Elliptic solutions of the Yang-Baxter equation

Dmitry Chicherin

Johannes Gutenberg University, Mainz, Germany

20 March 2017, Vienna

Based on work in collaboration with
S.É. Derkachov and V.P. Spiridonov
Comm. Math. Phys. (2016) **345**

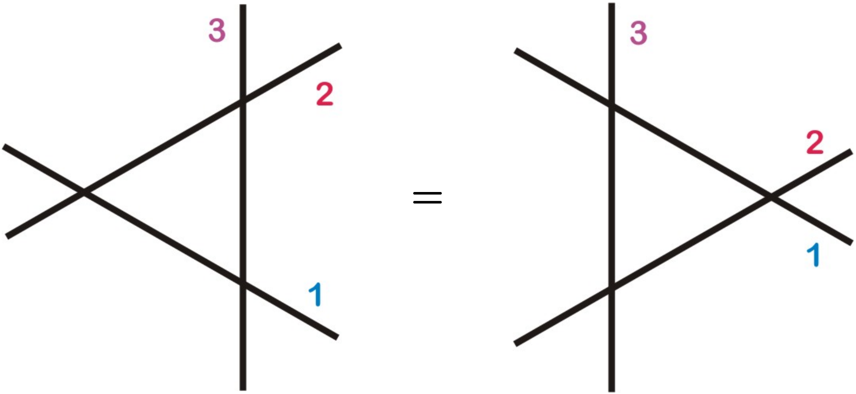
Overview

- The Yang-Baxter equation
- The master solution (integral operator with elliptic hypergeometric kernel)
- Continuous spin lattice models
- The star-triangle relation
- Factorization of the integral R-operator
- Finite-dimensional reductions

The Yang-Baxter equation

$$\mathbb{R}_{ik}(u) \ : \ V_i \otimes V_k \ \to \ V_i \otimes V_k$$

$$\mathbb{R}_{12}(u-v)\, \mathbb{R}_{13}(u)\, \mathbb{R}_{23}(v) \ = \ \mathbb{R}_{23}(v)\, \mathbb{R}_{13}(u)\, \mathbb{R}_{12}(u-v)$$



The Yang-Baxter equation

$$\mathbb{R}_{ik}(u) : V_i \otimes V_k \to V_i \otimes V_k$$

Master solution	inf.dim. $\otimes$ inf.dim.	Integral operator with elliptic hypergeometric kernel
	$\downarrow$	
	fin.dim. $\otimes$ inf.dim.	Matrix of finite-difference operators
	$\downarrow$	
	fin.dim. $\otimes$ fin.dim.	Matrix with numerical entries

- Fin. dim. irrep. – physical discrete spin systems on lattices (Ising model)
- Inf. dim. irrep. – noncompact spin chains, continuous spin systems

Basic definitions

The infinite q-product

$$(z;q)_\infty \equiv \prod_{k=0}^{\infty}(1-zq^k) \quad , \quad |q|<1$$

The shorten theta function

$$\theta(z;p) \equiv (z;p)_\infty (p\,z^{-1};p)_\infty = \frac{1}{(p;p)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k p^{\frac{k(k-1)}{2}} z^k$$

Quasi-periodicity

$$\theta(p\,z;p) = -z^{-1}\,\theta(z;p)$$

The elliptic gamma function

$$\Gamma(z;p,q) = \prod_{i,j=0}^{\infty}\frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j} \quad , \quad |p|<1\,,\ |q|<1$$

[S.N.M. Ruijsenaars '97]

Two quasi periods

$$\Gamma(q\,z;p,q) = \theta(z;p)\,\Gamma(z;p,q) \quad , \quad \Gamma(p\,z;p,q) = \theta(z;q)\,\Gamma(z;p,q)$$

The elliptic beta integral

Shorthand notation $\Gamma(tz^{\pm 1}) \equiv \Gamma(tz)\Gamma(tz^{-1})$, $\Gamma(tx^{\pm 1}z^{\pm 1}) \equiv \Gamma(txz^{\pm 1})\Gamma(tx^{-1}z^{\pm 1})$

Exactly computable elliptic hypergeometric integral

$$\kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^{6} \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{\mathrm{i}z} = \prod_{1 \le j < k \le 6} \Gamma(t_j t_k; p, q)$$

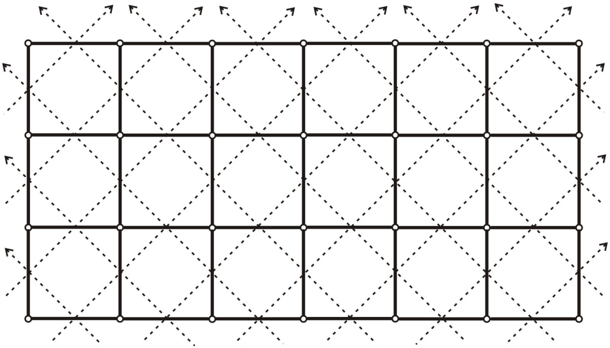
[V.Spiridonov '01]

balancing condition $\displaystyle\prod_{j=1}^{6} t_j = pq$

$$\kappa \equiv \frac{(p;p)_\infty (q;q)_\infty}{4\pi} \ , \ |t_j| < 1 \ , \ j = 1, \dots, 6$$

The most general exact univariate integration formula; generalization of the Euler beta integral $\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

Solvable 2d lattice models



Partition function

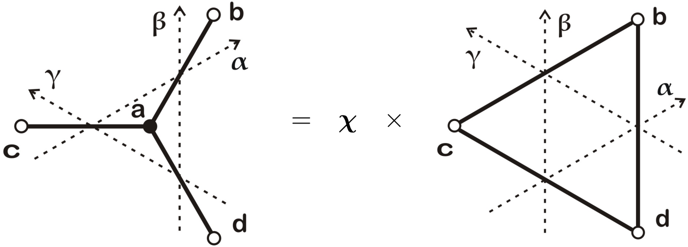
$$\mathcal{Z} = \int\limits_{\underbrace{\mathbb{T}\times\ldots\times\mathbb{T}}_{\#\ \text{of vertex}}} \prod_{\substack{\text{horizontal}\\ \text{edges}\\ <i,j>}} W_{\alpha_i-\beta_j}(a_i, a_j) \prod_{\substack{\text{vertical}\\ \text{edges}\\ <k,l>}} \overline{W}_{\alpha_k-\beta_l}(a_k, a_l) \prod_{\substack{\text{vertices}\\ m}} \rho(a_m)\, da_m$$

Continuous spin variables $a_m \in \mathbb{T}$. Rapidities α_i, β_j

Star-triangle relation and Z-invariance

$$\int_0^1 \rho(a) W_{\xi-(\alpha-\beta)}(a,b)\, W_{\alpha-\gamma}(c,a) W_{\xi-(\beta-\gamma)}(d,a)\, da$$
$$= \chi \cdot W_{\alpha-\beta}(c,d)\, W_{\xi-(\alpha-\gamma)}(d,b)\, W_{\beta-\gamma}(c,b)$$

[V.Bazhanov, S.Sergeev '10]



Spin variables $a, b, c, d, \ldots$ Rapidities $\alpha, \beta, \gamma, \ldots$

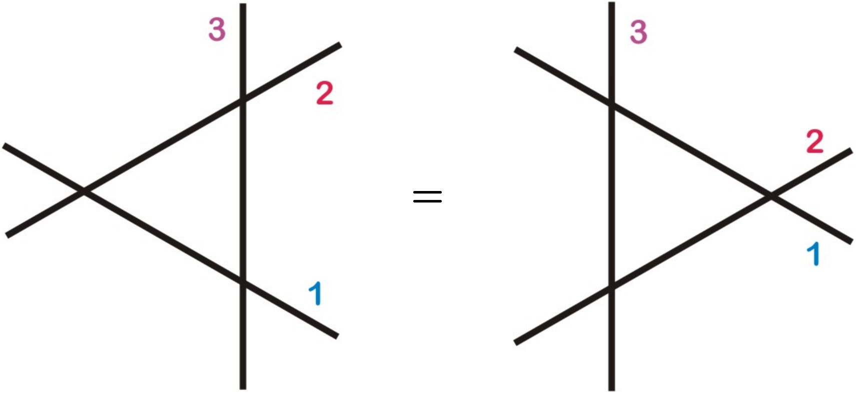
Edge Boltzmann weights $W_\alpha(a,b) = W_\alpha(b,a)$, $\overline{W}_\alpha(a,b) = W_{\xi-\alpha}(a,b)$

