Spin chains and Gustafson's integrals

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Integrals $\rightleftarrows$ $SL(2,\mathbb{R})$ spin magnet

$$\left(\prod_{n=1}^{N}\int_{-i\infty}^{i\infty}\frac{dz_n}{2\pi i}\right)\frac{\prod_{k=1}^{N+1}\prod_{j=1}^{N}\Gamma(\alpha_k-z_j)\Gamma(\beta_k+z_j)}{\prod_{k<j}\Gamma(z_k-z_j)\Gamma(z_j-z_k)}=\frac{N!\prod_{k,j=1}^{N+1}\Gamma(\alpha_k+\beta_j)}{\Gamma\left(\sum_{k=1}^{N+1}(\alpha_k+\beta_k)\right)}$$

$$\left(\prod_{n=1}^{N}\int_{-i\infty}^{i\infty}\frac{dz_n}{2\pi i}\right)\frac{\prod_{k=1}^{2N+2}\prod_{j=1}^{N}\Gamma(\alpha_k\pm z_j)}{\prod_{k=1}^{N}\Gamma(\pm 2z_k)\prod_{k<j}\Gamma(z_k\pm z_j)\Gamma(-z_k\pm z_j)}=\frac{2^N N!\prod_{k<j}\Gamma(\alpha_k+\alpha_j)}{\Gamma\left(\sum_{k=1}^{2N+2}\alpha_k\right)}$$

$\Gamma(\alpha\pm\beta)\equiv\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)$ and the integration contours separate the series of poles of Γ functions

$$\left(\prod_{n=1}^{N}\int\frac{dz_n}{2\pi i}\right)\frac{\prod_{j=1}^{N}\left(\prod_{k=1}^{N+1}\Gamma(\alpha_k-z_j)\right)\left(\prod_{m=1}^{N}\Gamma(z_j\pm\beta_m)\right)}{\prod_{k<j}\Gamma(z_k\pm z_j)\Gamma(z_j-z_k)}=\frac{N!\prod_{j=1}^{N}\prod_{k=1}^{N+1}\Gamma(\alpha_k\pm\beta_j)}{\prod_{j<k}^{N+1}\Gamma(\alpha_j+\alpha_k)}$$

the series of poles $\{\alpha_k+n_k\}$ and $\{\pm\beta_k-n_k\}$ are separated by the integration contours.

Integrals $\rightleftarrows$ $SL(2,\mathbb{R})$ spin magnet

Robert A. Gustafson, *Some q-beta and Mellin-Barnes integrals on compact Lie groups and Lie algebras,* Trans. AMS Volume 341, Number 1, 1994

Some q-beta and Mellin-Barnes integrals with many parameters associated to the classical groups, SIAM. J. Math. Anal., vol. 23, No. 3, 525, 1992.

- $SL(2,\mathbb{R})$ spin magnet
  - discrete series representations of $\mathrm{SL}(2,\mathbb{R})$
    I. M. Gelfand, M. I. Graev, N. Ya. Vilenkin (1966)
    Generalized functions. Vol. 5: Integral geometry and representation theory
  - monodromy matrices for closed and open spin chains
    E.K. Sklyanin J.Phys. A21 (1988) 2375-289
    Boundary Conditions for Integrable Quantum Systems
- Closed spin chain
  - R-operators
  - Iterative construction of eigenfunctions
  - Feynman diagrams
  - Calculation of the scalar products and matrix elements of the shift operator
- Open spin chain
  based on
  S.E. Derkachov, A.N. Manashov e-Print: arXiv:1611.09593 [math-ph]
  S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, JHEP **0307** (2003) 047 , JHEP **0310** (2003) 053

Integrals $\rightleftarrows$ $SL(2,\mathbb{C})$ spin magnet

$$\left(\prod_{n=1}^{N}\int_{-i\infty}^{i\infty}\frac{dz_n}{2\pi i}\right)\frac{\prod_{k=1}^{N+1}\prod_{j=1}^{N}\Gamma(\alpha_k-z_j)\Gamma(\beta_k+z_j)}{\prod_{k<j}\Gamma(z_k-z_j)\Gamma(z_j-z_k)}=\frac{N!\prod_{k,j=1}^{N+1}\Gamma(\alpha_k+\beta_j)}{\Gamma\left(\sum_{k=1}^{N+1}(\alpha_k+\beta_k)\right)}$$

$$\frac{1}{N!}\left(\prod_{k=1}^{N}\sum_{n_k=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\nu_k}{2\pi}\right)\frac{\prod_{k=1}^{N+1}\prod_{j=1}^{N}a(1+i(x_k-u_j))\,a(1+i(u_j-x'_k))}{\prod_{m<j}a(1+i(u_j-u_m))a(1+i(u_m-u_j))}=$$

$$=\frac{N!\prod_{k,j=1}^{N+1}a(1+i(x_k-x'_j))}{a\left(1+i\sum_{k=1}^{N+1}(x_k-x'_k)\right)}$$

$a(\alpha)=\Gamma(1-\bar{\alpha})/\Gamma(\alpha)$ is a function of two complex variables such that $\alpha-\bar{\alpha}=n$. It is related to the gamma function for the complex field $\mathbb{C}$ defined by Gelfand, Graev and Retakh: $\Gamma(\alpha,\bar{\alpha})=i^{\alpha-\bar{\alpha}}a(1-\bar{\alpha})$.
$u_k=-in_k/2+\nu_k,\quad \bar{u}_k=in_k/2+\nu_k$, where $n_k\in\mathbb{Z}\,,\nu_k\in\mathbb{R}$

$$x_k=-\frac{im_k}{2}+\mu_k,\quad \bar{x}_k=\frac{im_k}{2}+\mu_k,\quad x'_k=-\frac{im'_k}{2}+\mu'_k,\quad \bar{x}_k=\frac{im'_k}{2}+\mu'_k\,,$$

where m_k, m'_k are integers and μ_k and μ'_k are complex numbers such that $\operatorname{Im}\mu_k>0$ and $\operatorname{Im}\mu'_k<0$. By this conditions the ν-poles of the functions $a(1+i(x_k-u_j))$ and $a(1+i(u_j-x'_k))$ are separated by the integration contour.

Integrals $\rightleftarrows$ $SL(2,\mathbb{C})$ spin magnet

- $SL(2,\mathbb{C})$ spin magnet
  - principal series representations of $\mathrm{SL}(2,\mathbb{C})$
    I. M. Gelfand, M. I. Graev, N. Ya. Vilenkin (1966) Generalized functions. Vol. 5: Integral geometry and representation theory
  - monodromy matrix
  - operators $A(u)$ and $B(u)$
- Closed spin chain
  - R-matrices and iterative construction of eigenfunctions
  - Feynman diagrams, scalar product and matrix elements

based on

S.E. Derkachov, A.N. Manashov, P.A. Valinevich e-Print: arXiv:1612.00727 [math-ph]

S. Derkachov, A.Manashov J.Phys. A47 (2014) 305204 Iterative construction of eigenfunctions of the monodromy matrix for SL(2,C) magnet

S. Derkachov, G. Korchemsky, A. Manashov Nucl.Phys. B617 (2001) 375-440 Noncompact Heisenberg spin magnets from high-energy QCD: 1. Baxter Q operator and separation of variables

$SL(2,\mathbb{R})$ spin magnet

The quantum $SL(2,\mathbb{R})$ spin magnet is a straightforward generalization of the standard XXX_s spin chain.

XXX_s spin chain

The Hilbert space of the XXX_s model is given by the tensor product of the $(2s+1)$-dimensional representations of the $SU(2)$ group

$$\mathbb{H}_N = \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \cdots \otimes \mathbb{V}_N, \qquad \mathbb{V}_k = \mathbb{C}^{2s+1}, \qquad k = 1, \ldots, N.$$

To each site k we associate the quantum L-operator acting nontrivially on the $k-$th space in the tensor product

$$\mathrm{L}_k(u) = \begin{pmatrix} u + iS^{(k)} & iS_-^{(k)} \\ iS_+^{(k)} & u - iS^{(k)} \end{pmatrix} \; ; \; [S_+^{(k)}, S_-^{(k)}] = 2S^{(k)}, [S^{(k)}, S_\pm^{(k)}] = \pm S_\pm^{(k)}$$

The monodromy matrices are defined as a products of $L-$operators

closed spin chain

$$\mathrm{T}(u) = \mathrm{L}_1(u)\mathrm{L}_2(u)\cdots\mathrm{L}_N(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix}$$

open spin chain

$$\mathbb{T}(u) = \mathrm{L}_1(u)\mathrm{L}_2(u)\cdots\mathrm{L}_N(u)\,\mathrm{L}_N(u)\mathrm{L}_{N-1}(u)\cdots\mathrm{L}_1(u) = \begin{pmatrix} \mathbb{A}_N(u) & \mathbb{B}_N(u) \\ \mathbb{C}_N(u) & \mathbb{D}_N(u) \end{pmatrix}$$

