilde{y} + (1-k^2\mathrm{s}_o^2\widehat{\mathrm{sz}}^2)\widehat{\mathrm{cz}}\,\widehat{\mathrm{dz}}\,x}$$

where

$$\begin{array}{llll}
\mathrm{sz} = \mathrm{sn}\,(z)\,, & \widehat{\mathrm{sz}} = \mathrm{sn}\,(z+\gamma)\,, & \mathrm{s}_e = \mathrm{sn}\,(\gamma_\mathrm{e})\,, & \mathrm{s}_o = \mathrm{sn}\,(\gamma_\mathrm{o})\,,\\
\mathrm{cz} = \mathrm{cn}\,(z)\,, & \widehat{\mathrm{cz}} = \mathrm{cn}\,(z+\gamma)\,, & \mathrm{c}_e = \mathrm{cn}\,(\gamma_\mathrm{e})\,, & \mathrm{c}_o = \mathrm{cn}\,(\gamma_\mathrm{o})\,,\\
\mathrm{dz} = \mathrm{dn}\,(z)\,, & \widehat{\mathrm{dz}} = \mathrm{dn}\,(z+\gamma)\,, & \mathrm{d}_e = \mathrm{dn}\,(\gamma_\mathrm{e})\,, & \mathrm{d}_o = \mathrm{dn}\,(\gamma_\mathrm{o})\,,\\
y = y(z), & x = x(z), & \gamma = \gamma_\mathrm{e}+\gamma_\mathrm{o}, & \tilde{}\,: z \mapsto z+2\gamma,
\end{array}$$

sn, cn, dn are the Jacobian elliptic functions, and k is the modulus.

Geometry of the RCG equation

In [2], the space of initial values of RCG equation was investigated.

The eight base points are given by

$$p_1 : (x,y) = \left(\mathrm{cd}\left(\gamma_{\mathrm{o}} + 2K + \mathrm{i}K'\right), \mathrm{cd}\left(z_0 - \gamma_{\mathrm{e}} - \gamma_{\mathrm{o}} + 2K + \mathrm{i}K'\right)\right),$$
$$p_2 : (x,y) = \left(\mathrm{cd}\left(\gamma_{\mathrm{o}} + \mathrm{i}K'\right), \mathrm{cd}\left(z_0 - \gamma_{\mathrm{e}} - \gamma_{\mathrm{o}} + \mathrm{i}K'\right)\right),$$
$$p_3 : (x,y) = \left(\mathrm{cd}\left(\gamma_{\mathrm{o}} + 2K\right), \mathrm{cd}\left(z_0 - \gamma_{\mathrm{e}} - \gamma_{\mathrm{o}} + 2K\right)\right),$$
$$p_4 : (x,y) = \left(\mathrm{cd}\left(\gamma_{\mathrm{o}}\right), \mathrm{cd}\left(z_0 - \gamma_{\mathrm{e}} - \gamma_{\mathrm{o}}\right)\right),$$
$$p_5 : (x,y) = \left(\mathrm{cd}\left(z_0 + 2K + \mathrm{i}K'\right), \mathrm{cd}\left(\gamma_{\mathrm{e}} + 2K + \mathrm{i}K'\right)\right),$$
$$p_6 : (x,y) = \left(\mathrm{cd}\left(z_0 + \mathrm{i}K'\right), \mathrm{cd}\left(\gamma_{\mathrm{e}} + \mathrm{i}K'\right)\right),$$
$$p_7 : (x,y) = \left(\mathrm{cd}\left(z_0 + 2K\right), \mathrm{cd}\left(\gamma_{\mathrm{e}} + 2K\right)\right),$$
$$p_8 : (x,y) = \left(\mathrm{cd}\left(z_0\right), \mathrm{cd}\left(\gamma_{\mathrm{e}}\right)\right),$$

where $\mathrm{cd} = \mathrm{cn}/\mathrm{dn}$ and $K = K(k)$, $K' = K'(k)$ are complete elliptic integrals, which lie on the bi-degree $(2,2)$-curve

$$x^2 + y^2 = \mathrm{sn}\left(z_0 - \gamma_{\mathrm{e}}\right)^2\left(1 + k^2x^2y^2\right) + 2\mathrm{cn}\left(z_0 - \gamma_{\mathrm{e}}\right)\mathrm{dn}\left(z_0 - \gamma_{\mathrm{e}}\right)xy.$$