$W_\alpha(a,b) = \Gamma\left(\sqrt{pq}\, e^{2\pi \mathrm{i}(\alpha \pm a \pm b)}\right)$ Self-interaction $\rho(a) = \kappa/\Gamma\left(e^{\pm 4\pi \mathrm{i} a}\right)$

Normalization $\chi = \chi(\alpha, \beta, \gamma)$ Crossing-parameter ξ , $e^{-2\xi} \equiv pq$

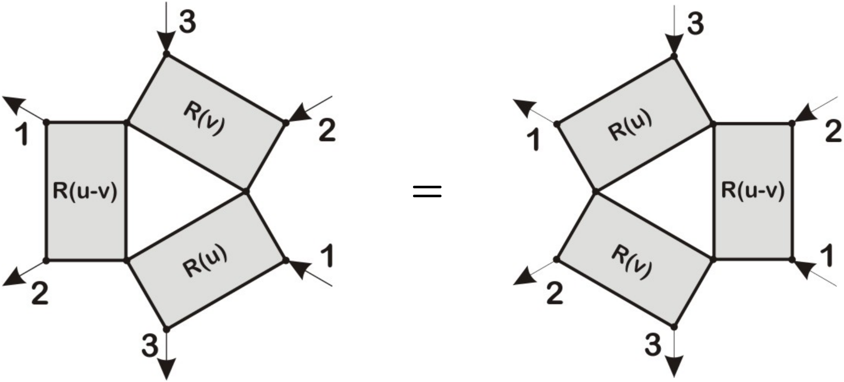
The Yang-Baxter equation

$$\mathbb{R}_{12}(u-v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{23}(v) = \mathbb{R}_{23}(v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{12}(u-v)$$



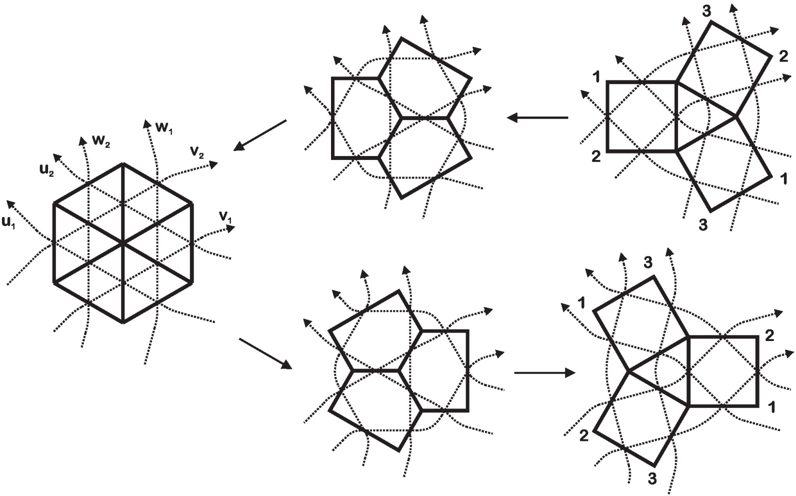
The Yang-Baxter equation

$$\mathbb{R}_{12}(u-v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{23}(v) = \mathbb{R}_{23}(v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{12}(u-v)$$



The Yang-Baxter equation

$$\mathbb{R}_{12}(u_1, u_2 | v_1, v_2)\, \mathbb{R}_{13}(u_1, u_2 | w_1, w_2)\, \mathbb{R}_{23}(v_1, v_2 | w_1, w_2)$$
$$= \mathbb{R}_{23}(v_1, v_2 | w_1, w_2)\, \mathbb{R}_{13}(u_1, u_2 | w_1, w_2)\, \mathbb{R}_{12}(u_1, u_2 | v_1, v_2)$$



From the lattice model to the master R-operator

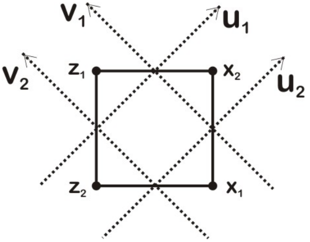
Boltzmann weights

$$W_u(x,z) = \Gamma(\sqrt{pq}e^{2\pi i(u\pm x\pm z)})$$

$$\overline{W}_u(x,z) = W_{\xi-u}(x,z)$$

Self-interaction

$$\rho(x) = \kappa/\Gamma(e^{\pm 4\pi i x})$$



$$\Big[\mathbb{R}_{12}(u-v|g_1,g_2)\,\Phi\Big](z_1,z_2) = W_{u_1-v_2}(z_1,z_2)\times$$

$$\times\int \overline{W}_{u_1-v_1}(z_1,x_2)\,\overline{W}_{u_2-v_2}(z_2,x_1)\,W_{u_2-v_1}(x_1,x_2)\,\rho(x_1)\,\rho(x_2)\,\Phi(x_1,x_2)\,dx_1\,dx_2$$

[S.Derkachov, V.Spiridonov '12]

The R-operator $\in \mathrm{End}(V_{g_1}\otimes V_{g_2})$

Spectral parameters u, v, g-spins g_1, g_2

Partition function of an elementary cell

Rapidities $u_1,\ u_2,\ v_1,\ v_2$

$$u_1 = \frac{u+g_1}{2}\ ,\ u_2 = \frac{u-g_1}{2}\ ,\ v_1 = \frac{v+g_2}{2}\ ,\ v_2 = \frac{v-g_2}{2}$$

From the lattice model to the master R-operator

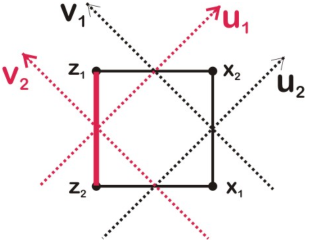
Boltzmann weights

$$W_u(x,z) = \Gamma(\sqrt{pq}e^{2\pi \mathrm{i}(u\pm x\pm z)})$$

$$\overline{W}_u(x,z) = W_{\xi-u}(x,z)$$

Self-interaction

$$\rho(x) = \kappa/\Gamma(e^{\pm 4\pi \mathrm{i} x})$$



$$\Big[\mathbb{R}_{12}(u-v|g_1,g_2)\,\Phi\Big](z_1,z_2) = \mathbf{W}_{\mathbf{u_1-v_2}}(\mathbf{z_1},\mathbf{z_2})\times$$

$$\times\int \overline{W}_{u_1-v_1}(z_1,x_2)\,\overline{W}_{u_2-v_2}(z_2,x_1)\,W_{u_2-v_1}(x_1,x_2)\,\rho(x_1)\,\rho(x_2)\,\Phi(x_1,x_2)\,dx_1\,dx_2$$

[S.Derkachov, V.Spiridonov '12]

The R-operator $\in \mathrm{End}(V_{g_1}\otimes V_{g_2})$

Spectral parameters u, v, g-spins g_1, g_2

Partition function of an elementary cell

Rapidities $u_1,\ u_2,\ v_1,\ v_2$

$$u_1 = \frac{u+g_1}{2}\ ,\ u_2 = \frac{u-g_1}{2}\ ,\ v_1 = \frac{v+g_2}{2}\ ,\ v_2 = \frac{v-g_2}{2}$$