It is shown in the QISM that the entries of the monodromy matrix $T(u)$ form commuting families

$$[A_N(u), A_N(v)] = [B_N(u), B_N(v)] = 0\,.$$

For the open spin chains this property holds for the off-diagonal elements only

$$[\mathbb{B}_N(u), \mathbb{B}_N(v)] = 0.$$

XXX_s spin chain $\longrightarrow$ $SL(2,\mathbb{R})$ spin magnet

$(2s+1)$-dimensional representations of the $SU(2)$ $\longrightarrow$ irreducible discrete series representation of the $SL(2,R)$

$$\mathbb{V}_k = \mathbb{C}^{2s+1} \to \mathbb{V}_k = D_s^+$$

generalized eigenfunctions

$$A_N(u)\Psi_A^{(N)}(x_1,\ldots,x_N) = (u-x_1)\ldots(u-x_N)\,\Psi_A^{(N)}(x_1,\ldots,x_N)$$

$$B_N(u)\Psi_B^{(N)}(p,x_1,\ldots,x_{N-1}) = p(u-x_1)\ldots(u-x_{N-1})\,\Psi_B^{(N)}(p,x_1,\ldots,x_{N-1})$$

$$S_-\Psi_B^{(N)}(p,x_1,\ldots,x_{N-1}) = -ip\,\Psi_B^{(N)}(p,x_1,\ldots,x_{N-1})$$

$$\mathbb{B}_N(u)\Psi_{\mathbb{B}}^{(N)}(p,x_1,\ldots,x_{N-1}) =$$

$$p(2iu+1)(u^2-x_1^2)\ldots(u^2-x_{N-1}^2)\,\Psi_{\mathbb{B}}^{(N)}(p,x_1,\ldots,x_{N-1})$$

$$S_- = \sum_{k=1}^{N} S_-^{(k)} \quad ; \quad [S_-\,, B_N(u)] = [S_-\,, \mathbb{B}_N(u)] = 0$$

discrete series representation of the $SL(2,R)$

Discrete series representation D_s^+ is realized on the space of functions holomorphic in the upper complex half-plane (spin s being a positive integer)

$$[T(g)\Phi](z) = \frac{1}{(cz+d)^{2s}}\Phi\left(\frac{az+b}{cz+d}\right),$$

where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R)$. Invariant scalar product

$$\langle \Phi \,|\, \Psi \rangle = \int \mathcal{D}z\, \overline{\Phi(z)}\, \Psi(z) \quad , \quad \langle T(g)\Phi \,|\, T(g)\Psi \rangle = \langle \Phi \,|\, \Psi \rangle$$

The integration goes over the upper half-plane, $y \geq 0$, $(z = x + iy)$ and the integration measure is defined as

$$\mathcal{D}z = \frac{2s-1}{\pi}(2y)^{2s-2}dxdy\,.$$

generators

$$S_+ = z^2\partial_z + 2sz \;,\; S = z\partial_z + s \;,\; S_- = -\partial_z$$

conjugation rules

$$S_\pm^\dagger = -S_\pm \quad , \quad S^\dagger = -S$$

$SL(2,R)$ spin magnet

The Hilbert space of the model is given by the direct product of vector spaces of the representation D_s^+ in each site, $\mathcal{H}_N = \prod_{k=1}^N \otimes V_s$. Thus the space $\mathcal{H}_N$ is the space of functions of N complex variables holomorphic in each variable in the upper half-plane and equipped with the invariant scalar product

$$\langle \Phi \,|\Psi\rangle = \prod_{k=1}^{N} \int \mathcal{D}z_k \, \overline{\Phi(z_1,\ldots,z_N)}\, \Psi(z_1,\ldots,z_N)\,.$$

The scalar product is invariant under the $SL(2,R)$ transformations

$$[T(g)\Phi](z_1,\ldots,z_N) = \prod_{k=1}^{N} \frac{1}{(cz_k+d)^{2s}} \Phi\left(\frac{az_1+b}{cz_1+d},\ldots,\frac{az_N+b}{cz_N+d}\right)\,,$$

generators

$$S_+^{(k)} = z_k^2\partial_k + 2sz_k \;\;,\;\; S^{(k)} = z_k\partial_k + s \;\;,\;\; S_-^{(k)} = -\partial_k$$

L-operators: $\mathrm{L}_k(u) \to \mathrm{L}_k(u_1,u_2)$; $\mathbf{u_1 = s - iu - 1}$, $\mathbf{u_2 = -s - iu}$

$$\mathrm{L}_k(u) = \begin{pmatrix} u + iS^{(k)} & iS_-^{(k)} \\ iS_+^{(k)} & u - iS^{(k)} \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ z_k & 1 \end{pmatrix} \begin{pmatrix} s - iu - 1 & -\partial_k \\ 0 & -s - iu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z_k & 1 \end{pmatrix}$$

Iterative construction of eigenfunctions: closed spin chain

$$A_N(u)\Psi_A^{(N)}(x_1,\ldots,x_N) = (u-x_1)\ldots(u-x_N)\,\Psi_A^{(N)}(x_1,\ldots,x_N)$$
$$B_N(u)\Psi_B^{(N)}(p,x_1,\ldots,x_{N-1}) = p(u-x_1)\ldots(u-x_{N-1})\,\Psi_B^{(N)}(p,x_1,\ldots,x_{N-1})$$

The main building block is the operator $\mathrm{R}_{12}(x)$

$$\left[\mathrm{R}_{12}(x)\Phi\right](z_1,z_2) = \int \mathcal{D}w\,(z_1-\bar{w})^{-s+ix}(z_2-\bar{w})^{-s-ix}\,\Phi(w,z_2)$$

The operator $\mathrm{R}_{12}(x)$ interchanges the second parameters $u_2=-s-iu$ and $v_2=ix-iu$ in the product of two L-operators so that the defining equation for $\mathrm{R}_{12}(x)$ has the form

$$\mathrm{R}_{12}(x)\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(u_1,v_2) = \mathrm{L}_1(u_1,v_2)\,\mathrm{L}_2(u_1,u_2)\,\mathrm{R}_{12}(x)$$

where

$$u_1 = s-1-iu \quad ; \quad u_2=-s-iu \quad ; \quad v_2 = ix-iu\,.$$

The operator

$$\mathrm{R}_{12\ldots N}(x) = \mathrm{R}_{12}(x)\mathrm{R}_{23}(x)\cdots\mathrm{R}_{N-1,N}(x)$$

moves $v_2=ix-iu$ from the right hand side of the product of L-operators to the left hand side

$$\mathrm{R}_{12\ldots N}(x)\,\mathrm{L}_1(u_1,u_2)\cdots\mathrm{L}_{N-1}(u_1,u_2)\,\mathrm{L}_N(u_1,ix-iu) =$$
$$= \mathrm{L}_1(u_1,ix-iu)\,\mathrm{L}_2(u_1,u_2)\cdots\mathrm{L}_N(u_1,u_2)\,\mathrm{R}_{12\ldots N}(x)$$

Iterative construction of eigenfunctions: closed spin chain

Relations for two elements in the first row

$$i\mathrm{R}_{12\ldots N}(x)\begin{pmatrix}\mathrm{A}_{N-1}(u) & \mathrm{B}_{N-1}(u)\end{pmatrix}\begin{pmatrix} s-iu+z_N\partial_N & -\partial_N \\ z_N^2\partial_N+(s-ix)\,z_N & ix-iu-z_N\partial_N\end{pmatrix}=$$
$$=\begin{pmatrix}\mathrm{A}_N(u) & \mathrm{B}_N(u)\end{pmatrix}\mathrm{R}_{12\ldots N}(x)$$

B-system

$$\mathrm{B}_N(u)\,\mathrm{R}_{12\ldots N}(x)=i\mathrm{R}_{12\ldots N}(x)\,\big(-\mathrm{A}_{N-1}(u)\,\partial_N+\mathrm{B}_{N-1}(u)\,(ix-iu-z_N\partial_N)\big)$$
$$\mathrm{B}_N(u)\,\mathrm{R}_{12\ldots N}(x)\,\Psi(z_1\ldots z_{N-1})=(u-x)\,\mathrm{R}_{12\ldots N}(x)\;\mathrm{B}_{N-1}(u)\Psi(z_1\ldots z_{N-1})$$

Next we continue up to the last step

$$\mathrm{B}_N(u)\,\mathrm{R}_{12\ldots N}(x_1)\cdots\mathrm{R}_{12\ldots k}(x_{k-1})\cdots\mathrm{R}_{12}(x_{N-1})\Psi(z_1)=$$
$$=(u-x_1)\cdots(u-x_{k-1})\cdots(u-x_{N-1})$$
$$\mathrm{R}_{12\ldots N}(x_1)\cdots\mathrm{R}_{12\ldots k}(x_{k-1})\cdots\mathrm{R}_{12}(x_{N-1})\mathrm{B}_1(u)\Psi(z_1)\;,$$

where $\mathrm{R}_{12\ldots k}(x)=\mathrm{R}_{12}(x)\mathrm{R}_{23}(x)\cdots\mathrm{R}_{k-1,k}(x)$. The eigenfunction of the last operator $\mathrm{B}_1(u)=-i\partial_1$ is e^{ipz_1} so that we have

$$\mathrm{B}_N(u)\,\mathrm{R}_{12\ldots N}(x_1)\cdots\mathrm{R}_{12}(x_{N-1})\mathrm{e}^{ipz_1}=$$
$$p\,(u-x_1)\cdots(u-x_{N-1})\,\mathrm{R}_{12\ldots N}(x_1)\cdots\mathrm{R}_{12}(x_{N-1})\,\mathrm{e}^{ipz_1}\;,$$

$$\Psi_B^{(N)}(p,x_1,\ldots,x_{N-1}|z_1,\ldots,z_N)=\mathrm{R}_{12\ldots N}(x_1)\cdots\mathrm{R}_{12}(x_{N-1})\,\mathrm{e}^{ipz_1}$$