Elliptic $A_0^{(1)}$ type $\Rightarrow$ RCG equation is an elliptic Painlevé equation

Moreover, the expression of the time evolution of the RCG equation in the affine Weyl group of type $E_8^{(1)}$ were found in [2].

Let $\phi : (x, y) \mapsto (\tilde{x}, \tilde{y})$ be the time evolution of the RCG equation. Then, it can be expressed by the element of $W\left(E_8^{(1)}\right) = \langle s_0, \ldots, s_8 \rangle$ as the following:

$$\phi = s_{56453483706756452348321564534837067564523483217067348304 68},$$

where $s_{i_1 \cdots i_m} = s_{i_1} \ldots s_{i_m}$, $i_1 \cdots i_m \in \{0, \ldots, 8\}$, $m \in \mathbb{Z}_{>0}$.

The time iteration ϕ turns out not to be given by translation on the $E_8^{(1)}$ lattice. However, its square (i.e., composition with itself) is a translation. The iteration ϕ^2 corresponds to next-nearest-neighbor-connecting vectors (NNVs) whose squared length is 4 on the weight-lattice of $E_8^{(1)}$.
Therefore, in this sense the birational action of ϕ^2 gives a different elliptic Painlevé equation from the well known elliptic Painlevé equation [8], which corresponds to nearest-neighbor-connecting vectors (NVs) whose squared length is 2.

Aim of this work

Although the geometry of the RCG equation has been clarified, its realization from the birational action of the affine Weyl group was missing since its base points are parametrized by the Jacobian elliptic function, and birational actions of the affine Weyl group on such setting were not explicitly known.

The present study fills this gap, that is, our main result provides the realization of the RCG equation as a half-translation of the affine Weyl group of type $E_8^{(1)}$. Moreover, we explicitly show the generic version of the RCG equation, which corresponds to NNVs.

Generalized base points

Base points: $p_i : (x, y) = \big(\mathrm{cd}\,(c_i + \eta)\,, \mathrm{cd}\,(\eta - c_i)\,\big), \quad i = 1, \ldots, 8$

(2, 2)-curve: $x^2 + y^2 = \mathrm{sn}\,(2\eta)^2\,(1 + k^2x^2y^2) + 2\mathrm{cn}\,(2\eta)\,\mathrm{dn}\,(2\eta)\,xy$

Note that the following transformations do not change the base points:

$$\iota_1 : (c_1, \ldots, c_8, \eta, x, y) \mapsto \left(c_1 - \frac{\mathrm{i}K'}{2}, \ldots, c_8 - \frac{\mathrm{i}K'}{2}, \eta - \frac{\mathrm{i}K'}{2}, \frac{1}{kx}, y\right),$$
$$\iota_2 : (c_1, \ldots, c_8, \eta, x, y) \mapsto \left(c_1 - \frac{\mathrm{i}K'}{2}, \ldots, c_8 - \frac{\mathrm{i}K'}{2}, \eta + \frac{\mathrm{i}K'}{2}, x, \frac{1}{ky}\right),$$
$$\iota_3 : (c_1, \ldots, c_8, \eta, x, y) \mapsto (c_1 - K, \ldots, c_8 - K, \eta - K, -x, y)\,,$$
$$\iota_4 : (c_1, \ldots, c_8, \eta, x, y) \mapsto (c_1 - K, \ldots, c_8 - K, \eta + K, x, -y)\,.$$

Specialization $\Big\Downarrow$
$$c_2 = c_1 + 2K, \quad c_3 = c_1 + \mathrm{i}K', \quad c_4 = c_1 + \kappa$$
$$c_6 = c_5 + 2K, \quad c_7 = c_5 + \mathrm{i}K', \quad c_8 = c_5 + \kappa$$
$$z_0 = \eta + c_5 + \kappa, \quad \gamma_{\mathrm{e}} = c_5 - \eta + \kappa, \quad \gamma_{\mathrm{o}} = \eta + c_1 + \kappa$$

The base points and (2, 2)-curve for the RCG equation.

