Series solutions of the non-stationary Heun equation

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Aim:

Solve a special PDE known under many different names, *e.g.*,

– **non-stationary Heun equation** [2, 3]
– **quantum Painlevé VI** [4]
– **KZB heat equation** [5]

given by

$$\begin{aligned}\left(\frac{\mathrm{i}}{\pi}\kappa\frac{\partial}{\partial\tau} + H(x;\{g_\nu\}_{\nu=0}^{3})\right)\psi(x) = E\psi(x),\\ H(x;\{g_\nu\}_{\nu=0}^{3}) = -\frac{\partial^2}{\partial x^2} + \sum_{\nu=0}^{3} g_\nu(g_\nu-1)\wp(x+\omega_\nu)\end{aligned} \qquad (1)$$

where $\wp(x)$ is the Weierstrass elliptic function with periods $(2\pi, 2\pi\tau)$ [8],

$$\omega_0 = 0,\ \omega_1 = \pi,\ \omega_2 = -\pi - \pi\tau,\ \omega_3 = \pi\tau \quad (\Im(\tau) > 0).$$

Has appeared in context of

– quantum statistical physics (*e.g.*, 8VSOS [3])
– conformal field theory (*e.g.*, quantum Liouville theory [6])
– representation theory (*e.g.*, $\widehat{\mathfrak{sl}}_2$ [7])
…

Solutions defined by the following conditions:

- Of the form

$$\begin{aligned}\psi_n(x) &= (2q^{\frac14})^{-g_0-g_1}\left(\prod_{\nu=0}^{3}\theta_{\nu+1}(\tfrac12 x)^{g_\nu}\right)\mathcal{P}_n(\cos(x))\\ E_n &= \kappa^2\left(\frac{1}{12} - \frac{\eta_1}{\pi}\right) - \sum_{\nu=0}^{3} g_\nu(g_\nu-1)\frac{\eta_1}{\pi} + \mathcal{E}_n\end{aligned} \qquad (2)$$

where θ_ν are the Jacobi theta functions [8].

- $\mathcal{P}_n(z)$ (resp. $\mathcal{E}_n$) analytic function of z and the nomé $q = \exp(\mathrm{i}\pi\tau)$ (resp. q) with expansions

$$\mathcal{P}_n(z) = \sum_{\ell=0}^{\infty}\mathcal{P}_n^{(\ell)}(z)q^\ell, \quad \mathcal{E}_n = \sum_{\ell=0}^{\infty}\mathcal{E}_n^{(\ell)}q^\ell$$

where $\mathcal{P}_n^{(\ell)}(z)$ are polynomials of degree $n+\ell$.

- $\mathcal{P}_n$ reduces to the Jacobi polynomials in the limit $q \to 0$, *i.e.*, $\mathcal{P}_n^{(0)} = P_n^{(g_0-\frac12, g_1-\frac12)}$ and $\mathcal{E}_n^{(0)} = (n + \frac12(g_0+g_1))^2$.

$\Rightarrow$ **Elliptic generalization of the Jacobi polynomials.**

Tool: Generalized kernel function

- Removing the balancing conditions for elliptic kernel function identities yields non-stationary generalizations.
- The generalized kernel function identity for the non-stationary Heun equation [2]:

$$\left(\frac{\mathrm{i}}{\pi}\kappa\frac{\partial}{\partial\tau} + H(x;\{g_\nu\}_{\nu=0}^{3}) - H(y;\{\tilde g_\nu\}_{\nu=0}^{3}) - C_{1,1}\right)\mathcal{K}(x,y) = 0$$

where $\tilde g_\nu = \lambda - g_\nu$ $(\nu = 0,1,2,3)$, $\lambda = \frac12(g_0+g_1+g_2+g_3-\kappa)$, and $C_{1,1}$ an analytic function of q.

- The kernel function [2]:

$$\mathcal{K}(x,y) = \frac{\prod_{\nu=0}^{3}\theta_{\nu+1}(\frac12 x)^{g_\nu}\theta_{\nu+1}(\frac12 y)^{\tilde g_\nu}}{\left(\theta_1(\frac12(x+y))\theta_1(\frac12(x-y))\right)^{\lambda}}$$

Result 1: Explicit solutions in special cases

Let $n \in \mathbb{Z}$, $\tilde g_\nu \in \{0,1\}$, and $\kappa = 2\lambda - \sum_{\nu=0}^{3}\tilde g_\nu$ such that $-\lambda \notin \mathbb{N}_0$ for $n > 0$ and $-(g_0+g_1) \notin \mathbb{N}_0$ for $n < 0$ ($g_\nu = \lambda - \tilde g_\nu$ for $\nu = 0,1,2,3$). Then the non-stationary Heun equation has solutions $\psi_n(x)$ and E_n in (2) with

$$\mathcal{P}_n(z) = \mathcal{N}_n\oint_{|\xi|\uparrow 1}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}\xi}\xi^{-n}\frac{\prod_{\nu=0}^{3}\Theta_{\nu+1}(\xi)^{\tilde g_\nu}}{\Theta(z,\xi)^{\lambda}}, \quad \mathcal{E}_n = (n + \tfrac12(g_0+g_1))^2.$$