Iterative construction of eigenfunctions: closed spin chain

A-system

$$\mathrm{A}_N(u)\,\mathrm{R}_{12\dots N}\,(x) = \\ i\mathrm{R}_{12\dots N}\,(x)\,\left(\mathrm{A}_{N-1}(u)\,(s-iu+z_N\partial_N)+\mathrm{B}_{N-1}(u)\left(z_N^2\partial_N+(s-ix)\,z_N\right)\right)$$

$$\mathrm{A}_N(u)\,\mathrm{R}_{12\dots N}\,(x)\,\Psi(z_1\dots z_{N-1})z_N^{-s+ix} = \\ (u-x)\,\mathrm{R}_{12\dots N}\,(x)\,z_N^{-s+ix}\,\mathrm{A}_{N-1}(u)\Psi(z_1\dots z_{N-1})$$

Next we continue up to the last step

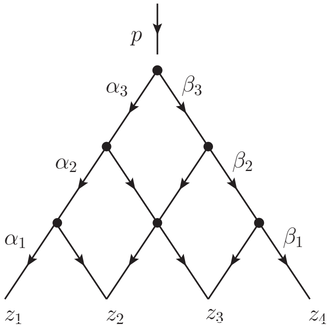
$$\mathrm{A}_N(u)\,\mathrm{R}_{12\dots N}\,(x_1)\,z_N^{-s+ix_1}\cdots\mathrm{R}_{12}\,(x_{N-1})\,z_2^{-s+ix_{N-1}}\Psi(z_1) = \\ = (u-x_1)\cdots(u-x_{N-1})\,\mathrm{R}_{12\dots N}\,(x_1)\,z_N^{-s+ix_1}\cdots\mathrm{R}_{12}\,(x_{N-1})\,z_2^{-s+ix_{N-1}}\mathrm{A}_1(u)\cdot\Psi(z_1)$$

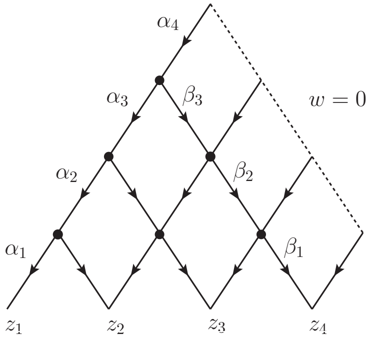
The eigenfunction of the last operator $\mathrm{A}_1 = u+i(s+z_1\partial_1)$ is $z_1^{-s+ix_N}$ so that we obtain

$$\mathrm{A}_N(u)\,\mathrm{R}_{12\dots N}\,(x_1)\,z_N^{-s+ix_1}\cdots\mathrm{R}_{12}\,(x_{N-1})\,z_2^{-s+ix_{N-1}}\,z_1^{-s+ix_N} = \\ = (u-x_1)\cdots(u-x_N)\,\mathrm{R}_{12\dots N}\,(x_1)\,z_N^{-s+ix_1}\cdots\mathrm{R}_{12}\,(x_{N-1})\,z_2^{-s+ix_{N-1}}\,z_1^{-s+ix_N}\,,$$

$$\Psi_A^{(N)}(x_1,\dots,x_N|z_1,\dots,z_N) = \mathrm{R}_{12\dots N}\,(x_1)\,z_N^{-s+ix_1}\cdots\mathrm{R}_{12}\,(x_{N-1})\,z_2^{-s+ix_{N-1}}\,z_1^{-s+ix_N}$$

Diagrammatic representation for eigenfunctions: closed spin chain





$$\alpha_{\mathbf{k}} = \mathbf{s} - \mathbf{ix_k} \quad , \quad \beta_{\mathbf{k}} = \mathbf{s} + \mathbf{ix_k}$$

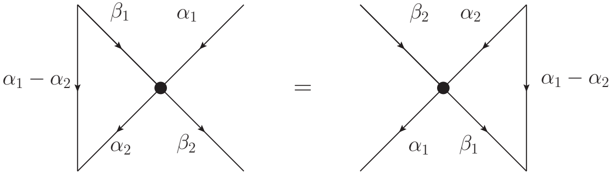
The propagator

$$D_\alpha(z,\bar{w}) = \left(\frac{i}{z-\bar{w}}\right)^\alpha = \frac{1}{\Gamma(\alpha)}\int_0^\infty dp\, e^{ip\,(z-\bar{w})}\, p^{\alpha-1}$$

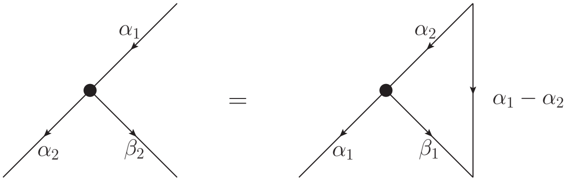
is shown by the arrow directed from $\bar{w}$ to z with the index α attached to it. The arrows attached to the dashed lines start from the point $w=0$.

Symmetry of eigenfunctions

It is symmetric under permutation of the variables $\{x_1, \ldots, x_{N-1}\}$.
Permutation relation:

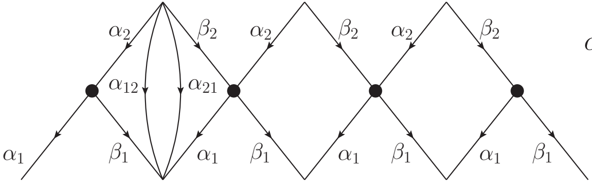


Reduced permutation relation:

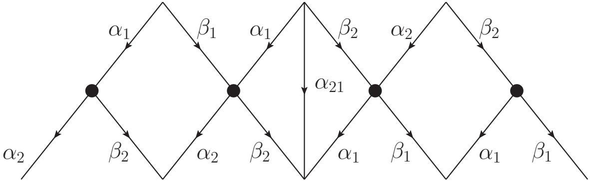


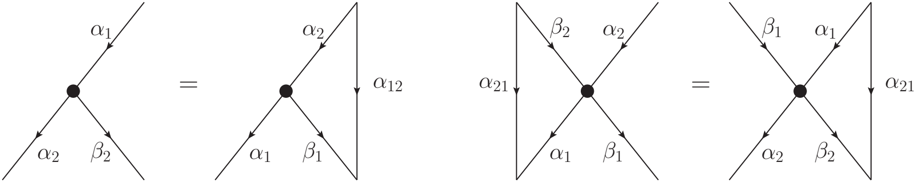
$$\alpha_1 = s - ix_1\,,\ \beta_1 = s + ix_1\,,\ \alpha_2 = s - ix_2\,,\ \beta_2 = s + ix_2$$

Symmetry of eigenfunctions



$$\alpha_{12} = \alpha_1 - \alpha_2$$





Scalar products

The operators $A_N(u)$ and $B_N(u)$ are self-adjoint for real u

$$(A_N(u))^\dagger = A_N(u) \quad , \quad (B_N(u))^\dagger = B_N(u)$$

The eigenfunctions are mutually orthogonal

$$\left\langle \Psi_B^{(N)}(p',\vec{x}') \,\middle|\, \Psi_B^{(N)}(p,\vec{x}) \right\rangle = (2\pi)^{N-1}\,\delta(p-p')\,\delta_{N-1}\left(\vec{x}-\vec{x}'\right)\,\frac{p^{1-2Ns}\,(N-1)!\,\prod_{j\neq k}\Gamma(i(x_k-x_j))}{\prod_{k=1}^{N-1}\left[\Gamma(s-ix_k)\Gamma(s+ix_k)\right]^N}$$

where the separated variables $\vec{x}=\{x_1,\dots,x_{N-1}\}$ are real numbers and $p\geq 0$.

$$\left\langle \Psi_A^{(N)}(\vec{x}') \,\middle|\, \Psi_A^{(N)}(\vec{x}) \right\rangle = (2\pi)^N\,\delta_N\left(\vec{x}-\vec{x}'\right)\,\frac{N!\,\prod_{j\neq k}\Gamma(i(x_k-x_j))}{\prod_{k=1}^{N}\left[\Gamma(s-ix_k)\Gamma(s+ix_k)\right]^N}\,,$$

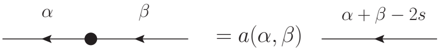
where $\vec{x}=\{x_1,\dots,x_N\}$ are real numbers.
The symmetric delta-function is defined as follows

$$\delta_N(\vec{x}-\vec{x}') \equiv \frac{1}{N!}\sum_{s\in\mathcal{S}_N}\delta(x_1-x'_{s(1)})\dots\delta(x_N-x'_{s(N)})$$

and the sum goes over all permutations.