Here, $\kappa = 2K + \mathrm{i}K'$.

Birational action of the affine Weyl group

Using the geometric approach investigated by Sakai [8], we obtain the following birational action of $W\left(E_8^{(1)}\right) = \langle s_0, \ldots, s_8 \rangle$ on the coordinates (x, y) and parameters c_i, $i = 1, \ldots, 8$, and η.

$$s_1(x) = y, \quad s_1(y) = x,$$

$$\left(\frac{s_2(y) - \mathrm{cd}\left(2\eta - \frac{c_1 - c_2}{2}\right)}{s_2(y) - \mathrm{cd}\left(2\eta + \frac{c_1 - c_2}{2}\right)}\right)\left(\frac{x - \mathrm{cd}\,(\eta + c_1)}{x - \mathrm{cd}\,(\eta + c_2)}\right)\left(\frac{y - \mathrm{cd}\,(\eta - c_2)}{y - \mathrm{cd}\,(\eta - c_1)}\right)$$

$$= \left(\frac{1 - \frac{\mathrm{cd}(\eta - c_2)}{\mathrm{cd}(\eta)}}{1 - \frac{\mathrm{cd}(\eta - c_1)}{\mathrm{cd}(\eta)}}\right)\left(\frac{1 - \frac{\mathrm{cd}(\eta + c_1)}{\mathrm{cd}(\eta)}}{1 - \frac{\mathrm{cd}(\eta + c_2)}{\mathrm{cd}(\eta)}}\right)\left(\frac{1 - \frac{\mathrm{cd}\left(2\eta - \frac{c_1 - c_2}{2}\right)}{\mathrm{cd}\left(\frac{c_1 + c_2}{2}\right)}}{1 - \frac{\mathrm{cd}\left(2\eta + \frac{c_1 - c_2}{2}\right)}{\mathrm{cd}\left(\frac{c_1 + c_2}{2}\right)}}\right),$$

$$s_0(c_7) = c_8, \quad s_0(c_8) = c_7, \quad s_1(\eta) = -\eta, \quad s_2(\eta) = \eta - \frac{2\eta + c_1 + c_2}{4},$$

$$s_2(c_i) = c_i - \frac{3(2\eta + c_1 + c_2)}{4}, \quad i = 1, 2, \quad s_2(c_j) = c_j + \frac{2\eta + c_1 + c_2}{4}, \quad j \neq 1, 2,$$

$$s_k(c_{k-1}) = c_k, \quad s_k(c_k) = c_{k-1}, \quad k = 3, \ldots, 7, \quad s_8(c_1) = c_2, \quad s_8(c_2) = c_1.$$

Note that $\lambda = \sum_{i=1}^{8} c_i$ is invariant under the action of $W\left(E_8^{(1)}\right)$.

Moreover, by adding the transformations ι_i, $i = 1, \ldots, 4$, $W\big(E_8^{(1)}\big)$ can be extended to $\widetilde{W}\big(E_8^{(1)}\big) = \langle \iota_1, \iota_2, \iota_3, \iota_4 \rangle \rtimes W\big(E_8^{(1)}\big)$.

