String Theory and Integrable Lattice Models

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Quantum Field Theories
Integrable lattice models can be embedded in String Theory.

What does this buy us?

A lot!

Today:

- Construct an integrable lattice model that unifies
  - Belavin model
  - Jimbo–Miwa–Okado model group
  - Bazhanov–Sergeev model
- Relate it to 4d SUSY QFTs.
Fix

- $N \in \mathbb{N}, \ N \geq 2$
- $\tau, \gamma \in \mathbb{C}, \ \text{Im} \tau, \text{Im} \gamma > 0$

Let

- $V = \mathbb{C}^N$
- $\mathbb{V} = \{\text{meromorphic symmetric functions of } (z_1, \ldots, z_N)\}$
**Belavin model** [Belavin '81]

Belavin model is a lattice model in Statistical Mechanics:

It's a **vertex model**: spins live on edges, interact at vertices.

Spin variables $i, j, k, l, \ldots \in \{1, \ldots, N\}$.

Each line carries a **spectral parameter** $u \in \mathbb{C}$.

Lattice can be drawn on any surface $\Sigma$; in this talk $\Sigma = T^2$. 
Interaction is governed by the **R-matrix** $R^B(u) \in \text{End}(V \otimes V)$.

**Local Boltzmann weight:**

\[ u_{12} = u_1 - u_2. \]

**Partition function**

\[ Z = \sum_{\text{spin configs}} \prod_{\text{vertices}} \text{local Boltzmann weight}. \]
Explicitly,

\[ R^B(u)_{ij}^{kl} = \delta_{i+j,k+l} \frac{\theta_1(\gamma)}{\theta_1(u + \gamma)} \frac{\theta^{(k-l)}(u + \gamma)}{\theta^{(k-i)}(\gamma) \theta^{(i-l)}(u)} \frac{\prod_{m=0}^{N-1} \theta^{(m)}(u)}{\prod_{n=1}^{N-1} \theta^{(n)}(0)} , \]

where \( i, j, k, l \) are treated mod \( N \) and

\[ \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (u | \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+a)^2 \tau + 2\pi i (n+a)(u+b)} , \]

\[ \theta^{(j)}(u) = \theta \left[ \begin{array}{c} 1/2 - j/N \\ 1/2 \end{array} \right] (u | N\tau) , \]

\[ \theta_1(u) = -\theta \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (u | \tau) . \]

When \( N = 2 \), we get the 8-vertex model.
$R^B$ satisfies the Yang–Baxter equation

$$R^B_{12}(u_{12})R^B_{13}(u_{13})R^B_{23}(u_{23}) = R^B_{23}(u_{23})R^B_{13}(u_{13})R^B_{12}(u_{12}).$$

Graphically,

It follows the model is integrable.

This is quantum integrability:

2d lattice model $\leftrightarrow$ 1d quantum spin chain.

Belavin model $\leftrightarrow \mathfrak{sl}_N$ generalization of XYZ spin chain.
Jimbo–Miwa–Okado model [JMO ’87]

JMO model is an IRF model, i.e. spins live on faces:

Spin variables $\lambda \in \mathfrak{h} \subset \mathfrak{sl}_N$: $\lambda = (\lambda_1, \ldots, \lambda_N)$, $\sum_{i=1}^{N} \lambda_i = 0$.

Allowed configs: to each edge we can assign $i \in \{1, \ldots, N\}$ s.t.

$\lambda_i \rightarrow$, $\mu = \lambda - \gamma w_i$.

$w_i = e_i - \frac{1}{N} \sum_{j=1}^{N} e_j$ is the weight of $e_i \in V = \mathbb{C}^N$, $(e_i)_j = \delta_{ij}$.
Alternatively, think of $i, j, \ldots$ as spin variables living on edges.

We get a vertex model with a **dynamical variable** $\lambda$.

Specify $\lambda$ on one face. Spin config determines it on the rest:

$$
\begin{align*}
\lambda & \quad \overset{\lambda}{\longrightarrow} \quad \lambda - \gamma \omega_i \\
\lambda - \gamma \omega_i & \quad \overset{\lambda}{\longrightarrow} \quad \lambda - \gamma (\omega_i + \omega_j) \\
& = \lambda - \gamma (\omega_k + \omega_l)
\end{align*}
$$

This vertex model is described by **Felder’s R-matrix** $R^F(u, \lambda)$:

$$
(R^F)_{ii}^{ii} = 1, \quad (R^F)_{ij}^{ij} = \frac{\theta_1(u)\theta_1(\lambda_{ij} + \gamma)}{\theta_1(u + \gamma)\theta_1(\lambda_{ij})}, \quad (R^F)_{ji}^{ij} = \frac{\theta_1(\gamma)\theta_1(u + \lambda_{ij})}{\theta_1(u + \gamma)\theta_1(\lambda_{ij})}
$$

[Felder ‘94, Felder–Varchenko ‘97].