Chain rule

Chain rule:

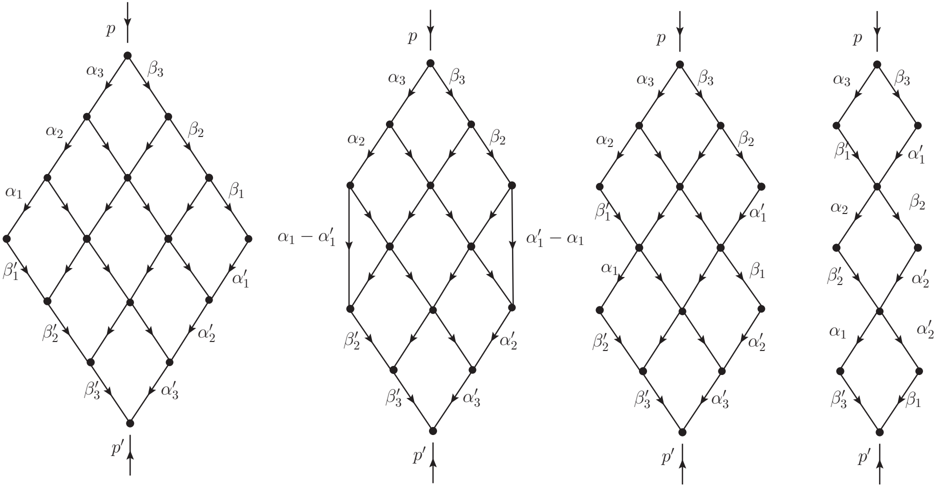


$$\int \mathcal{D}w\, D_\alpha(z,\bar{w})D_\beta(w,\bar{\zeta}) = a(\alpha,\beta)D_{\alpha+\beta-2s}(z,\bar{\zeta})\,,$$

where

$$a(\alpha,\beta) = \frac{\Gamma(2s)\Gamma(\alpha+\beta-2s)}{\Gamma(\alpha)\Gamma(\beta)}\,.$$

Scalar product of eigenfunctions of B-operator



Mixed scalar product and the matrix element of the shift operator

mixed scalar product

$$\left\langle \Psi_B^{(N)}(p,\vec{u}) \,\middle|\, \Psi_A^{(N)}(\vec{x}) \right\rangle = p^{-Ns}\, p^{-iX} \prod_{k=1}^{N} \frac{1}{\Gamma(s-ix_k)} \prod_{j=1}^{N-1} \frac{\Gamma(i(u_j-x_k))}{\Gamma(s-ix_k)\Gamma(s+iu_j)}$$

shift operator $T_\gamma = \exp\{-\gamma S_-\}$, where $S_- = \sum_{k=1}^{N} S_-^{(k)}$

$$T_\gamma f(z_1,\ldots,z_N) = f(z_1+\gamma,\ldots,z_N+\gamma)$$

Action on $\Psi_B^{(N)}(p,\vec{x})$ is diagonal

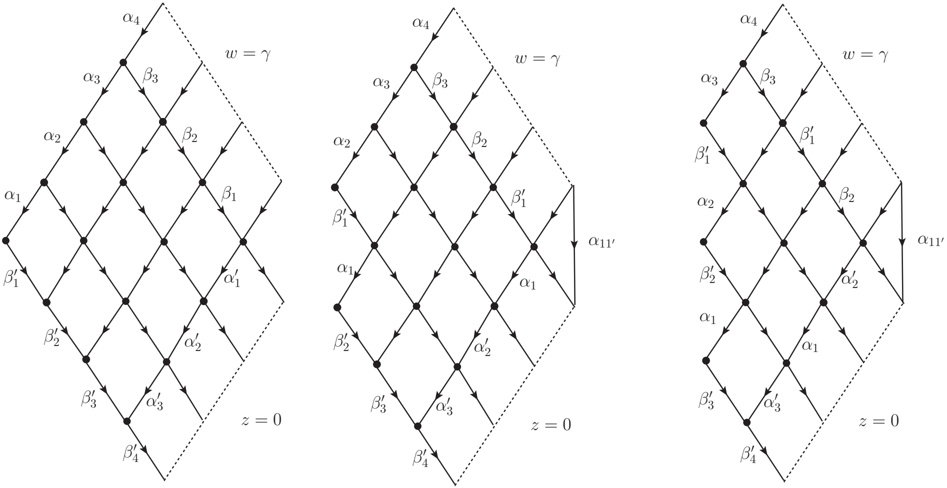
$$S_-\Psi_B^{(N)}(p,\vec{x}) = -ip\,\Psi_B^{(N)}(p,\vec{x})\,,$$
$$T_\gamma\Psi_B^{(N)}(p,\vec{x}) = e^{i\gamma p}\,\Psi_B^{(N)}(p,\vec{x})$$

and for $\Psi_A^{(N)}(\vec{x})$ the direct calculation results in the following expression

$$\left\langle \Psi_A^{(N)}(\vec{x}') \,\middle|\, T_\gamma\, \Psi_A^{(N)}(\vec{x}) \right\rangle = \gamma^{i(X-X')} e^{\frac{\pi}{2}(X-X')} \prod_{k,j} \frac{\Gamma(i(x_j'-x_k))}{\Gamma(s-ix_j)\Gamma(s+ix_k')}$$

$$X = \sum_{k=1}^{N} x_k \quad , \quad X' = \sum_{k=1}^{N} x_k'$$

Matrix elements of the shift operator



$$\left\langle \Psi_A^{(N)}(\vec{x}') \,\middle|\, T_\gamma \Psi_A^{(N)}(\vec{x}) \right\rangle = \gamma^{i(X-X')} e^{\frac{\pi}{2}(X-X')} \prod_{k,j} \frac{\Gamma(i(x'_j - x_k))}{\Gamma(s - ix_j)\Gamma(s + ix'_k)}$$

The first Gustafson's integral

Expanding the eigenfunctions $\Psi_A^{(N)}$ over $\Psi_B^{(N)}$(B-system) one gets the following integral representation for the matrix element of the shift operator

$$\left\langle \Psi_A^{(N)}(\vec{x}') \,\middle|\, T_\gamma \Psi_A^{(N)}(\vec{x}) \right\rangle = \frac{1}{(N-1)!}\int_0^\infty dp\, e^{i\gamma p}\int_{-\infty}^{\infty}\prod_{k=1}^{N-1}\frac{du_k}{2\pi}\,\mu_N(p,\vec{u}) \left\langle \Psi_A^{(N)}(\vec{x}') \,\middle|\, \Psi_B^{(N)}(p,\vec{u}) \right\rangle \left\langle \Psi_B^{(N)}(p,\vec{u}) \,\middle|\, \Psi_A^{(N)}(\vec{x}) \right\rangle,$$

where the measure is defined as follows

$$\mu_N(p,\vec{u}) = p^{2Ns-1}\,\frac{\prod_{k=1}^{N-1}\left[\Gamma(s+iu_k)\Gamma(s-iu_k)\right]^N}{\prod_{j\neq k}\Gamma(i(u_k-u_j))}\,.$$

Calculating the momentum integral and canceling common factors one gets the identity

$$\frac{1}{(N-1)!}\left(\prod_{n=1}^{N-1}\int\frac{du_n}{2\pi}\right)\frac{\prod_{k=1}^{N}\prod_{j=1}^{N-1}\Gamma(i(x_k'-u_j))\Gamma(i(u_j-x_k))}{\prod_{k<j}\Gamma(i(u_k-u_j))\Gamma(i(u_j-u_k))} = \frac{\prod_{k,j=1}^{N}\Gamma(i(x_k'-x_j))}{\Gamma(i\sum_{k=1}^{N-1}(x_k'-x_k))}\,,$$

which is equivalent to the first Gustafson's integral.

Iterative construction of eigenfunctions: open spin chain

The main building blocks are two operators. The operator $\mathrm{R}_{12}(x)$

$$[\mathrm{R}_{12}(x)\Phi]\,(z_1,z_2) = \int \mathcal{D}w\,(z_1-\bar{w})^{-s+ix}(z_2-\bar{w})^{-s-ix}\,\Phi(w,z_2)$$

interchanges parameters in the product of two L-operators in a special way

$$\mathrm{R}_{12}(x)\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(u_1,v_2) = \mathrm{L}_1(u_1,v_2)\,\mathrm{L}_2(u_1,u_2)\,\mathrm{R}_{12}(x)\,,$$
$$\mathrm{R}_{12}(x)\,\mathrm{L}_2(v_1,u_2)\,\mathrm{L}_1(u_1,u_2) = \mathrm{L}_2(u_1,u_2)\,\mathrm{L}_1(v_1,u_2)\,\mathrm{R}_{12}(x)\,,$$

where

$$u_1 = s-1-iu \quad ; \quad u_2 = -s-iu \quad ; \quad v_1 = -1-ix-iu \quad ; \quad v_2 = ix-iu\,.$$

The operator $\mathrm{R}_{21}(x)$

$$[\mathrm{R}_{21}(x)\Phi]\,(z_1,z_2) = \int \mathcal{D}w\,(z_2-\bar{w})^{-s+ix}(z_1-\bar{w})^{-s-ix}\,\Phi(z_1,w)$$

interchanges parameters in a similar way but $1 \rightleftarrows 2$

$$\mathrm{R}_{21}(x)\,\mathrm{L}_2(u_1,u_2)\,\mathrm{L}_1(u_1,v_2) = \mathrm{L}_2(u_1,v_2)\,\mathrm{L}_1(u_1,u_2)\,\mathrm{R}_{21}(x)\,,$$
$$\mathrm{R}_{21}(x)\,\mathrm{L}_1(v_1,u_2)\,\mathrm{L}_2(u_1,u_2) = \mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,u_2)\,\mathrm{R}_{21}(x)$$