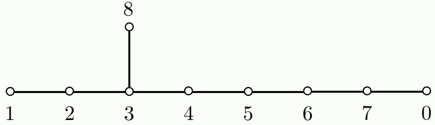
The following fundamental relations hold:

$$(s_i s_j)^{l_{ij}} = (\iota_i \iota_j)^{m_{ij}} = 1,$$
$$\iota_i s_j = s_j \iota_i, \quad i = 1,2,3,4,\ j \neq 1,2, \quad \iota_{\{1,2,3,4\}} s_1 = s_1 \iota_{\{2,1,4,3\}},$$
$$\iota_1 s_2 = s_2 \iota_1 \iota_2, \quad \iota_2 s_2 = s_2 \iota_2, \quad \iota_3 s_2 = s_2 \iota_3 \iota_4, \quad \iota_4 s_2 = s_2 \iota_4,$$

where

$$l_{ij} = \begin{cases} 1, & i = j \\ 3, & i = j-1 \quad (j = 2, \ldots, 7), \quad \text{or if} \quad (i,j) = (3,8), (7,0) \\ 2, & \text{otherwise}, \end{cases}$$

$$m_{ij} = \begin{cases} 1, & i = j \\ 2, & \text{otherwise}. \end{cases}$$



Derivations of the elliptic Painlevé equations

Let

$$\phi = s_{5645348370675645234832156453483706756452348321706734830468}\,\iota_4\iota_3\iota_2\iota_1.$$

The action of ϕ on the parameter space is not translational, but when the parameters take special values

$$c_2 = c_1 + 2K, \quad c_3 = c_1 + \mathrm{i}K', \quad c_4 = c_1 + 2K + \mathrm{i}K',$$
$$c_6 = c_5 + 2K, \quad c_7 = c_5 + \mathrm{i}K', \quad c_8 = c_5 + 2K + \mathrm{i}K',$$

the action of ϕ becomes the translational motion in the parameter subspace:

$$\phi : (\gamma_{\mathrm{e}}, \gamma_{\mathrm{o}}, z) \mapsto (\gamma_{\mathrm{e}}, \gamma_{\mathrm{o}}, z + 2(\gamma_{\mathrm{e}} + \gamma_{\mathrm{o}})),$$

where

$$z = \eta + c_5 + \kappa, \quad \gamma_{\mathrm{e}} = c_5 - \eta + \kappa, \quad \gamma_{\mathrm{o}} = \eta + c_1 + \kappa.$$

Then, the action of ϕ on the coordinates $\phi : (x, y) \mapsto (\tilde{x}, \tilde{y})$ gives the RCG equation.

The action of $\phi^2 : (c_i, \eta, x, y) \mapsto (\overline{c_i}, \eta + \lambda - 2\kappa, \overline{x}, \overline{y})$ gives the generic version of the RCG equation:

$$\left(\frac{k\,\mathrm{cd}\,(\eta - c_8 + \kappa)\,\overline{y} + 1}{k\,\mathrm{cd}\,(\eta - c_7 + \kappa)\,\overline{y} + 1}\right)\left(\frac{\tilde{x} - \mathrm{cd}\left(\eta - c_7 + \frac{c_{5678}}{2} + \lambda + \kappa\right)}{\tilde{x} - \mathrm{cd}\left(\eta - c_8 + \frac{c_{5678}}{2} + \lambda + \kappa\right)}\right)$$
$$= G_{\frac{c_{5678}-2c_5+\lambda}{2}, \frac{c_{5678}-2c_6+\lambda}{2}, \frac{c_{5678}-2c_7+\lambda}{2}, \frac{c_{5678}-2c_8+\lambda}{2}, \eta+\frac{\lambda}{2}+\kappa}$$
$$\times \frac{P_{\frac{c_{5678}-2c_5+\lambda}{2}, \frac{c_{5678}-2c_6+\lambda}{2}, \frac{c_{5678}-2c_7+\lambda}{2}, \eta+\frac{\lambda}{2}+\kappa}(\tilde{x}, \tilde{y})}{P_{\frac{c_{5678}-2c_5+\lambda}{2}, \frac{c_{5678}-2c_6+\lambda}{2}, \frac{c_{5678}-2c_8+\lambda}{2}, \eta+\frac{\lambda}{2}+\kappa}(\tilde{x}, \tilde{y})},$$