For $N = 2$, we get the 8VSOS model [Baxter ‘73].
Graphically represent $R^F$ as

$$R^F(u_{12}, \lambda) = u_1 \rightarrow u_2 \lambda.$$

This time, we have the dynamical YBE:

or

$$R^F_{12}(u_{12}, \lambda - \gamma h_3)R^F_{13}(u_{13}, \lambda)R^F_{23}(u_{23}, \lambda - \gamma h_1)$$

$$= R^F_{23}(u_{23}, \lambda)R^F_{13}(u_{13}, \lambda - \gamma h_2)R^F_{12}(u_{12}, \lambda).$$

$h_a$: the weight of the state on the $a$th line.
**Bazhanov–Sergeev model** [BS '10, '11]

**BS model** is defined on a tricolor checkerboard lattice:

Lines carry multiplicative spectral parameters $a, b, \ldots$.

Spin variables: multiplicative dynamical variables $z, w, \ldots$ on uncolored faces. They are all independent.

Boltzmann weights involve the elliptic gamma function

$$\Gamma(z) = \prod_{m,n=0}^{\infty} \frac{1 - p^{m+1}q^{n+1}/z}{1 - p^m q^n z}; \quad p = e^{2\pi i \tau}, \; q = e^{2\pi i \gamma}. $$
We assign Boltzmann weights

\[
\begin{align*}
M \left( \frac{a_2}{a_1}; z, w \right) &= M \left( \frac{b_2}{b_1}; w, z \right), \\
D \left( \frac{b}{a}; z, w \right) &= D \left( \frac{a}{b}; w, z \right),
\end{align*}
\]

where

\[
M(a; z, w) = \prod_{i,j} \frac{\Gamma \left( a \frac{w_i}{z_j} \right)}{\Gamma (a^N)}, \quad D(a; z, w) = \prod_{i,j} \frac{1}{\Gamma \left( \sqrt{pq} \frac{1}{a} \frac{w_i}{z_j} \right)}.
\]

Each spin \( z \) is integrated over \( \mathbb{T}^{N-1} \subset \text{SU}(N) \) with measure

\[
\frac{(p; p)_\infty^{N-1}(q; q)_\infty^{N-1}}{N!} \frac{\prod_{k=1}^{N-1} \frac{dz_k}{2\pi i z_k}}{\prod_{i \neq j} \frac{1}{\Gamma(z_i/z_j)}}; \quad (p; p)_\infty = \prod_{k=1}^{\infty} (1-p^k).
\]

\( M(a; z, w) \) defines the elliptic Fourier transform on \( f(z_1, \ldots, z_N) \)

[Spiridonov '03, Spiridonov–Warnaar '05].
We can reformulate the BS model as a vertex model.

Introduce double line notation \((a, b) \xrightarrow{\quad} = \frac{b}{a} \xrightarrow{\quad}\).

BS R-operator \(R^{BS}((a_1, b_1), (a_2, b_2)) \in \text{End}(\mathbb{V} \otimes \mathbb{V})\) is given by

\[
R^{BS}((a_1, b_1), (a_2, b_2)) = (a_1, b_1) \xrightarrow{\quad} = \frac{b_1}{a_1} \xrightarrow{\quad} = b_2a_2.
\]

Lattice made from \(R^{BS}\) gives a tricolor checkerboard pattern.

\(R^{BS}\) is an \(\infty\)-dim R-matrix; it’s an integral operator [Derkachov–Spiridonov ’12, Maruyoshi–Y ’16].

YBE follows from an integral identity for \(\Gamma\) [Spiridonov ’03, Rains ’10].
Summary of the three models

Belavin:

\[ R^B(u_{12}) = u_1 \begin{array}{c|c}
\uparrow & \downarrow \\
\hline 
\end{array} \begin{array}{c|c}
\downarrow & \uparrow \\
\hline 
\end{array} \in \text{End}(V \otimes V), \quad \widetilde{R}^B = u_1 \begin{array}{c|c}
\uparrow & \downarrow \\
\hline 
\end{array} \begin{array}{c|c}
\downarrow & \uparrow \\
\hline 
\end{array} = (R^B)^T \]

Jimbo–Miwa–Okado/Felder:

\[ R^F(u_{12}, \lambda) = u_1 \begin{array}{c|c}
\uparrow & \downarrow \\
\hline 
\end{array} \begin{array}{c|c}
\downarrow & \uparrow \\
\hline 
\end{array} \lambda \in \text{End}(V \otimes V) \]