Iterative construction of eigenfunctions: open spin chain

open spin chain

$$\mathbb{T}(u) = \mathrm{L}_1(u_1,u_2)\cdots\mathrm{L}_N(u_1,u_2)\,\mathrm{L}_N(u_1,u_2)\cdots\mathrm{L}_1(u_1,u_2) = \begin{pmatrix} \mathbb{A}_N(u) & \mathbb{B}_N(u) \\ \mathbb{C}_N(u) & \mathbb{D}_N(u) \end{pmatrix}$$

$$\mathrm{R}_{12\ldots N}(x) = \mathrm{R}_{12}(x)\mathrm{R}_{23}(x)\cdots\mathrm{R}_{N-1,N}(x)$$

$$\mathrm{L}_1(u_1,v_2)\cdots\mathrm{L}_N(u_1,u_2)\,\mathrm{L}_N(u_1,u_2)\cdots\mathrm{L}_1(v_1,u_2)\,\mathrm{R}_{12\ldots N}(x) =$$
$$= \mathrm{R}_{12\ldots N}(x)\mathrm{L}_1(u_1,u_2)\cdots\mathrm{L}_N(u_1,v_2)\,\mathrm{L}_N(v_1,u_2)\cdots\mathrm{L}_1(u_1,u_2)$$

$$\mathrm{L}_N(u_1,v_2)\,\mathrm{L}_N(v_1,u_2) = \mathrm{L}_N(v_1,u_2)\,\mathrm{L}_N(u_1,v_2)$$

$$\mathrm{R}_{N\ldots 21}(x) = \mathrm{R}_{N,N-1}(x)\cdots\mathrm{R}_{32}(x)\mathrm{R}_{21}(x)$$

$$\mathrm{L}_1(u_1,u_2)\cdots\mathrm{L}_N(v_1,u_2)\,\mathrm{L}_N(u_1,v_2)\cdots\mathrm{L}_1(u_1,u_2)\,\mathrm{R}_{N\ldots 21}(x) =$$
$$= \mathrm{R}_{N\ldots 21}(x)\mathrm{L}_1(v_1,u_2)\cdots\mathrm{L}_N(u_1,u_2)\,\mathrm{L}_N(u_1,u_2)\cdots\mathrm{L}_1(u_1,v_2)$$

$$\mathbb{R}_{12\ldots N}(x) = \mathrm{R}_{12\ldots N}(x)\mathrm{R}_{N\ldots 21}(x)$$

$$\mathrm{L}_1(u_1,ix-iu)\cdots\mathrm{L}_N(u_1,u_2)\,\mathrm{L}_N(u_1,u_2)\cdots\mathrm{L}_1(-1-ix-iu,u_2)\,\mathbb{R}_{12\ldots N}(x) =$$
$$= \mathbb{R}_{12\ldots N}(x)\,\mathrm{L}_1(-1-ix-iu,u_2)\cdots\mathrm{L}_N(u_1,u_2)\,\mathrm{L}_N(u_1,u_2)\cdots\mathrm{L}_1(u_1,ix-iu)$$

Iterative construction of eigenfunctions: open spin chain

$$\mathbb{B}_N(u)\,\mathbb{R}_{12\dots N}(x) = -\mathbb{R}_{12\dots N}(x)$$

$$\begin{pmatrix} -ix-iu+z_1\partial_1 & -\partial_1 \end{pmatrix}\begin{pmatrix} \mathbb{A}_{N-1}(u) & \mathbb{B}_{N-1}(u) \\ \mathbb{C}_{N-1}(u) & \mathbb{D}_{N-1}(u) \end{pmatrix}\begin{pmatrix} -\partial_1 \\ ix-iu-z_1\partial_1 \end{pmatrix}$$

$$\mathbb{B}_N(u)\,\mathbb{R}_{12\dots N}(x)\Psi(z_2,\dots,z_N) =$$
$$(u+x)(u-x)\mathbb{R}_{12\dots N}(x)\mathbb{B}_{N-1}(u)\Psi(z_2,\dots,z_N)$$

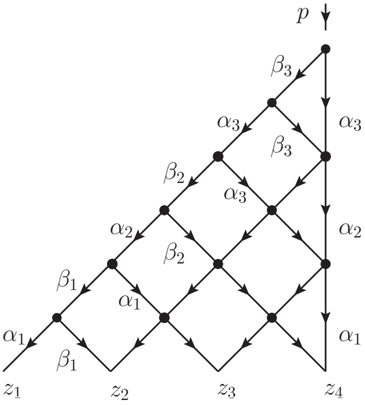
Next we continue up to the last step

$$\mathbb{B}_N(u)\,\mathbb{R}_{12\dots N}(x_1)\cdots\mathbb{R}_{12\dots k}(x_{k-1})\cdots\mathbb{R}_{12}(x_{N-1})\Psi(z_N) =$$
$$= (u^2-x_1^2)\cdots(u^2-x_{k-1}^2)\cdots(u^2-x_{N-1}^2)$$
$$\mathbb{R}_{12\dots N}(x_1)\cdots\mathbb{R}_{12\dots k}(x_{k-1})\cdots\mathbb{R}_{12}(x_{N-1})\mathbb{B}_1(u)\Psi(z_N)\ ,$$

where $\mathbb{R}_{12\dots k}(x) = \mathrm{R}_{12}(x)\mathrm{R}_{23}(x)\cdots\mathrm{R}_{k-1,k}(x)\mathrm{R}_{k,k-1}(x)\cdots\mathrm{R}_{32}(x)\mathrm{R}_{21}(x)$. The eigenfunction of the last operator $\mathrm{B}_1(u) = -(2iu+1)\partial_N$ is e^{ipz_N} so that we have

$$\mathbb{B}_N(u)\,\mathbb{R}_{12\dots N}(x_1)\cdots\mathbb{R}_{12}(x_{N-1})\mathrm{e}^{ipz_N} =$$
$$p(2iu+1)\,(u^2-x_1^2)\cdots(u^2-x_{N-1}^2)\,\mathrm{R}_{12\dots N}(x_1)\cdots\mathrm{R}_{12}(x_{N-1})\,\mathrm{e}^{ipz_1}\ ,$$
$$\Psi^{(N)}_{\mathbb{B}}(p,x_1,\dots,x_{N-1}|z_1,\dots,z_N) = \mathbb{R}_{12\dots N}(x_1)\cdots\mathbb{R}_{12}(x_{N-1})\,\mathrm{e}^{ipz_N}$$

Diagrammatic representation for eigenfunctions: open spin chain



$\alpha_k = s - ix_k$, $\beta_k = s + ix_k$

The diagrammatic representation of the eigenfunction $\Psi^{(4)}_{\mathbb{B}}(p, x_1, x_2, x_3 | z_1, z_2, z_3, z_4)$.

Scalar products

The scalar product: $\mathbb{B}$-system (*open chain*) $x_k \geq 0$ and $x_k' \geq 0$

$$\left\langle \Psi_{\mathbb{B}}^{(N)}(p',\vec{x}') \,\middle|\, \Psi_{\mathbb{B}}^{(N)}(p,\vec{x}) \right\rangle = (2\pi)^{N-1}\,\delta(p-p')\,\delta_{N-1}\left(\vec{x}-\vec{x}'\right)\; p^{1-2Ns}$$

$$\prod_{n=1}^{N-1}\Gamma(2ix_n)\Gamma(-2ix_n)\,\frac{\prod_{j<k}\Gamma(i(x_k\pm x_j))\Gamma(-i(x_k\pm x_j))}{\prod_{k=1}^{N-1}\left[\Gamma(s-ix_k)\Gamma(s+ix_k)\right]^{2N}}\,.$$

The scalar product: $\mathbb{B}$-system (*open chain*) and A-system (*closed chain*)

$$\left\langle \Psi_{\mathbb{B}}^{(N)}(p,\vec{u}) \,\middle|\, \Psi_{A}^{(N)}(\vec{x}) \right\rangle = p^{-Ns}\,p^{-iX}$$

$$\frac{\prod_{k=1}^{N}\prod_{j=1}^{N-1}\Gamma(\pm iu_j - ix_k)}{\left(\prod_{k=1}^{N-1}\Gamma(s-iu_k)\Gamma(s+iu_k)\prod_{k=1}^{N}\Gamma(s-ix_k)\right)^{N}\prod_{k<j}\Gamma(-i(x_k+x_j))}$$

The scalar product: $\mathbb{B}$-system (*open chain*) and B-system (*closed chain*)

$$\left\langle \Psi_{\mathbb{B}}^{(N)}(p,\vec{u}) \,\middle|\, \Psi_{B}^{(N)}(q,\vec{x}) \right\rangle = \delta(p-q)\,p^{1-2Ns}$$

$$\frac{\prod_{k=1}^{N-1}\prod_{j=1}^{N-1}\Gamma(\pm iu_j - ix_k)}{\left(\prod_{k=1}^{N-1}\Gamma(s-iu_k)\Gamma(s+iu_k)\Gamma(s-ix_k)\right)^{N}\prod_{k<j}\Gamma(-i(x_k+x_j))}\,.$$