$$\left(\frac{k\,\mathrm{cd}\,(\eta + c_4 + \kappa)\,\overline{x} + 1}{k\,\mathrm{cd}\,(\eta + c_3 + \kappa)\,\overline{x} + 1}\right)\left(\frac{k\,\mathrm{cd}\,(\eta - c_3 + 2\lambda + \kappa)\,\overline{y} + 1}{k\,\mathrm{cd}\,(\eta - c_4 + 2\lambda + \kappa)\,\overline{y} + 1}\right)$$
$$= G_{\eta-c_1+\frac{c_{1234}}{4}+\lambda, \eta-c_2+\frac{c_{1234}}{4}+\lambda, \eta-c_3+\frac{c_{1234}}{4}+\lambda, \eta-c_4+\frac{c_{1234}}{4}+\lambda, \frac{c_{5678}+2\lambda}{4}+\kappa}$$
$$\times \frac{P_{\eta-c_1+\frac{c_{1234}}{4}+\lambda, \eta-c_2+\frac{c_{1234}}{4}+\lambda, \eta-c_3+\frac{c_{1234}}{4}+\lambda, \frac{c_{5678}+2\lambda}{4}+\kappa}\left(\frac{-1}{k\overline{y}}, \tilde{x}\right)}{P_{\eta-c_1+\frac{c_{1234}}{4}+\lambda, \eta-c_2+\frac{c_{1234}}{4}+\lambda, \eta-c_4+\frac{c_{1234}}{4}+\lambda, \frac{c_{5678}+2\lambda}{4}+\kappa}\left(\frac{-1}{k\overline{y}}, \tilde{x}\right)},$$

where $\overline{c_i} = c_i - \lambda$, $\overline{c_{i+4}} = c_{i+4} + \lambda + 4\kappa$, $i = 1, \ldots, 4$, $c_{j_1 \cdots j_n} = \sum_{i=1}^{n} c_{j_i}$ and $\tilde{x}$, $\tilde{y}$ are given by

$$\left(\frac{k\,\mathrm{cd}\left(\eta + c_8 - \frac{c_{5678}}{2}\right)\tilde{y} + 1}{k\,\mathrm{cd}\left(\eta + c_7 - \frac{c_{5678}}{2}\right)\tilde{y} + 1}\right)\left(\frac{x - \mathrm{cd}\,(\eta + c_7)}{x - \mathrm{cd}\,(\eta + c_8)}\right) = G_{c_5, c_6, c_7, c_8, \eta}\,\frac{P_{c_5, c_6, c_7, \eta}(x, y)}{P_{c_5, c_6, c_8, \eta}(x, y)},$$

$$\left(\frac{k\,\mathrm{cd}\left(\eta - c_4 + \frac{c_{1234}}{2}\right)\tilde{x} + 1}{k\,\mathrm{cd}\left(\eta - c_3 + \frac{c_{1234}}{2}\right)\tilde{x} + 1}\right)\left(\frac{k\,\mathrm{cd}\left(\eta + c_3 + \frac{c_{5678}}{2}\right)\tilde{y} + 1}{k\,\mathrm{cd}\left(\eta + c_4 + \frac{c_{5678}}{2}\right)\tilde{y} + 1}\right)$$
$$= G_{\eta+c_1+\frac{c_{5678}}{4}, \eta+c_2+\frac{c_{5678}}{4}, \eta+c_3+\frac{c_{5678}}{4}, \eta+c_4+\frac{c_{5678}}{4}, \frac{c_{5678}}{4}}$$
$$\times \frac{P_{\eta+c_1+\frac{c_{5678}}{4}, \eta+c_2+\frac{c_{5678}}{4}, \eta+c_3+\frac{c_{5678}}{4}, \frac{c_{5678}}{4}}\left(\frac{-1}{k\tilde{y}}, x\right)}{P_{\eta+c_1+\frac{c_{5678}}{4}, \eta+c_2+\frac{c_{5678}}{4}, \eta+c_4+\frac{c_{5678}}{4}, \frac{c_{5678}}{4}}\left(\frac{-1}{k\tilde{y}}, x\right)}.$$