Bazhanov–Sergeev:

\[ R^{BS}( (a_1, b_1), (a_2, b_2) ) = (a_1, b_1) \begin{array}{c|c}
\uparrow & \downarrow \\
\hline 
\end{array} (a_2, b_2) = \frac{b_1}{a_1} \begin{array}{c|c}
\uparrow & \downarrow \\
\hline 
\end{array} b_2a_2 \in \text{End}(V \otimes V) \]
String Theory tells me the three models can be unified [Y ’17].
If different kinds of lines can coexist, we have more crossings:

\[
\begin{align*}
  c \quad \stackrel{z}{\rightarrow} & \quad = S\left(\frac{c}{a}; z\right) , \\
  c \quad \stackrel{z}{\rightarrow} & \quad = S'\left(\frac{c}{b}; z\right) , \\
  c \quad \stackrel{z}{\rightarrow} & \quad = \tilde{S}\left(\frac{c}{b}; z\right) , \\
  c \quad \stackrel{z}{\rightarrow} & \quad = \tilde{S}'\left(\frac{c}{a}; z\right) .
\end{align*}
\]

These intertwining operators are matrix-valued functions. They must solve many Yang–Baxter equations!
Yang–Baxter equations with one dashed line such as

allow us to determine the intertwining operators:

\[ S(a; z) = a^{-N/2} \Psi(u, \lambda), \quad \tilde{S}'(a; z) = a^{-N/2} Z^{N/2} \Phi(u, \lambda)^T, \]

\[ S'(a; z) = S(\tilde{a}; z)^{-1}, \quad \tilde{S}(a; z) = (-1)^{N-1} \tilde{S}'(\tilde{a}; z)^{-1} Z^N, \]

with \( a = e^{2\pi i u/N}, \ z_j = e^{2\pi i \lambda_j}, \ \tilde{a} = q^{-1/N} \sqrt{pqa}, \) and

\[ Z = \text{diag}(z_1, \ldots, z_N), \]

\[ \Phi(u, \lambda)^j_i = \theta^{(j)} \left( u - N \lambda_i + \frac{N-1}{2} \right), \]

\[ \Psi(u, \lambda)^j_i = \Phi(u, -\lambda)^j_i / \prod_{k(\neq i)} \theta_1(\lambda_{ki}). \]

Similar analysis in [Sergeev '92, Quano–Fujii '93, Derkachov–Spiridonov]
Yang–Baxter equations with two dashed lines such as

\[ \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{image1}} = \text{\includegraphics[width=0.2\textwidth]{image2}} \quad \text{and} \quad \text{\includegraphics[width=0.2\textwidth]{image3}} = \text{\includegraphics[width=0.2\textwidth]{image4}}
\end{array} \]

or

\[ R^B S_1 S_2 = S_2 S_1 R^F \quad \text{and} \quad R^F S'_1 S'_2 = S'_2 S'_1 R^B \]

describe the vertex–face correspondence [Baxter ‘73, JMO]. They relate \( R^B \) and \( R^F \), vertex and IRF models.

Those in different colors,

\[ \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{image5}} = \text{\includegraphics[width=0.2\textwidth]{image6}} \quad \text{and} \quad \text{\includegraphics[width=0.2\textwidth]{image7}} = \text{\includegraphics[width=0.2\textwidth]{image8}}
\end{array} \]

also hold.
We can also construct an L-operator

\[ L^B = \begin{array}{c}
\uparrow \\
\downarrow \\
\uparrow 
\end{array} = \begin{array}{c}
\uparrow \\
\downarrow \\
\uparrow 
\end{array} \in \text{End}(V \otimes V) .
\]

It’s a matrix of difference operators [Hasegawa ’90, Sergeev, Quano–Fujii].

It satisfies two RLL relations, one with \( R^B \) and another with \( R^{BS} \):

\[ L_1^B L_2^B R^{BS} = R^{BS} L_2^B L_1^B \quad \text{and} \quad R^B L_1^B L_2^B = L_2^B L_1^B R^B . \]

This is the elliptic lift of the chiral Potts/six-vertex relation [Bazhanov–Stroganov ‘90, Bazhanov–Kashaev–Mangazeev–Stroganov ’91].
Another L-operator

\[ L^F = \begin{array}{c}
\end{array} \]

satisfies an RLL relation with Felder’s R-matrix:

\[ \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} \]

or

\[ R^F L_1^F L_2^F = L_2^F L_1^F R^F. \]

\( L^F \) defines an \( \infty \)-dimensional representation of Felder’s elliptic quantum group \( E_{\tau, \gamma/2}(\mathfrak{sl}_N) \).