($\mathcal{N}_n$ a normalization constant,

$$\Theta_1(\xi) = (1-\xi)\prod_{n\in\mathbb{N}}(1-q^{2n}\xi)(1-q^{2n}\xi^{-1}), \quad \Theta_2(\xi) = \Theta_1(-\xi)$$

$$\Theta_3(\xi) = \prod_{n\in\mathbb{N}}(1+q^{2n-1}\xi)(1+q^{2n-1}\xi^{-1}), \quad \Theta_4(\xi) = \Theta_3(-\xi)$$

$$\Theta(z,\xi) = (1-2z\xi+\xi^2)\prod_{n\in\mathbb{N}}(1-2q^{2n}\xi z+q^{4n}\xi^2)(1-2q^{2n}\xi^{-1}z+q^{4n}\xi^{-2}).)$$

Result 2: Series solutions for general parameters

- Construct an unconventional basis using the kernel function

$$F_m(x) = \int_{-\pi+\mathrm{i}0^+}^{\pi+\mathrm{i}0^+}\frac{\mathrm{d}y}{2\pi}\,\mathcal{K}(x,y)\,\mathrm{e}^{-\mathrm{i}(m+\frac12(g_0+g_1))y} \quad (m\in\mathbb{Z}).$$

- Then

$$\psi_n(x) = \mathcal{N}_n\sum_{m\in\mathbb{Z}}\alpha_n(m)F_m(x),$$

is a solution of (1) if the coefficients $\alpha_n(m)$ satisfy the differential-difference equation

$$\begin{aligned}\left[\frac{\mathrm{i}}{\pi}\kappa\frac{\partial}{\partial\tau} + \mathcal{E}_n^{(0)} - \mathcal{E}_n\right]\alpha_n(m) &= \sum_{\mu=1}^{\infty}\mu\gamma_0^\mu\alpha_n(m+\mu)\\ &+ \sum_{\mu=1}^{\infty}\frac{\mu q^\mu}{1-q^{2\mu}}(\gamma_0^\mu q^\mu + \gamma_1^\mu)\left[\alpha_n(m+\mu)+\alpha_n(m-\mu)\right]\end{aligned}$$

with shorthand notation

$$\gamma_k^\mu = \begin{cases}\tilde g_0(\tilde g_0-1) + (-1)^\mu\tilde g_1(\tilde g_1-1) & \text{if } \frac12 k\in\mathbb{N}_0\\ (-1)^\mu\tilde g_2(\tilde g_2-1) + \tilde g_3(\tilde g_3-1) & \text{if } \frac12(k-1)\in\mathbb{N}_0\end{cases} \quad (\mu\in\mathbb{N}).$$

Solving PDE in (1) $\Leftrightarrow$ solving a differential-difference equation.

- Expanding $\alpha_n(m) = \sum_\ell\alpha_n^{(\ell)}(m)q^\ell$ yields recursive solution algorithm:

$$\begin{aligned}((m-n)(m+n+g_0+g_1)-\kappa\ell)\alpha_n^{(\ell)}(m) &= \sum_{\ell'=1}^{\ell}\mathcal{E}_n^{(\ell')}\alpha_n^{(\ell-\ell')}(m)\\ &+ \sum_{\mu=1}^{n-m+\ell}\mu\gamma_0^\mu\alpha_n^{(\ell)}(m+\mu) + \sum_{\ell'=0}^{\ell-1}\sum_{\mu=1}^{\ell-\ell'}\sum_{k=1}^{\lfloor\frac{\ell-\ell'}{\mu}\rfloor}\mu\gamma_k^\mu\delta_{\ell,\ell'+k\mu}\left[\alpha_n^{(\ell')}(m+\mu)+\alpha_n^{(\ell')}(m-\mu)\right],\end{aligned}$$

where $n\in\mathbb{Z}$, $-\lambda\notin\mathbb{N}_0$ for $n>0$ and $-(g_0+g_1)\notin\mathbb{N}_0$ for $n<0$, together with

$$\alpha_n^{(0)}(n) = 1,\ \alpha_n^{(\ell)}(n) = 0\ \forall\ell\geq 1,\ \alpha_n^{(\ell)}(m) = 0\ \forall m > n+\ell, \ell\geq 0,$$

$$\mathcal{E}_n^{(\ell)} = -\sum_{\mu=1}^{\ell}\mu\gamma_0^\mu\alpha_n^{(\ell)}(n+\mu) - \sum_{\ell'=0}^{\ell-1}\sum_{\mu=1}^{\ell-\ell'}\sum_{k=1}^{\lfloor\frac{\ell-\ell'}{\mu}\rfloor}\mu\gamma_k^\mu\delta_{\ell,\ell'+k\mu}\left[\alpha_n^{(\ell')}(n+\mu)+\alpha_n^{(\ell')}(n-\mu)\right]$$

(see Section 5 in [1]).

- The coefficients $\alpha_n^{(\ell)}(m)$ can be obtained to all orders by solving a particular combinatorial problem: see Theorem 6.3 in [1].

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