Here,

$$G_{a_1,a_2,a_3,a_4,b} = \left(\frac{1 - \frac{\mathrm{cd}\left(a_4 + \frac{a_1+a_2}{2}\right)}{\mathrm{cd}\left(a_2 + \frac{a_1+a_2}{2}\right)}}{1 - \frac{\mathrm{cd}\left(a_3 + \frac{a_1+a_2}{2}\right)}{\mathrm{cd}\left(a_2 + \frac{a_1+a_2}{2}\right)}} \right) \left(\frac{1 - \frac{\mathrm{cd}(b-a_4)}{\mathrm{cd}(b-a_1)}}{1 - \frac{\mathrm{cd}(b-a_3)}{\mathrm{cd}(b-a_1)}} \right) \left(\frac{1 - \frac{\mathrm{cd}\left(b + a_4 - \frac{a_1+a_2+a_3+a_4}{2}\right)}{\mathrm{cd}\left(b + a_2 + \frac{a_1+a_2+a_3+a_4}{2}\right)}}{1 - \frac{\mathrm{cd}\left(b + a_3 - \frac{a_1+a_2+a_3+a_4}{2}\right)}{\mathrm{cd}\left(b + a_2 + \frac{a_1+a_2+a_3+a_4}{2}\right)}} \right)$$
$$\times \left(\frac{1 - \frac{\mathrm{cd}\left(a_3 + \frac{a_1+a_2}{2}\right)}{\mathrm{cd}\left(2b + a_2 - \frac{a_1+a_2}{2}\right)}}{1 - \frac{\mathrm{cd}\left(a_4 + \frac{a_1+a_2}{2}\right)}{\mathrm{cd}\left(2b + a_2 - \frac{a_1+a_2}{2}\right)}} \right),$$

$$\begin{aligned}
&Q_{a_1,a_2,a_3,a_4,a_5,b}(X) \\
&\quad = \left(\mathrm{cd}\left(b + a_3 - \frac{a_5}{2}\right) - \mathrm{cd}\left(b + a_2 + \frac{a_5}{2}\right)\right)\left(\mathrm{cd}\left(b + a_1 + \frac{a_5}{2}\right) - \mathrm{cd}\left(b + a_4 + \frac{a_5}{2}\right)\right) \\
&\qquad \times \Big(\mathrm{cd}(b+a_4)\,\mathrm{cd}(b+a_1) + \mathrm{cd}(b+a_2)\,X\Big) + \left(\mathrm{cd}\left(b + a_3 - \frac{a_5}{2}\right) - \mathrm{cd}\left(b + a_1 + \frac{a_5}{2}\right)\right) \\
&\qquad \times \left(\mathrm{cd}\left(b + a_4 + \frac{a_5}{2}\right) - \mathrm{cd}\left(b + a_2 + \frac{a_5}{2}\right)\right)\Big(\mathrm{cd}(b+a_4)\,\mathrm{cd}(b+a_2) + \mathrm{cd}(b+a_1)\,X\Big) \\
&\quad - \left(\mathrm{cd}\left(b + a_3 - \frac{a_5}{2}\right) - \mathrm{cd}\left(b + a_4 + \frac{a_5}{2}\right)\right)\left(\mathrm{cd}\left(b + a_1 + \frac{a_5}{2}\right) - \mathrm{cd}\left(b + a_2 + \frac{a_5}{2}\right)\right) \\
&\qquad \times \Big(\mathrm{cd}(b+a_1)\,\mathrm{cd}(b+a_2) + \mathrm{cd}(b+a_4)\,X\Big),
\end{aligned}$$