From the lattice model to the master R-operator

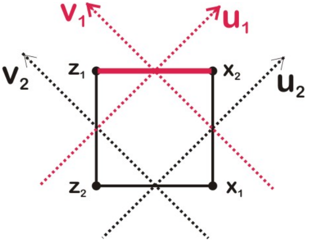
Boltzmann weights

$$W_u(x,z) = \Gamma(\sqrt{pq}e^{2\pi i(u\pm x\pm z)})$$

$$\overline{W}_u(x,z) = W_{\xi-u}(x,z)$$

Self-interaction

$$\rho(x) = \kappa/\Gamma(e^{\pm 4\pi i x})$$



$$\Big[\mathbb{R}_{12}(u-v|g_1,g_2)\,\Phi\Big](z_1,z_2) = W_{u_1-v_2}(z_1,z_2)\times$$

$$\times\int \mathbf{W}_{u_1-v_1}(z_1,x_2)\,\overline{W}_{u_2-v_2}(z_2,x_1)\,W_{u_2-v_1}(x_1,x_2)\,\rho(x_1)\,\rho(x_2)\,\Phi(x_1,x_2)\,dx_1dx_2$$

[S.Derkachov, V.Spiridonov '12]

The R-operator $\in \mathrm{End}(V_{g_1}\otimes V_{g_2})$

Spectral parameters u, v, g-spins g_1, g_2

Partition function of an elementary cell

Rapidities $u_1,\ u_2,\ v_1,\ v_2$

$$u_1 = \frac{u+g_1}{2}\ ,\quad u_2 = \frac{u-g_1}{2}\ ,\quad v_1 = \frac{v+g_2}{2}\ ,\quad v_2 = \frac{v-g_2}{2}$$

From the lattice model to the master R-operator

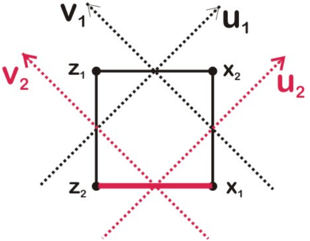
Boltzmann weights

$$W_u(x,z) = \Gamma(\sqrt{pq}e^{2\pi \mathrm{i}(u\pm x\pm z)})$$

$$\overline{W}_u(x,z) = W_{\xi-u}(x,z)$$

Self-interaction

$$\rho(x) = \kappa/\Gamma(e^{\pm 4\pi \mathrm{i} x})$$



$$\Big[\mathbb{R}_{12}(u-v|g_1,g_2)\,\Phi\Big](z_1,z_2) = W_{u_1-v_2}(z_1,z_2)\times$$

$$\times\int \overline{W}_{u_1-v_1}(z_1,x_2)\,\overline{W}_{u_2-v_2}(z_2,x_1)\,W_{u_2-v_1}(x_1,x_2)\,\rho(x_1)\,\rho(x_2)\,\Phi(x_1,x_2)\,dx_1dx_2$$

[S.Derkachov, V.Spiridonov '12]

The R-operator $\in \mathrm{End}(V_{g_1}\otimes V_{g_2})$

Spectral parameters u, v, g-spins g_1, g_2

Partition function of an elementary cell

Rapidities $u_1,\ u_2,\ v_1,\ v_2$

$$u_1 = \frac{u+g_1}{2}\ ,\quad u_2 = \frac{u-g_1}{2}\ ,\quad v_1 = \frac{v+g_2}{2}\ ,\quad v_2 = \frac{v-g_2}{2}$$

From the lattice model to the master R-operator

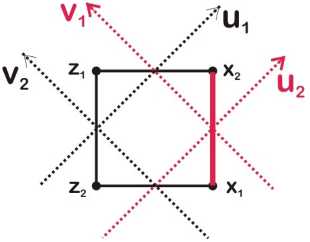
Boltzmann weights

$W_u(x,z) = \Gamma(\sqrt{pq}e^{2\pi \mathrm{i}(u \pm x \pm z)})$

$\overline{W}_u(x,z) = W_{\xi - u}(x,z)$

Self-interaction

$\rho(x) = \kappa / \Gamma(e^{\pm 4\pi \mathrm{i} x})$



$$\left[\mathbb{R}_{12}(u-v|g_1,g_2)\,\Phi\right](z_1,z_2) = W_{u_1-v_2}(z_1,z_2)\times$$

$$\times \int \overline{W}_{u_1-v_1}(z_1,x_2)\,\overline{W}_{u_2-v_2}(z_2,x_1)\,\mathbf{W}_{u_2-v_1}(\mathbf{x}_1,\mathbf{x}_2)\,\rho(x_1)\,\rho(x_2)\,\Phi(x_1,x_2)\,dx_1\,dx_2$$

[S.Derkachov, V.Spiridonov '12]

The R-operator $\in \mathrm{End}(V_{g_1} \otimes V_{g_2})$

Spectral parameters u, v, g-spins g_1, g_2

Partition function of an elementary cell

Rapidities $u_1,\ u_2,\ v_1,\ v_2$

$$u_1 = \frac{u+g_1}{2}\ ,\quad u_2 = \frac{u-g_1}{2}\ ,\quad v_1 = \frac{v+g_2}{2}\ ,\quad v_2 = \frac{v-g_2}{2}$$

From the lattice model to the master R-operator

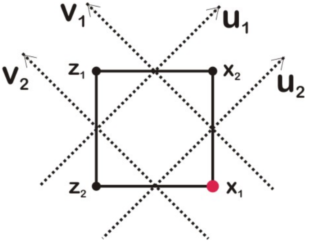
Boltzmann weights

$$W_u(x,z) = \Gamma(\sqrt{pq}e^{2\pi i(u\pm x\pm z)})$$

$$\overline{W}_u(x,z) = W_{\xi-u}(x,z)$$

Self-interaction

$$\rho(x) = \kappa/\Gamma(e^{\pm 4\pi i x})$$



$$\Big[\mathbb{R}_{12}(u-v|g_1,g_2)\,\Phi\Big](z_1,z_2) = W_{u_1-v_2}(z_1,z_2)\times$$

$$\times\int \overline{W}_{u_1-v_1}(z_1,x_2)\,\overline{W}_{u_2-v_2}(z_2,x_1)\,W_{u_2-v_1}(x_1,x_2)\,\rho(x_1)\,\rho(x_2)\,\Phi(x_1,x_2)\,dx_1\,dx_2$$

[S.Derkachov, V.Spiridonov '12]

The R-operator $\in \mathrm{End}(V_{g_1}\otimes V_{g_2})$

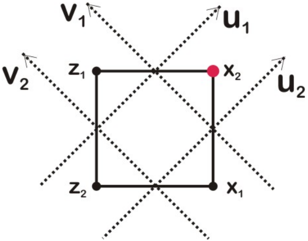
Spectral parameters u, v, g-spins g_1, g_2

Partition function of an elementary cell

Rapidities $u_1,\ u_2,\ v_1,\ v_2$

$$u_1 = \frac{u+g_1}{2}\ ,\quad u_2 = \frac{u-g_1}{2}\ ,\quad v_1 = \frac{v+g_2}{2}\ ,\quad v_2 = \frac{v-g_2}{2}$$

From the lattice model to the master R-operator



Boltzmann weights

$$W_u(x,z) = \Gamma(\sqrt{pq} e^{2\pi i(u \pm x \pm z)})$$

$$\overline{W}_u(x,z) = W_{\xi - u}(x,z)$$

Self-interaction

$$\rho(x) = \kappa / \Gamma(e^{\pm 4\pi i x})$$

$$\Big[\mathbb{R}_{12}(u-v|g_1,g_2)\,\Phi\Big](z_1,z_2) = W_{u_1-v_2}(z_1,z_2) \times$$

$$\times \int \overline{W}_{u_1-v_1}(z_1,x_2)\,\overline{W}_{u_2-v_2}(z_2,x_1)\,W_{u_2-v_1}(x_1,x_2)\,\rho(x_1)\,\rho(x_2)\,\Phi(x_1,x_2)\,dx_1\,dx_2$$