Matrix elements

The second Gustafson's integral is also related to the matrix element of the shift operator

$$\left\langle \Psi_A^{(N)}(\vec{x}') \,\middle|\, T_\gamma \Psi_A^{(N)}(\vec{x}) \right\rangle = \frac{1}{(N-1)!} \int_0^\infty dp\, e^{i\gamma p} \int_{-\infty}^{\infty} \prod_{k=1}^{N-1} \frac{du_k}{4\pi}\, \widetilde{\mu}_N(p,\vec{u}) \left\langle \Psi_A^{(N)}(\vec{x}') \,\middle|\, \Psi_{\mathbb{B}}^{(N)}(p,\vec{u}) \right\rangle \left\langle \Psi_{\mathbb{B}}^{(N)}(p,\vec{u}) \,\middle|\, \Psi_A^{(N)}(\vec{x}) \right\rangle,$$

The measure is defined as follows

$$\widetilde{\mu}_N(p,\vec{u}) = p^{2Ns-1} \frac{\prod_{k=1}^{N-1} \left[\Gamma(s+iu_k)\Gamma(s-iu_k)\right]^{2N}}{\prod_{j<k} \Gamma(i(u_k \pm u_j))\Gamma(-i(u_k \pm u_j))}$$

Calculating the momentum integral and canceling common factors one gets the identity

$$\frac{1}{(N-1)!} \left(\prod_{n=1}^{N-1} \int \frac{du_n}{4\pi} \right) \frac{\prod_{k=1}^{N} \prod_{j=1}^{N-1} \Gamma(i(x_k' \pm u_j))\Gamma(-i(x_k \pm u_j))}{\prod_{k=1}^{N-1} \Gamma(2iu_k)\Gamma(-2iu_k) \prod_{k<j}^{N-1} \Gamma(i(u_k \pm u_j))\Gamma(-i(u_k \pm u_j))}$$

$$= \Gamma^{-1}(i(X'-X)) \prod_{k,j=1}^{N} \Gamma(i(x_k'-x_j)) \prod_{k<j}^{N} \Gamma(i(x_k'+x_j'))\Gamma(-i(x_k+x_j))$$

This integral, after redefinition $\alpha_k = ix_k'$, $\alpha_{N+k} = -ix_k$ and $N-1 \to N$, coincides with the second Gustafson's integral.

Matrix elements

The scalar product $\left\langle \Psi^{(N)}_{\mathbb{B}}(p,\vec{u}) \,\middle|\, \Psi^{(N)}_{A}(\vec{x}) \right\rangle$ in the $\Psi^{(N)}_{B}(q,\vec{u})$ basis

$$\left\langle \Psi^{(N)}_{\mathbb{B}}(p,\vec{u}) \,\middle|\, \Psi^{(N)}_{A}(\vec{x}) \right\rangle = \frac{1}{(N-1)!}\int_0^\infty dq \left(\prod_{k=1}^{N-1}\int \frac{du_k}{2\pi}\right)\mu_N(q,\vec{u}) \left\langle \Psi^{(N)}_{\mathbb{B}}(p,\vec{u}) \,\middle|\, \Psi^{(N)}_{B}(q,\vec{x}) \right\rangle \left\langle \Psi^{(N)}_{B}(q,\vec{u}) \,\middle|\, \Psi^{(N)}_{A}(\vec{x}) \right\rangle,$$

where the measure $\mu_N(q,\vec{u})$ is

$$\mu_N(q,\vec{u}) = q^{2Ns-1}\,\frac{\prod_{k=1}^{N-1}\left[\Gamma(s+iu_k)\Gamma(s-iu_k)\right]^N}{\prod_{j\neq k}\Gamma(i(u_k-u_j))}$$

After some redefinitions and simplifications one gets the identity

$$\left(\prod_{n=1}^{N}\int\frac{dz_n}{2\pi i}\right)\frac{\prod_{j=1}^{N}\left(\prod_{k=1}^{N+1}\Gamma(\alpha_k-z_j)\right)\left(\prod_{m=1}^{N}\Gamma(z_j\pm\beta_m)\right)}{\prod_{k<j}\Gamma(z_k\pm z_j)\Gamma(z_j-z_k)} = \frac{N!\prod_{j=1}^{N}\prod_{k=1}^{N+1}\Gamma(\alpha_k\pm\beta_j)}{\prod_{j<k}^{N+1}\Gamma(\alpha_j+\alpha_k)}$$

$SL(2,\mathbb{C})$ spin magnet

unitary principal series representations of $\mathrm{SL}(2,\mathbb{C})$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \;\; ; \;\; g \to \mathrm{T}^{(s,\bar{s})}(g) : L_2(\mathbb{C}) \to L_2(\mathbb{C})$$

$$\left[\mathrm{T}^{(s,\bar{s})}(g)\,\phi\right](z,\bar{z}) = (d-bz)^{-2s}\,\left(\bar{d}-\bar{b}\bar{z}\right)^{-2\bar{s}}\,\phi\left(\frac{-c+az}{d-bz},\frac{-\bar{c}+\bar{a}\bar{z}}{\bar{d}-\bar{b}\bar{z}}\right)$$

$$\langle\phi\,|\psi\rangle = \int d^2z\,\overline{\phi(z,\bar{z})}\,\psi(z,\bar{z}) \;\; ; \;\; \langle\, T^{(s,\bar{s})}(g)\phi\,|\,T^{(s,\bar{s})}(g)\psi\,\rangle = \langle\phi\,|\psi\rangle$$

The spins s and $\bar{s}$ are parameterized as follows ($n_s \in \mathbb{Z}\,,\nu_s \in \mathbb{R}$)

$$s = \frac{1+n_s}{2} + i\nu_s \;\; ; \;\; \bar{s} = \frac{1-n_s}{2} + i\nu_s$$

generators:

$$S_- = -\partial_z \;\; , \;\; S = z\partial_z + s \;\; , \;\; S_+ = z^2\partial_z + 2s\,z$$
$$\bar{S}_- = -\partial_{\bar{z}} \;\; , \;\; \bar{S} = \bar{z}\partial_{\bar{z}} + \bar{s} \;\; , \;\; \bar{S}_+ = \bar{z}^2\partial_{\bar{z}} + 2\bar{s}\,\bar{z}$$

commutation relations:

$$[S_+,S_-] = 2S \;\; , \;\; [S,S_\pm] = \pm S_\pm \;\; , \;\; [\bar{S}_+,\bar{S}_-] = 2\bar{S} \;\; , \;\; [\bar{S},\bar{S}_\pm] = \pm\bar{S}_\pm$$

conjugation:

$$S_\pm^\dagger = -\bar{S}_\pm \;\; , \;\; S^\dagger = -\bar{S}$$

$SL(2,\mathbb{C})$ spin magnet

The Hilbert space of the $SL(2,\mathbb{C})$ spin magnet is given by the tensor product of the unitary principal series representations of the $SL(2,\mathbb{C})$ group

$$\mathbb{H}_N = \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \cdots \otimes \mathbb{V}_N, \qquad \mathbb{V}_k = L_2(\mathbb{C})\,, \qquad k = 1,\ldots,N.$$

The space $\mathbb{H}_N$ is the space of functions $\Psi(z_1,\bar{z}_1\ldots,z_N,\bar{z}_N)$ equipped with the invariant scalar product

$$\langle\Phi\,|\Psi\rangle = \prod_{k=1}^{N}\int d^2 z_k\,\overline{\Phi(z_1,\bar{z}_1\ldots,z_N,\bar{z}_N)}\,\Psi(z_1,\bar{z}_1\ldots,z_N,\bar{z}_N)$$

To each site k we associate the pair of quantum L-operators with subscript k acting nontrivially on the $k-$th space in the tensor product

$$L_k(u) = \begin{pmatrix} u + iS^{(k)} & iS_-^{(k)} \\ iS_+^{(k)} & u - iS^{(k)} \end{pmatrix}, \qquad \bar{L}_k(\bar{u}) = \begin{pmatrix} \bar{u} + i\bar{S}^{(k)} & i\bar{S}_-^{(k)} \\ i\bar{S}_+^{(k)} & \bar{u} - i\bar{S}^{(k)} \end{pmatrix},$$

where $u\in\mathbb{C}$ and $\bar{u}\in\mathbb{C}$ are two independent spectral parameters. The monodromy matrices $T_N(u)$ and $\bar{T}_N(\bar{u})$ are defined as a product of L operators

$$T(u) = L_1(u)L_2(u)\ldots L_N(u)\,, \qquad \bar{T}(\bar{u}) = \bar{L}_1(\bar{u})\bar{L}_2(\bar{u})\ldots\bar{L}_N(\bar{u})$$

$$T(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix}, \qquad \bar{T}(\bar{u}) = \begin{pmatrix} \bar{A}_N(\bar{u}) & \bar{B}_N(\bar{u}) \\ \bar{C}_N(\bar{u}) & \bar{D}_N(\bar{u}) \end{pmatrix}$$

A-system and B-system

A-system

$$\begin{aligned} A_N(u)\,\Psi_A(\boldsymbol{x}|\boldsymbol{z}) &= (u-x_1)\cdots(u-x_N)\,\Psi_A(\boldsymbol{x}|\boldsymbol{z}) \\ \bar{A}_N(\bar{u})\,\Psi_A(\boldsymbol{x}|\boldsymbol{z}) &= (\bar{u}-\bar{x}_1)\cdots(\bar{u}-\bar{x}_N)\,\Psi_A(\boldsymbol{x}|\boldsymbol{z}) \end{aligned}$$