$$P_{a_1,a_2,a_3,b}(X,Y) = C_1XY + C_2X + C_3Y + C_4,$$

where

$$\begin{aligned}
C_1 =& \Big(\mathrm{cd}(b-a_3) - \mathrm{cd}(b-a_2)\Big)\mathrm{cd}(b+a_1) + \Big(\mathrm{cd}(b-a_1) - \mathrm{cd}(b-a_3)\Big)\mathrm{cd}(b+a_2) \\
&+ \Big(\mathrm{cd}(b-a_2) - \mathrm{cd}(b-a_1)\Big)\mathrm{cd}(b+a_3),
\end{aligned}$$

$$\begin{aligned}
C_2 =& \Big(\mathrm{cd}(b-a_2) - \mathrm{cd}(b-a_3)\Big)\mathrm{cd}(b-a_1)\,\mathrm{cd}(b+a_1) \\
&+ \Big(\mathrm{cd}(b-a_3) - \mathrm{cd}(b-a_1)\Big)\mathrm{cd}(b-a_2)\,\mathrm{cd}(b+a_2) \\
&+ \Big(\mathrm{cd}(b-a_1) - \mathrm{cd}(b-a_2)\Big)\mathrm{cd}(b-a_3)\,\mathrm{cd}(b+a_3),
\end{aligned}$$

$$\begin{aligned}
C_3 =& \Big(\mathrm{cd}(b+a_3) - \mathrm{cd}(b+a_2)\Big)\mathrm{cd}(b-a_1)\,\mathrm{cd}(b+a_1) \\
&+ \Big(\mathrm{cd}(b+a_1) - \mathrm{cd}(b+a_3)\Big)\mathrm{cd}(b-a_2)\,\mathrm{cd}(b+a_2) \\
&+ \Big(\mathrm{cd}(b+a_2) - \mathrm{cd}(b+a_1)\Big)\mathrm{cd}(b-a_3)\,\mathrm{cd}(b+a_3),
\end{aligned}$$

$$\begin{aligned}
C_4 =& \Big(\mathrm{cd}(b+a_2)\,\mathrm{cd}(b-a_3) - \mathrm{cd}(b-a_2)\,\mathrm{cd}(b+a_3)\Big)\mathrm{cd}(b-a_1)\,\mathrm{cd}(b+a_1) \\
&+ \Big(\mathrm{cd}(b+a_3)\,\mathrm{cd}(b-a_1) - \mathrm{cd}(b-a_3)\,\mathrm{cd}(b+a_1)\Big)\mathrm{cd}(b-a_2)\,\mathrm{cd}(b+a_2) \\
&+ \Big(\mathrm{cd}(b+a_1)\,\mathrm{cd}(b-a_2) - \mathrm{cd}(b-a_1)\,\mathrm{cd}(b+a_2)\Big)\mathrm{cd}(b-a_3)\,\mathrm{cd}(b+a_3).
\end{aligned}$$

V. E. Adler.
Bäcklund transformation for the Krichever-Novikov equation.
Internat. Math. Res. Notices, (1):1–4, 1998.

J. Atkinson, P. Howes, N. Joshi, and N. Nakazono.
Geometry of an elliptic difference equation related to Q4.
J. Lond. Math. Soc. (2), 93(3):763–784, 2016.

É. Brézin and V. A. Kazakov.
Exactly solvable field theories of closed strings.
Phys. Lett. B, 236(2):144–150, 1990.

B. Grammaticos, A. Ramani, and V. Papageorgiou.
Do integrable mappings have the Painlevé property?
Phys. Rev. Lett., 67(14):1825–1828, 1991.

C. Itzykson and J. B. Zuber.
The planar approximation. II.
J. Math. Phys., 21(3):411–421, 1980.

K. Okamoto.
Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé.
Japan. J. Math. (N.S.), 5(1):1–79, 1979.

A. Ramani, A. S. Carstea, and B. Grammaticos.
On the non-autonomous form of the Q_4 mapping and its relation to elliptic Painlevé equations.
J. Phys. A, 42(32):322003, 8, 2009.

H. Sakai.
Rational surfaces associated with affine root systems and geometry of the Painlevé equations.
Comm. Math. Phys., 220(1):165–229, 2001.