\( L^B \) gives a vertex-type elliptic algebra. Sklyanin algebra for \( N = 2 \).
Our model can be constructed from branes in String Theory:

We have

\[ Z \text{ of the brane system} = Z \text{ of our model}. \]

YBEs become brane movements.

A 2d TQFT with “extra dimensions” underlies the correspondence [Costello ’13, Y ’15, ’16].
Correspondence with 4d SUSY QFTs

The branes allow us to map our model to 4d SUSY QFT.

BS model $\leftrightarrow$ quiver gauge theory [Spiridonov ‘10, Yamazaki ‘13]:

Now $\circ$ is an SU($N$) gauge group, $\rightarrow$ is a matter field.

$Z$ of this theory on $S^3 \times_{p,q} S^1 = Z_{BS}$.

YBE = invariance of $Z$ under Seiberg duality (quiver mutation).
Introduce an operator supported on a $T^2 \subset S^3 \times S^1$. It acts on $Z$ by a difference operator [Gadde–Gukov, Gaiotto–Rastelli–Razamat, Gaiotto–Razamat,…].

In the lattice model, it appears as a dashed lines [Maruyoshi–Y, Y ’17]:

Recalling $L^B = \uparrow \downarrow \uparrow \downarrow$ and $L^F = \uparrow \downarrow \uparrow \downarrow$, we can write it as

$$\text{Tr}(L^B \cdots L^B) = \text{Tr}(L^F \cdots L^F).$$

This is a transfer matrix constructed from $L^B$ or $L^F$.

Two choices for $T^2$ compatible with symmetries related by $p \leftrightarrow q$ lead to the elliptic modular double [Faddeev, Spiridonov].
By fusion (OPE of surface ops), we can construct a dashed line associated with any irrep $R$ of $\mathfrak{sl}_N$:

$$R \rightarrow$$

\[\text{Tr}(L^F_R \cdots L^F_R)\] is a surface op for class-$S_k$ theories [Gaiotto–Razamat].

For $R = \bigwedge^n V$ and $k = 1$, we get Ruijsenaars’ ops [Hasegawa ’95]. The proof uses the theta function identity

$$\sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{r=1}^n \left[ \Phi \left( v+(r-1)\gamma, \lambda-\gamma \sum_{s=1}^{r-1} \omega_{i_s} \right) \right]^{-1} \Phi \left( u+(r-1)\gamma, \lambda-\gamma \sum_{s=1}^{r-1} \omega_{i_{\sigma(s)}} \right) \right]^{i_r}$$

$$= \frac{\theta_1(v+(u-v)n/N)}{\theta_1(v)} \prod_{i \in I} \frac{\theta_1(\lambda_{ki}+(u-v)/N)}{\theta_1(\lambda_{ki})} \prod_{k \notin I} \frac{\theta_1(\lambda_{ki}+u-v)}{\theta_1(\lambda_{ki})}; \quad I = \{i_1, \ldots, i_n\} \subset \{1, \ldots, N\}.$$ 

Reproduces the QFT result [Bullimore–Fluder–Hollands–Richmond ’14].
The transfer matrices match QFT results [Maruyoshi–Y, Y ’17] for

- \( R = V, k = 1 \) [Gaiotto–Rastelli–Razamat, Gadde–Gukov]
- \( R = V, k > 1 \) [Gaiotto–Razamat, Maruyoshi–Y, Ito–Yoshida]
- \( R = \bigwedge^n V, n > 1, k = 1 \) [Bullimore et al.]

Comparison in progress [Vaško–Y]:

- \( R = S^n V, n > 1, k = 1 \) [Gaiotto–Rastelli–Razamat, Gadde–Gukov]
- \( R = S^n V, n > 1, k > 1 \): partial results [Ito–Yoshida]

Need the symmetric analogue of Hasegawa’s formula.

Other cases: no QFT results yet.
Conclusion

String Theory allows us to construct integrable lattice models and relate them to 4d SUSY QFTs.

Further directions:

- Change $S^3 \times S^1$ to $M_3 \times S^1$ to get new R-matrices: $M_3 = S^3 / \mathbb{Z}_r$ [Yamazaki, Kels], $S^2 \times S^1, \Sigma \times S^1, \ldots$

- Dimensional reduction [Y ’15, Yamazaki–Wen, Gahramanov–Spiridonov, Gahramanov–Rosengren, Gahramanov–Kels]

- Relation to the works of Costello & Nekrasov–Shatashvili [Costello–Y, in progress]

- Zamolodchikov’s tetrahedron equation [Y ’15]

- Chiral Potts model and monopoles [Atiyah ’91]

- Geometric Langlands and AGT correspondences

- Categorification of lattice models

- Little String Theory, AdS/CFT correspondence, …