[S.Derkachov, V.Spiridonov '12]

The R-operator $\in \mathrm{End}(V_{g_1} \otimes V_{g_2})$

Spectral parameters u, v, g-spins g_1, g_2

Partition function of an elementary cell

Rapidities $u_1,\ u_2,\ v_1,\ v_2$

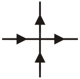
$$u_1 = \frac{u+g_1}{2}\ ,\quad u_2 = \frac{u-g_1}{2}\ ,\quad v_1 = \frac{v+g_2}{2}\ ,\quad v_2 = \frac{v-g_2}{2}$$

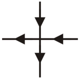
Baxter's R-matrix and 8-vertex model

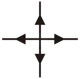
Matrix 4×4 solution of the YBE [R. Baxter '72]

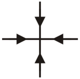
$$\mathbb{R}_{12}(u) = \sum_{\alpha=0}^{3} \frac{\theta_{\alpha+1}(u+\eta|\tau)}{\theta_{\alpha+1}(\eta|\tau)}\, \sigma_\alpha \otimes \sigma_\alpha$$

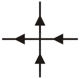
θ_α – Jacobi theta functions, σ_α – Pauli matrices, $\mathbf{p} \equiv \mathbf{e}^{2\pi i \tau}$, $\mathbf{q} \equiv \mathbf{e}^{4\pi i \eta}$

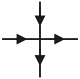

$= \mathbb{R}^{++}_{++}$

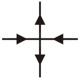

$= \mathbb{R}^{--}_{--}$

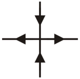

$= \mathbb{R}^{++}_{--}$


$= \mathbb{R}^{--}_{++}$


$= \mathbb{R}^{+-}_{+-}$


$= \mathbb{R}^{-+}_{-+}$


$= \mathbb{R}^{+-}_{-+}$


$= \mathbb{R}^{-+}_{+-}$

Sklyanin algebra and XYZ spin chain

Lax operator – Matrix 2×2 solution of the YBE with operator entries

$$\mathrm{L}(u) = \sum_{\alpha=0}^{3} \frac{\theta_{\alpha+1}(u+\eta|\tau)}{\theta_{\alpha+1}(\eta|\tau)}\, \sigma_\alpha \otimes \mathbf{S}^\alpha$$

Four generators $\mathbf{S}^0$, $\mathbf{S}^1$, $\mathbf{S}^2$, $\mathbf{S}^3$ respect commutation relations

$$\mathbf{S}^\alpha \mathbf{S}^\beta - \mathbf{S}^\beta \mathbf{S}^\alpha = \mathrm{i} \cdot \left(\mathbf{S}^0 \mathbf{S}^\gamma + \mathbf{S}^\gamma \mathbf{S}^0\right)$$
$$\mathbf{S}^0 \mathbf{S}^\alpha - \mathbf{S}^\alpha \mathbf{S}^0 = \mathrm{i}\, \mathbf{J}_{\beta\gamma} \cdot \left(\mathbf{S}^\beta \mathbf{S}^\gamma + \mathbf{S}^\gamma \mathbf{S}^\beta\right)$$

[E.Sklyanin '82]

where (α, β, γ) – arbitrary cyclic permutation of $(1, 2, 3)$

$\mathbf{J}_{\alpha\beta} = \mathbf{J}_{\alpha\beta}(2\eta, \tau)$ – structure constants

Sklyanin algebra and XYZ spin chain

Realization of the algebra by finite-difference operators

$$\mathbf{S}^{\alpha}(g) = e^{\pi i z^2/\eta} \frac{i^{\delta_{\alpha,2}} \theta_{\alpha+1}(\eta|\tau)}{\theta_1(2z|\tau)} \Big[\theta_{\alpha+1}\left(2z - g + \eta \middle| \tau\right) e^{\eta \partial_z} - \\ - \theta_{\alpha+1}\left(-2z - g + \eta \middle| \tau\right) e^{-\eta \partial_z} \Big] e^{-\pi i z^2/\eta}$$

[E.Sklyanin '83]

Label g of the representation

$$\textbf{Elliptic Modular Double} = \underset{2\eta,\,\tau}{\overset{\mathbf{S}^0, \mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3}{\text{Sklyanin algebra}}} \otimes \underset{\tau,\,2\eta}{\overset{\widetilde{\mathbf{S}}^0, \widetilde{\mathbf{S}}^1, \widetilde{\mathbf{S}}^2, \widetilde{\mathbf{S}}^3}{\text{Sklyanin algebra}}}$$

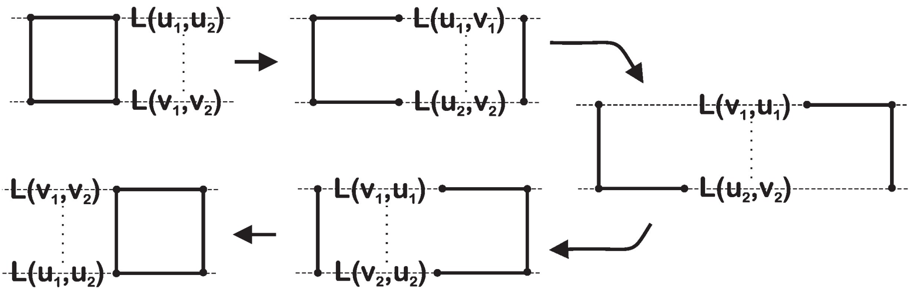
[L.Faddeev '99; V.Spiridonov '09]

Factorization of the master R-operator

Lax operator $\mathrm{L}(u|g): \ \mathbb{C}^2 \otimes V_g \ \rightarrow \ \mathbb{C}^2 \otimes V_g$

$$u_1 \leftrightarrows u_2 \quad \Leftrightarrow \quad g \leftrightarrows -g$$

$$\mathbb{R}_{12}(u-v|g_1,g_2)\,\mathrm{L}_1(u|g_1)\,\mathrm{L}_2(v|g_2) \;=\; \mathrm{L}_2(v|g_2)\,\mathrm{L}_1(u|g_1)\,\mathbb{R}_{12}(u-v|g_1,g_2)$$



The constituents of the master R-operator form the operator representation of S_4

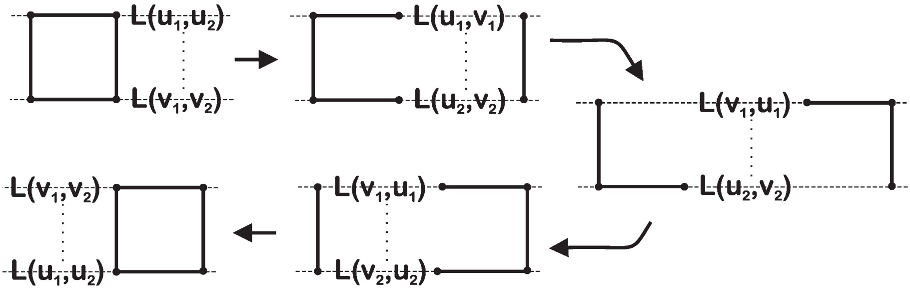
[Derkachov, Karakhanyan, Kirschner '07]

Factorization of the master R-operator

Lax operator $\mathrm{L}(u|g): \ \mathbb{C}^2 \otimes V_g \ \rightarrow \ \mathbb{C}^2 \otimes V_g$

$$u_1 \leftrightarrows u_2 \quad \Leftrightarrow \quad g \leftrightarrows -g$$

$$\mathbb{R}_{12}(u_1, u_2|v_1, v_2)\,\mathrm{L}_1(u_1, u_2)\,\mathrm{L}_2(v_1, v_2) \ = \ \mathrm{L}_2(v_1, v_2)\,\mathrm{L}_1(u_1, u_2)\,\mathbb{R}_{12}(u_1, u_2|v_1, v_2)$$



The constituents of the master R-operator form the operator representation of S_4