The eigenfunctions $\Psi_A(z_1,\bar{z}_1\ldots,z_N,\bar{z}_N)$ are labeled by zeroes of polynomial eigenvalues

$$\begin{aligned} \boldsymbol{x} &= \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N\}, & \boldsymbol{x}_k &= (x_k,\bar{x}_k) \\ \boldsymbol{z} &= \{\boldsymbol{z}_1,\ldots,\boldsymbol{z}_N\}, & \boldsymbol{z}_k &= (z_k,\bar{z}_k) \end{aligned}$$

The variables $\bar{x}_k$ are adjoint to x_k, $\bar{x}_k = x_k^*$ and parameterized as follows: $n_k\in\mathbb{Z}\,,\nu_k\in\mathbb{R}$

$$x_k = -\frac{in_k}{2}+\nu_k\,,\qquad \bar{x}_k = \frac{in_k}{2}+\nu_k$$

B-system

$$\begin{aligned} B_N(u)\,\Psi_B(\boldsymbol{p},\boldsymbol{x}|\boldsymbol{z}) &= p(u-x_1)\cdots(u-x_{N-1})\,\Psi_B(\boldsymbol{p},\boldsymbol{x}|\boldsymbol{z}) \\ \bar{B}_N(\bar{u})\,\Psi_B(\boldsymbol{p},\boldsymbol{x}|\boldsymbol{z}) &= \bar{p}(\bar{u}-\bar{x}_1)\cdots(\bar{u}-\bar{x}_{N-1})\,\Psi_B(\boldsymbol{p},\boldsymbol{x}|\boldsymbol{z}) \end{aligned}$$

$\Psi_B(\boldsymbol{p},\boldsymbol{x}|\boldsymbol{z})$ are parameterized by the momenta $p,\bar{p}$ – eigenvalues of the $S_-,\bar{S}_-$ operators and the roots $x_k,\bar{x}_k$, $k=1,\ldots,N-1$

$$\boldsymbol{p} = (p,\bar{p})\,,\boldsymbol{x} = \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{N-1}\}$$

simplest example $N=1$

$$L(u) = i\begin{pmatrix} s - iu + z\partial & -\partial \\ z^2\partial + 2sz & -s - iu - z\partial \end{pmatrix} = \begin{pmatrix} A_1(u) & B_1(u) \\ C_1(u) & D_1(u) \end{pmatrix}$$

A-system

$$\Psi_A(\boldsymbol{x}|\boldsymbol{z}) = [z]^{ix-s} \equiv z^{ix-s}\bar{z}^{i\bar{x}-\bar{s}}$$

$[z]^\alpha \equiv z^\alpha\, \bar{z}^{\bar{\alpha}}$ – single valued function in the complex plane provided that
$\alpha - \bar{\alpha} \in \mathbb{Z} \to (s - ix) - (\bar{s} - i\bar{x}) = -i(x - \bar{x}) + n_s \in \mathbb{Z} \to x = -\frac{in}{2} + \nu\,, \bar{x} = \frac{in}{2} + \nu$

$$A_1(u)\, z^{ix-s}\bar{z}^{i\bar{x}-\bar{s}} = i\,(s - iu + z\partial)\; z^{ix-s}\bar{z}^{i\bar{x}-\bar{s}} = (u - x)\, z^{ix-s}\bar{z}^{i\bar{x}-\bar{s}}$$
$$\bar{A}_1(\bar{u})\, z^{ix-s}\bar{z}^{i\bar{x}-\bar{s}} = i\left(\bar{s} - i\bar{u} + \bar{z}\bar{\partial}\right) z^{ix-s}\bar{z}^{i\bar{x}-\bar{s}} = (\bar{u} - \bar{x})\, z^{ix-s}\bar{z}^{i\bar{x}-\bar{s}}$$

orthogonality $x_1 = -\frac{in_1}{2} + \nu_1\,, \bar{x}_1 = \frac{in_1}{2} + \nu_1\,, x_2 = -\frac{in_2}{2} + \nu_2\,, \bar{x}_2 = \frac{in_2}{2} + \nu_2$

$$\int d^2z\, \overline{\Psi_A(\boldsymbol{x}_1|\boldsymbol{z})}\Psi_A(\boldsymbol{x}_2|\boldsymbol{z}) = 2\pi^2\delta^2(\boldsymbol{x}_1 - \boldsymbol{x}_2)\;\;,\;\; \delta^2(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \delta_{n_1 n_2}\,\delta(\nu_1 - \nu_2)$$

completeness $x = -\frac{in}{2} + \nu\,, \quad \bar{x} = \frac{in}{2} + \nu$

$$\int Dx\, \overline{\Psi_A(\boldsymbol{x}|\boldsymbol{z}_1)}\Psi_A(\boldsymbol{x}|\boldsymbol{z}_2) = 2\pi^2\delta^2(\boldsymbol{z}_1 - \boldsymbol{z}_2)\;\;,\;\; \int Dx = \sum_{n\in\mathbb{Z}}\int_{-\infty}^{+\infty} \mathrm{d}\nu$$

B-system: plane waves

$$\Psi_B(\boldsymbol{p}|\boldsymbol{z}) = e^{ipz+i\bar{p}\bar{z}}$$

$$B_1\, e^{ipz+i\bar{p}\bar{z}} = -i\partial\, e^{ipz+i\bar{p}\bar{z}} = p\, e^{ipz+i\bar{p}\bar{z}} \quad , \quad \bar{B}_1\, e^{ipz+i\bar{p}\bar{z}} = -i\bar{\partial}\, e^{ipz+i\bar{p}\bar{z}} = \bar{p}\, e^{ipz+i\bar{p}\bar{z}}$$

R-operator

factorization of L-operator $u_1 = s-1-iu \quad , \quad u_2 = -s-iu$

$$L(u_1,u_2) = i\begin{pmatrix} u_1+1+z\partial & -\partial \\ z^2\partial+(u_1-u_2+1)\,z & u_2-z\partial \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} u_1 & -\partial \\ 0 & u_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}$$

R-operator interchange $u_2 \leftrightarrow v_2$ in the product of $L-$operators

$$\begin{aligned} R_{12}\,L_1(u_1,u_2)L_2(v_1,v_2) &= L_1(u_1,v_2)L_2(v_1,u_2)\,R_{12} \\ R_{12}\,\bar{L}_1(\bar{u}_1,\bar{u}_2)\bar{L}_2(\bar{v}_1,\bar{v}_2) &= \bar{L}_1(\bar{u}_1,\bar{v}_2)\bar{L}_2(\bar{v}_1,\bar{u}_2)\,R_{12} \end{aligned}$$

integral operator

$$[R_{12}\,\Phi](z_1,z_2) = \int d^2w_1\,\frac{[z_1-z_2]^{u_2-u_1}}{[w_1-z_1]^{1+u_2-v_2}[w_1-z_2]^{v_2-u_1}}\,\Phi(w_1,z_2)$$

The main building block is the operator $R_{12}(x)$ which interchanges the second parameters $u_2 = -s-iu$ and $v_2 = ix-iu$ in the product of two L-operators

$$R_{12}(x)\,L_1(u_1,u_2)\,L_2(u_1,ix-iu) = L_1(u_1,ix-iu)\,L_2(u_1,u_2)\,R_{12}(x)$$

$$R_{12}(x)\,\bar{L}_1(\bar{u}_1,\bar{u}_2)\,\bar{L}_2(\bar{u}_1,i\bar{x}-i\bar{u}) = \bar{L}_1(\bar{u}_1,i\bar{x}-i\bar{u})\,\bar{L}_2(\bar{u}_1,\bar{u}_2)\,R_{12}(x)$$

Iterative construction of eigenfunctions

The operator

$$R_{12\dots N}(x) = R_{12}(x)R_{23}(x)\cdots R_{N-1\,N}(x)$$

moves $v_2 = ix - iu$ from the right hand side of the product of L-operators to the left hand side

$$R_{12\dots N}(x)L_1(u_1,u_2)\cdots L_{N-1}(u_1,u_2)\,L_N(u_1,ix-iu) =$$
$$= L_1(u_1,ix-iu)\,L_2(u_1,u_2)\cdots L_N(u_1,u_2)\,R_{12\dots N}(x)\,,$$
$$R_{12\dots N}(x)\bar{L}_1(\bar{u}_1,\bar{u}_2)\cdots \bar{L}_{N-1}(\bar{u}_1,\bar{u}_2)\,\bar{L}_N(\bar{u}_1,i\bar{x}-i\bar{u}) =$$
$$= \bar{L}_1(\bar{u}_1,i\bar{x}-i\bar{u})\,\bar{L}_2(\bar{u}_1,\bar{u}_2)\cdots \bar{L}_N(\bar{u}_1,\bar{u}_2)\,R_{12\dots N}(x)$$

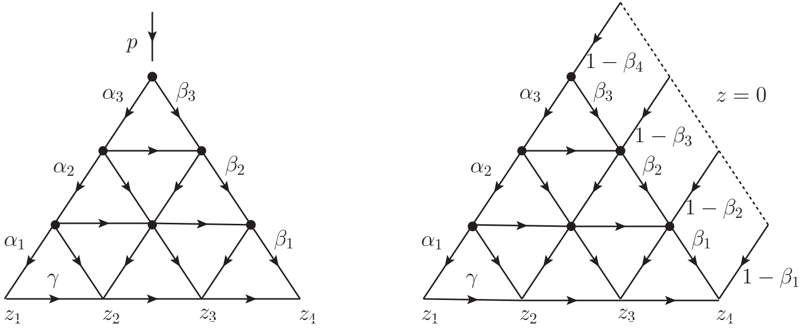
B-system

$$\Psi_B(\boldsymbol{x}|\boldsymbol{z}) = R_{12\dots N}(x_1)\cdots R_{12}(x_{N-1})\,\mathrm{e}^{ipz_1+i\bar{p}\bar{z}_1}$$