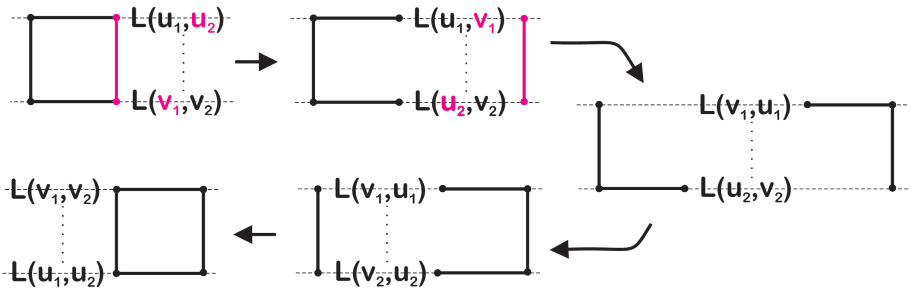
[Derkachov, Karakhanyan, Kirschner '07]

Factorization of the master R-operator

Lax operator $\mathrm{L}(u|g): \ \mathbb{C}^2 \otimes V_g \ \rightarrow \ \mathbb{C}^2 \otimes V_g$

$$u_1 \leftrightarrows u_2 \quad \Leftrightarrow \quad g \leftrightarrows -g$$

$$\mathbb{R}_{12}(u_1, u_2|v_1, v_2)\,\mathrm{L}_1(u_1, u_2)\,\mathrm{L}_2(v_1, v_2) \;=\; \mathrm{L}_2(v_1, v_2)\,\mathrm{L}_1(u_1, u_2)\,\mathbb{R}_{12}(u_1, u_2|v_1, v_2)$$



The constituents of the master R-operator form the operator representation of S_4

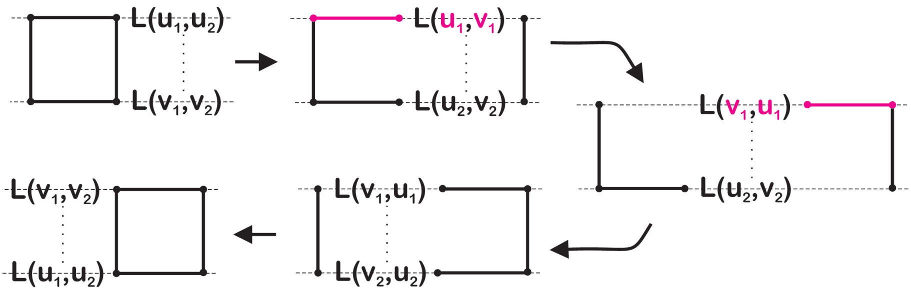
[Derkachov, Karakhanyan, Kirschner '07]

Factorization of the master R-operator

Lax operator $\mathrm{L}(u|g):\ \mathbb{C}^2\otimes V_g \ \to\ \mathbb{C}^2\otimes V_g$

$$u_1 \leftrightarrows u_2 \quad \Leftrightarrow \quad g \leftrightarrows -g$$

$$\mathbb{R}_{12}(u_1,u_2|v_1,v_2)\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,v_2) \;=\; \mathrm{L}_2(v_1,v_2)\,\mathrm{L}_1(u_1,u_2)\,\mathbb{R}_{12}(u_1,u_2|v_1,v_2)$$



The constituents of the master R-operator form the operator representation of S_4

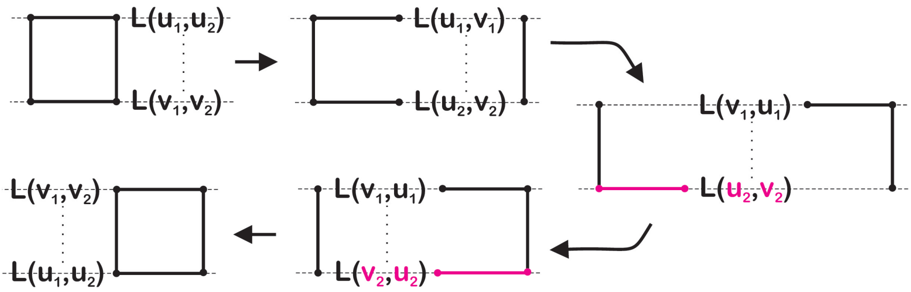
[Derkachov, Karakhanyan, Kirschner '07]

Factorization of the master R-operator

Lax operator $\mathrm{L}(u|g): \mathbb{C}^2 \otimes V_g \rightarrow \mathbb{C}^2 \otimes V_g$

$$u_1 \leftrightarrows u_2 \quad \Leftrightarrow \quad g \leftrightarrows -g$$

$$\mathbb{R}_{12}(u_1, u_2|v_1, v_2)\,\mathrm{L}_1(u_1, u_2)\,\mathrm{L}_2(v_1, v_2) \;=\; \mathrm{L}_2(v_1, v_2)\,\mathrm{L}_1(u_1, u_2)\,\mathbb{R}_{12}(u_1, u_2|v_1, v_2)$$



The constituents of the master R-operator form the operator representation of S_4

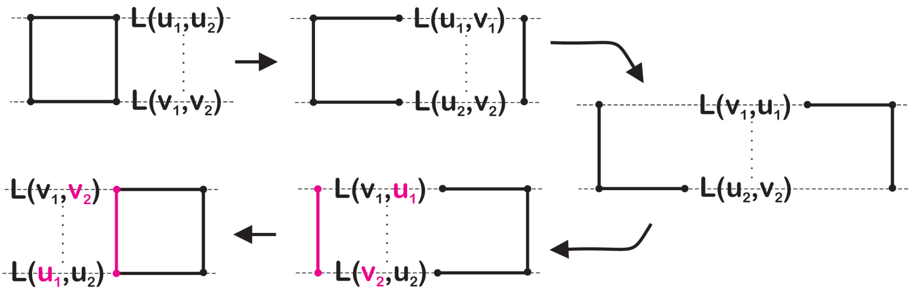
[Derkachov, Karakhanyan, Kirschner '07]

Factorization of the master R-operator

Lax operator $\mathrm{L}(u|g) : \mathbb{C}^2 \otimes V_g \to \mathbb{C}^2 \otimes V_g$

$$u_1 \leftrightarrows u_2 \quad \Leftrightarrow \quad g \leftrightarrows -g$$

$$\mathbb{R}_{12}(u_1, u_2|v_1, v_2)\,\mathrm{L}_1(u_1, u_2)\,\mathrm{L}_2(v_1, v_2) = \mathrm{L}_2(v_1, v_2)\,\mathrm{L}_1(u_1, u_2)\,\mathbb{R}_{12}(u_1, u_2|v_1, v_2)$$



The constituents of the master R-operator form the operator representation of S_4

[Derkachov, Karakhanyan, Kirschner '07]

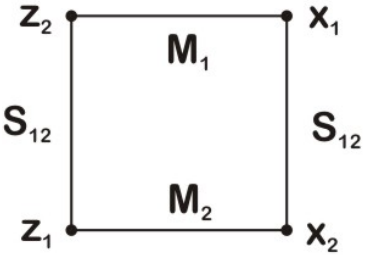
Factorization of the master R-operator

$$\mathbb{R}_{12}(u|g_1,g_2) = \mathrm{P}_{12}\,\mathrm{S}_{12}(u+\tfrac{g_1+g_2}{2})\,\mathrm{M}_2(u+\tfrac{g_2-g_1}{2})\,\mathrm{M}_1(u+\tfrac{g_1-g_2}{2})\,\mathrm{S}_{12}(u-\tfrac{g_1+g_2}{2})$$

Additive notations $\Gamma(e^{2\pi \mathrm{i}\, z}) \to \Gamma(z)$

- Permutation P_{12}
- Multiplication by

  $\mathrm{S}_{12}(u) \equiv \Gamma(\pm z_1 \pm z_2 + u + \eta + \frac{\tau}{2})$
- Elliptic Fourier transform



$$[\,\mathrm{M}(g)\,\Phi\,](z) \equiv \kappa \int_0^1 \frac{\Gamma(\pm z \pm x - g)}{\Gamma(-2g)\Gamma(\pm 2x)}\,\Phi(x)\,dx$$

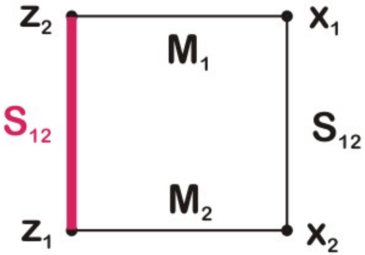
Equivalent representations with labels g and $-g$

$$\mathrm{M}(g)\,\mathbf{S}^{\alpha}(g) = \mathbf{S}^{\alpha}(-g)\,\mathrm{M}(g) \quad , \quad \mathrm{M}(g)\,\tilde{\mathbf{S}}^{\alpha}(g) = \tilde{\mathbf{S}}^{\alpha}(-g)\,\mathrm{M}(g)$$

Factorization of the master R-operator

$$\mathbb{R}_{12}(u|g_1,g_2) = \mathrm{P}_{12}\, \mathbf{S}_{12}(\mathbf{u}+\tfrac{\mathbf{g_1}+\mathbf{g_2}}{2})\, \mathrm{M}_2(u+\tfrac{g_2-g_1}{2})\, \mathrm{M}_1(u+\tfrac{g_1-g_2}{2})\, \mathrm{S}_{12}(u-\tfrac{g_1+g_2}{2})$$

Additive notations $\Gamma(e^{2\pi \mathrm{i}\, z}) \to \Gamma(z)$



- Permutation P_{12}
- Multiplication by

  $\mathrm{S}_{12}(u) \equiv \Gamma(\pm z_1 \pm z_2 + u + \eta + \frac{\tau}{2})$
- Elliptic Fourier transform

$$[\,\mathrm{M}(g)\,\Phi\,](z) \equiv \kappa \int_0^1 \frac{\Gamma(\pm z \pm x - g)}{\Gamma(-2g)\Gamma(\pm 2x)}\Phi(x)\,dx$$

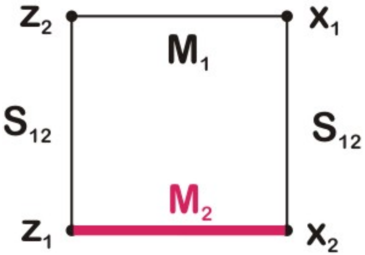
Equivalent representations with labels g and $-g$

$$\mathrm{M}(g)\,\mathbf{S}^{\alpha}(g) = \mathbf{S}^{\alpha}(-g)\,\mathrm{M}(g) \quad , \quad \mathrm{M}(g)\,\tilde{\mathbf{S}}^{\alpha}(g) = \tilde{\mathbf{S}}^{\alpha}(-g)\,\mathrm{M}(g)$$

Factorization of the master R-operator

$$\mathbb{R}_{12}(u|g_1,g_2) = \mathrm{P}_{12}\, \mathrm{S}_{12}(u+\tfrac{g_1+g_2}{2})\, \mathbf{M}_2(\mathbf{u}+\tfrac{\mathbf{g}_2-\mathbf{g}_1}{\mathbf{2}})\, \mathrm{M}_1(u+\tfrac{g_1-g_2}{2})\, \mathrm{S}_{12}(u-\tfrac{g_1+g_2}{2})$$

Additive notations $\Gamma(e^{2\pi \mathrm{i}\, z}) \to \Gamma(z)$



- Permutation P_{12}
- Multiplication by

  $\mathrm{S}_{12}(u) \equiv \Gamma(\pm z_1 \pm z_2 + u + \eta + \frac{\tau}{2})$
- Elliptic Fourier transform

$$[\,\mathrm{M}(g)\,\Phi\,](z) \equiv \kappa \int_0^1 \frac{\Gamma(\pm z \pm x - g)}{\Gamma(-2g)\Gamma(\pm 2x)}\Phi(x)\,dx$$

Equivalent representations with labels g and $-g$

$$\mathrm{M}(g)\,\mathbf{S}^\alpha(g) = \mathbf{S}^\alpha(-g)\,\mathrm{M}(g) \quad , \quad \mathrm{M}(g)\,\tilde{\mathbf{S}}^\alpha(g) = \tilde{\mathbf{S}}^\alpha(-g)\,\mathrm{M}(g)$$

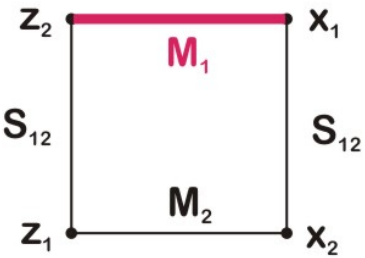
Factorization of the master R-operator

$$\mathbb{R}_{12}(u|g_1,g_2) = \mathrm{P}_{12}\,\mathrm{S}_{12}(u+\tfrac{g_1+g_2}{2})\,\mathrm{M}_2(u+\tfrac{g_2-g_1}{2})\,\mathbf{M}_1(\mathbf{u}+\tfrac{\mathbf{g}_1-\mathbf{g}_2}{2})\,\mathrm{S}_{12}(u-\tfrac{g_1+g_2}{2})$$

Additive notations $\Gamma(e^{2\pi \mathrm{i}\, z}) \to \Gamma(z)$

- Permutation P_{12}
- Multiplication by

  $\mathrm{S}_{12}(u) \equiv \Gamma(\pm z_1 \pm z_2 + u + \eta + \frac{\tau}{2})$
- Elliptic Fourier transform



$$[\,\mathrm{M}(g)\,\Phi\,](z) \equiv \kappa \int_0^1 \frac{\Gamma(\pm z \pm x - g)}{\Gamma(-2g)\Gamma(\pm 2x)}\,\Phi(x)\,dx$$

Equivalent representations with labels g and $-g$

$$\mathrm{M}(g)\,\mathbf{S}^{\alpha}(g) = \mathbf{S}^{\alpha}(-g)\,\mathrm{M}(g) \quad , \quad \mathrm{M}(g)\,\tilde{\mathbf{S}}^{\alpha}(g) = \tilde{\mathbf{S}}^{\alpha}(-g)\,\mathrm{M}(g)$$

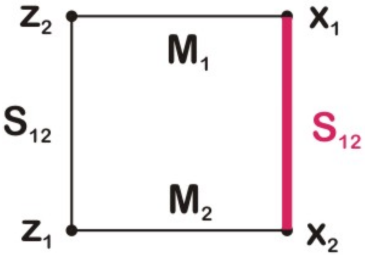
Factorization of the master R-operator

$$\mathbb{R}_{12}(u|g_1,g_2) = \mathrm{P}_{12}\,\mathrm{S}_{12}(u+\tfrac{g_1+g_2}{2})\,\mathrm{M}_2(u+\tfrac{g_2-g_1}{2})\,\mathrm{M}_1(u+\tfrac{g_1-g_2}{2})\,\mathbf{S}_{12}(\mathbf{u}-\tfrac{\mathbf{g}_1+\mathbf{g}_2}{2})$$

Additive notations $\Gamma(e^{2\pi \mathrm{i}\, z}) \to \Gamma(z)$

- Permutation P_{12}
- Multiplication by

  $\mathrm{S}_{12}(u) \equiv \Gamma(\pm z_1 \pm z_2 + u + \eta + \frac{\tau}{2})$
- Elliptic Fourier transform



$$[\,\mathrm{M}(g)\,\Phi\,](z) \equiv \kappa \int_0^1 \frac{\Gamma(\pm z \pm x - g)}{\Gamma(-2g)\Gamma(\pm 2x)}\,\Phi(x)\,dx$$

Equivalent representations with labels g and $-g$

$$\mathrm{M}(g)\,\mathbf{S}^{\alpha}(g) = \mathbf{S}^{\alpha}(-g)\,\mathrm{M}(g) \quad , \quad \mathrm{M}(g)\,\tilde{\mathbf{S}}^{\alpha}(g) = \tilde{\mathbf{S}}^{\alpha}(-g)\,\mathrm{M}(g)$$

The intertwining operator for Sklyanin algebra

Discrete lattice

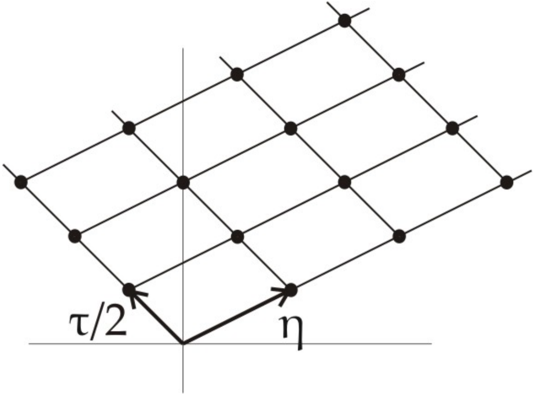
of g-spin

$$\boxed{g = n\,\eta + m\,\frac{\tau}{2}}$$

$n,\ m \in \mathbb{Z}_{\geq 0}$

Initial condition:

$\mathrm{M}(0) = \mathbb{1}$



Contiguous relations & **Factorization** of the intertwining operator

$$\mathrm{A}(g)\,\mathrm{M}(g) = \mathrm{M}(g+\eta)\,\theta_3\left(z|\tfrac{\tau}{2}\right) \qquad \mathrm{B}(g)\,\mathrm{M}(g) = \mathrm{M}\left(g+\tfrac{\tau}{2}\right)\,\theta_3\left(z|\eta\right)$$

The intertwining operator for Sklyanin algebra

Discrete lattice

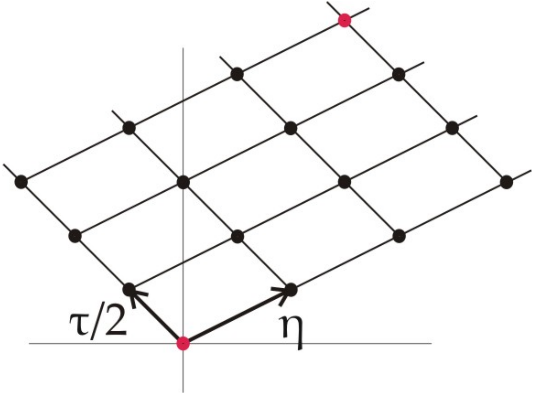
of g-spin

$$\boxed{g = n\,\eta + m\,\frac{\tau}{2}}$$

$n,\ m \in \mathbb{Z}_{\geq 0}$

Initial condition:

$\mathrm{M}(0) = \mathbb{1}$



Contiguous relations & Factorization of the intertwining operator

$$\mathrm{A}(g)\,\mathrm{M}(g) = \mathrm{M}(g+\eta)\,\theta_3\left(z|\tfrac{\tau}{2}\right) \qquad \mathrm{B}(g)\,\mathrm{M}(g) = \mathrm{M}\left(g+\tfrac{\tau}{2}\right)\,\theta_3\left(z|\eta\right)$$

The intertwining operator for Sklyanin algebra

Discrete lattice

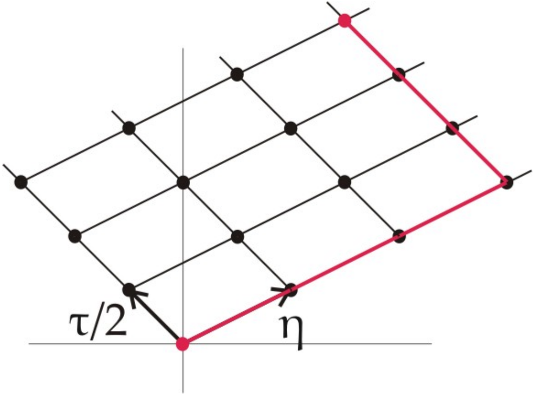
of g-spin

$$\boxed{g = n\,\eta + m\,\frac{\tau}{2}}$$

$n,\ m \in \mathbb{Z}_{\geq 0}$

Initial condition:

$\mathrm{M}(0) = \mathbb{1}$



Contiguous relations & **Factorization** of the intertwining operator

$$\mathrm{A}(g)\,\mathrm{M}(g) = \mathrm{M}(g+\eta)\,\theta_3\left(z|\tfrac{\tau}{2}\right) \qquad \mathrm{B}(g)\,\mathrm{M}(g) = \mathrm{M}\left(g+\tfrac{\tau}{2}\right)\,\theta_3\left(z|\eta\right)$$

The intertwining operator for Sklyanin algebra

Discrete lattice

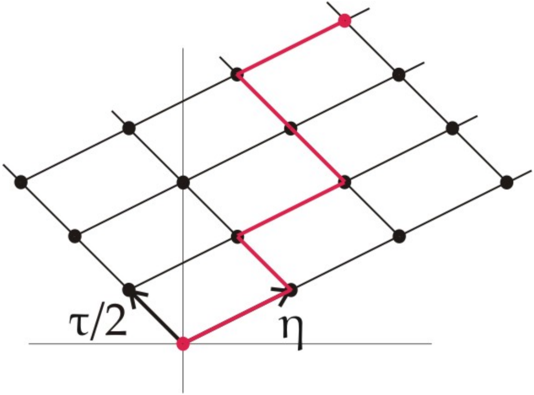
of g-spin

$$\boxed{g = n\,\eta + m\,\frac{\tau}{2}}$$

$n,\ m \in \mathbb{Z}_{\geq 0}$

Initial condition:

$\mathrm{M}(0) = \mathbb{1}$



Contiguous relations & Factorization of the intertwining operator

$$\mathrm{A}(g)\,\mathrm{M}(g) = \mathrm{M}(g+\eta)\,\theta_3\left(z|\tfrac{\tau}{2}\right) \qquad \mathrm{B}(g)\,\mathrm{M}(g) = \mathrm{M}\left(g+\tfrac{\tau}{2}\right)\,\theta_3\left(z|\eta\right)$$

The intertwining operator for Sklyanin algebra

Discrete lattice

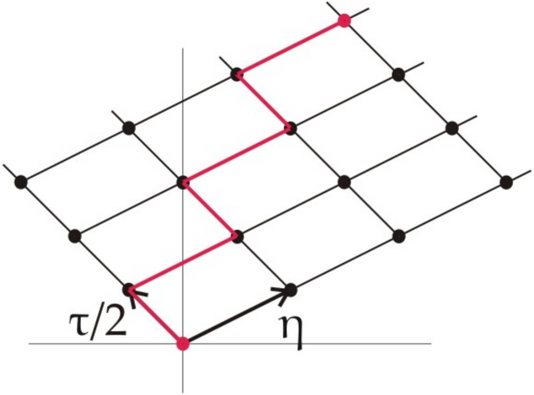
of g-spin

$$\boxed{g = n\,\eta + m\,\frac{\tau}{2}}$$

$n,\ m \in \mathbb{Z}_{\geq 0}$

Initial condition:

$\mathrm{M}(0) = \mathbb{1}$



Contiguous relations & Factorization of the intertwining operator

$$\mathrm{A}(g)\,\mathrm{M}(g) = \mathrm{M}(g+\eta)\,\theta_3\left(z|\tfrac{\tau}{2}\right) \qquad \mathrm{B}(g)\,\mathrm{M}(g) = \mathrm{M}\left(g+\tfrac{\tau}{2}\right)\,\theta_3\left(z|\eta\right)$$

The intertwining operator for Sklyanin algebra

Discrete lattice

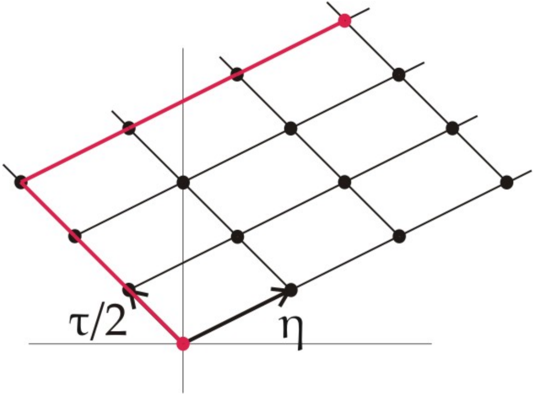
of g-spin

$$\boxed{g = n\,\eta + m\,\frac{\tau}{2}}$$

$n,\ m \in \mathbb{Z}_{\geq 0}$

Initial condition:

$\mathrm{M}(0) = \mathbb{1}$



Contiguous relations & Factorization of the intertwining operator

$$\mathrm{A}(g)\,\mathrm{M}(g) = \mathrm{M}(g+\eta)\,\theta_3\left(z|\tfrac{\tau}{2}\right) \qquad \mathrm{B}(g)\,\mathrm{M}(g) = \mathrm{M}\left(g+\tfrac{\tau}{2}\right)\,\theta_3\left(z|\eta\right)$$

The intertwining operator and finite-dimensional representations

Zero modes of $\mathrm{M}(g_{n,m})$ at $\boxed{g_{n,m} = (n+1)\,\eta + (m+1)\,\frac{\tau}{2}}$ $n,\ m \in \mathbb{Z}_{\geq 0}$

Due to intertwining relations

$$\mathrm{M}(g)\,\mathbf{S}^{\alpha}(g) = \mathbf{S}^{\alpha}(-g)\,\mathrm{M}(g)$$

$$\mathrm{M}_1(g_1)\;\mathbb{R}_{12}(\,u\,|\,g_1\,,\,g_2\,) = \mathbb{R}_{12}(\,u\,|\,-g_1\,,\,g_2\,)\;\mathrm{M}_1(g_1)$$

the space $\mathrm{Ker}\,\mathrm{M}(g_{n,m}) \bigcap \mathrm{Im}\,\mathrm{M}(-g_{n,m})$ is **invariant**

Generating function of the finite-dimensional representation,

$$\mathbf{\Gamma}(\pm\mathbf{z} \pm \mathbf{x} + \mathbf{g_{n,m}}) = \sum \varphi_{j,l}^{(n,m)}(z)\,\psi_{n-j,m-l}^{(n,m)}(x) \;\sim\; \Theta_{2n}^{+}(\tau) \otimes \Theta_{2m}^{+}(2\eta)$$

↰ auxiliary parameter

produces a pair of natural bases, $\dim = (n+1)(m+1)$,

$$\varphi_{j,l}^{(n,m)}(z) = \left[\theta_3\left(z|\tfrac{\tau}{2}\right)\right]^j \left[\theta_4\left(z|\tfrac{\tau}{2}\right)\right]^{n-j} [\,\theta_3(z|\eta)\,]^l\,[\,\theta_4(z|\eta)\,]^{m-l}$$

$$\psi_{j,l}^{(n,m)}(z) = \mathrm{Sym}\prod_{r=0}^{n-1} \theta_{3,4}\left(z+(n-1-2r)\eta\;\Big|\;\tfrac{\tau}{2}\right)\,\mathrm{Sym}\prod_{s=0}^{m-1} \theta_{3,4}\left(z+(m-1-2s)\tfrac{\tau}{2}\;\Big|\eta\right)$$

$j=0,1,2,\ldots,n\;,\quad l=0,1,2,\ldots,m$

Reduction of the general R-operator

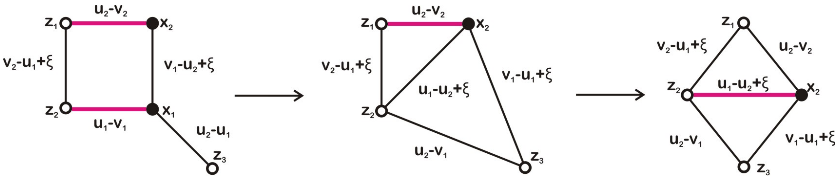
Reduce the master R-operator to finite-dimensional rep. in the first space

$$\mathbb{R}_{12}(u|g_{n,m}\,,\,g)\;:\;\mathbb{C}^{(n+1)(m+1)}\otimes V_g\to\mathbb{C}^{(n+1)(m+1)}\otimes V_g$$

Matrix with operator entries which are finite-difference operators of a high order

auxiliary parameter

$$\mathbb{R}_{12}(u|g_{n,m}\,,\,g)\,\Gamma(\mp z_1\mp z_3+g_{n,m})\,\Phi(z_2)$$
$$=\frac{\Gamma(\mp z_2\mp z_3+\frac{g_{n,m}+g-u}{2})}{\Gamma(\mp z_1\mp z_2-\frac{g_{n,m}+g+u}{2}+\eta+\frac{\tau}{2})}\,\mathrm{M}_2(n\,\eta+m\,\tfrac{\tau}{2})\,\frac{\Gamma(\mp z_1\mp z_2+\frac{g_{n,m}-g-u}{2})\,\Phi(z_2)}{\Gamma(\mp z_2\mp z_3+\frac{g-g_{n,m}-u}{2}+\eta+\frac{\tau}{2})}$$



Reduced R-operator is written in the pair of bases

$$\mathbb{R}_{12}(u|g_{n,m}\,,\,g)\,\psi^{(n,m)}_{j,l}(z_1) = \varphi^{(n,m)}_{r,s}(z_1)\left[\,\mathbb{R}\,\right]^{r,s}_{j,l}$$

All fin.dim. $\otimes$ inf.dim. solutions **factorize**

$$\mathbb{R}(u|g_{n,0}\,,\,g) = V(\tfrac{u+g}{2}, z)\, D(z,\partial)\, \mathbf{C}\, V^T(\tfrac{u-g}{2}, z)\, \mathbf{C}$$

$D(z,\partial)$ – diagonal matrix, $\mathbf{C}$ – constant numerical matrix

Example: the Lax operator, we recover the Sklyanin algebra

$$\mathbb{R}(u|g_{1,0}\,,\,g) = e^{\pi i z^2/\eta}\,\frac{1}{\theta_1(2z|\tau)} \begin{pmatrix} \theta_3\left(z-\frac{u+g}{2}|\frac{\tau}{2}\right) & -\theta_3\left(z+\frac{u+g}{2}|\frac{\tau}{2}\right) \\ -\theta_4\left(z-\frac{u+g}{2}|\frac{\tau}{2}\right) & \theta_4\left(z+\frac{u+g}{2}|\frac{\tau}{2}\right) \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} e^{\eta\partial} & 0 \\ 0 & e^{-\eta\partial} \end{pmatrix} \begin{pmatrix} \theta_4\left(z+\frac{u-g}{2}|\frac{\tau}{2}\right) & \theta_3\left(z+\frac{u-g}{2}|\frac{\tau}{2}\right) \\ \theta_4\left(z-\frac{u-g}{2}|\frac{\tau}{2}\right) & \theta_3\left(z-\frac{u-g}{2}|\frac{\tau}{2}\right) \end{pmatrix} e^{-\pi i z^2/\eta}$$

Summary and Conclusions

- The master solution (principal series rep.) of the YBE
- Continuous spin lattice models and the star-triangle
- Finite-dimensional reductions of the master solution
- Intertwining operators
- The factorized form of the integral R-operator and its finite-dimensional reductions (Explicit concise formulae)

Rational $\leftarrow$ Trigonometric $\leftarrow$ Elliptic

$\mathrm{SL}(2,\mathbb{C})$	$\leftarrow$	Modular double of $U_q(s\ell_2)$	$\leftarrow$	Hyperbolic modular double	$\leftarrow$	Elliptic modular double (Sklyanin algebra)