A-system

$$\Psi_A(\boldsymbol{x}|\boldsymbol{z}) = R_{12\dots N}(x_1)\cdots R_{12}(x_{N-1})\,[z_N]^{-s+ix_1}[z_{N-1}]^{-s+ix_2}\cdots[z_1]^{-s+ix_N}$$

Diagrammatic representation for eigenfunctions



$$\alpha_k = 1 - s - ix_k \quad , \quad \beta_k = 1 - s + ix_k \quad , \quad \gamma = 2s - 1$$

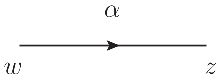
The propagator is given by the following expression ($\alpha - \bar{\alpha} = n_\alpha$ is integer)

$$\frac{1}{[z-w]^\alpha} \equiv \frac{1}{(z-w)^\alpha (\bar{z}-\bar{w})^{\bar{\alpha}}} = \frac{(\bar{z}-\bar{w})^{\alpha-\bar{\alpha}}}{|z-w|^{2\alpha}} = \frac{(-1)^{\alpha-\bar{\alpha}}}{[w-z]^\alpha},$$

and is shown by the arrow directed from w to z with the index α attached to it.

Diagram technique

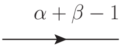
The diagrammatic representation of the propagator.

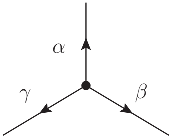


$$= [z - w]^{-\alpha}$$

The chain and star–triangle relations, $\alpha + \beta + \gamma = 2$.

$$= \pi(-1)^{\gamma-\bar{\gamma}} a(\alpha,\beta,\gamma)$$





$$= \pi\, a(\alpha,\beta,\gamma)$$

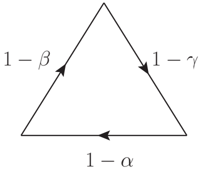
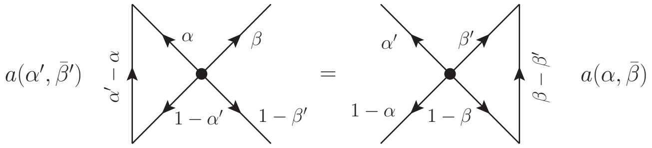


Diagram technique

The cross relation, $\alpha+\beta=\alpha'+\beta'$.



Here the notation $a(\alpha)$ is introduced for the function

$$a(\alpha)\equiv a(\alpha,\bar\alpha)=\frac{\Gamma(1-\bar\alpha)}{\Gamma(\alpha)}\quad,\quad a(\bar\alpha)=\frac{\Gamma(1-\alpha)}{\Gamma(\bar\alpha)}=(-1)^{\alpha-\bar\alpha}a(\bar\alpha)\,,$$

$$a(\alpha)a(1-\bar\alpha)=1\quad,\quad a(\alpha)a(1-\alpha)=(-1)^{\alpha-\bar\alpha}\quad,\quad a(1+\alpha)=-\frac{a(\alpha)}{\alpha\bar\alpha}\,,$$

$$a(\alpha,\beta,\gamma,\ldots)=a(\alpha)a(\beta)a(\gamma)\ldots$$

Orthogonality

orthogonality

$$\int d^{2N}\boldsymbol{z}\,\overline{\Psi_A(\boldsymbol{x}'|\boldsymbol{z})}\Psi_A(\boldsymbol{x}|\boldsymbol{z}) = \boldsymbol{\mu_A}^{-1}(\boldsymbol{x})\,\delta_N(\boldsymbol{x}-\boldsymbol{x}')$$

$$\int d^{2N}\boldsymbol{z}\,\overline{\Psi_B(\boldsymbol{p}',\boldsymbol{x}'|\boldsymbol{z})}\Psi_B(\boldsymbol{p},\boldsymbol{x}|\boldsymbol{z}) = \boldsymbol{\mu_B}^{-1}(\boldsymbol{p},\boldsymbol{x})\,\delta^2(\vec{p}-\vec{p}')\,\delta_{N-1}(\boldsymbol{x}-\boldsymbol{x}')$$

Here the delta function $\delta_N(\boldsymbol{x}-\boldsymbol{x}')$ is defined as follows:

$$\delta_N(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{N!}\sum_{S_N}\delta(\boldsymbol{x}_1-\boldsymbol{x}'_{k_1})\ldots\delta(\boldsymbol{x}_N-\boldsymbol{x}'_{k_N})\,,$$

where summation goes over all permutations of N elements and

$$\delta(\boldsymbol{x}_k-\boldsymbol{x}'_m)\equiv\delta_{n_k n'_m}\delta(\nu_k-\nu'_m)$$

Sklyanin measure

$$\boldsymbol{\mu}_A(\boldsymbol{x}) = (2\pi)^{-N}\pi^{-N^2}\prod_{k<j\le N}[x_k-x_j]$$

$$\boldsymbol{\mu}_B(\boldsymbol{x}) = 2(2\pi)^{-N}\pi^{-N^2}[p]^{N-1}\prod_{k<j\le N-1}[x_k-x_j]$$

$$[x_k-x_j] = (\nu_k-\nu_j)^2+\frac{1}{4}(n_k-n_j)^2$$

Completeness

completeness

$$\int \mathcal{D}_N \boldsymbol{x} \prod_{k<j}[x_k - x_j]\, \Psi_A(\boldsymbol{x}|\boldsymbol{z})\, \overline{\Psi_A(\boldsymbol{x}|\boldsymbol{z}')} = 2^N \pi^{N^2+N} \prod_{k=1}^{N} \delta^2(\vec{z}_k - \vec{z}'_k)$$

$$\int d^2p\, \mathcal{D}_{N-1}\boldsymbol{x}\, [p]^{N-1} \prod_{k<j}[x_k - x_j]\, \Psi_B(\boldsymbol{x}|\boldsymbol{z})\, \overline{\Psi_B(\boldsymbol{x}|\boldsymbol{z}')} = 2^{N-1} \pi^{N^2+N} \prod_{k=1}^{N} \delta^2(\vec{z}_k - \vec{z}'_k)$$

The symbol $\mathcal{D}_N \boldsymbol{x}$ stands for

$$\mathcal{D}_N \boldsymbol{x} = \prod_{k=1}^{N} \left(\sum_{n_k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu_k \right)$$

Mixed scalar product and matrix element of the shift operator

mixed scalar product

$$\left\langle \Psi_B(\boldsymbol{p},\boldsymbol{u}) \,\middle|\, \Psi_A(\boldsymbol{x}) \right\rangle = i^{[A_X]}\pi^N [p]^{-N}[p]^{A_X} \prod_{k=1}^{N} a(\bar{s}-i\bar{x}_k) \prod_{j=1}^{N-1} q(x_k,u_j)\,.$$

shift operator $T_{z_0} = \exp\{z_0 S_- + \bar{z}_0 \bar{S}_-\}$

$$T_{z_0}\Phi(\boldsymbol{z}) = \Phi(z_1-z_0,\bar{z}_1-\bar{z}_0,\ldots,z_N-z_0,\bar{z}_N-\bar{z}_0)$$

Action on $\Psi_B(\boldsymbol{p},\boldsymbol{x})$ is diagonal

$$T_{z_0}\Psi_B(\boldsymbol{p},\boldsymbol{x}) = e^{-ipz_0-i\bar{p}\bar{z}_0}\Psi_B(\boldsymbol{p},\boldsymbol{x})$$

and for $\Psi_A(\boldsymbol{x})$ the direct calculation results in the following expression

$$\left\langle \Psi_A(\boldsymbol{x}') \,\middle|\, T_{z_0}\Psi_A(\boldsymbol{x}) \right\rangle = (-1)^{[A_X]}[z_0]^{i(X-X')} \prod_{k,j=1}^{N} q(x_k,x_j')$$

The factor $q(x,x')$ is given by the following formula

$$q(x,x') = \pi a\big(1+i(x-x')\big)\,\frac{a(\bar{s}-i\bar{x})}{a(s-ix')}\,.$$

and we introduced the notations

$$X=\sum x_k,\quad \bar{X}=\sum \bar{x}_k,\quad A_X=\sum (s-ix_k),\quad \bar{A}_{\bar{X}}=\sum(\bar{s}-i\bar{x}_k).$$

The analog of the first Gustafson's integral

Using the completeness condition for the B–system one can represent the matrix element of the shift operator in the form

$$\left\langle \Psi_A(\boldsymbol{x}') \,\middle|\, T_{z_0}\Psi_A(\boldsymbol{x}) \right\rangle = \int d^2p\, e^{-ipz_0 - i\bar{p}\bar{z}_0} \int \mathcal{D}_{N-1}\boldsymbol{u}\, \boldsymbol{\mu}_N^{(B)}(\boldsymbol{p},\boldsymbol{u}) \left\langle \Psi_A(\boldsymbol{x}') \,\middle|\, \Psi_B(\boldsymbol{p},\boldsymbol{u}) \right\rangle \left\langle \Psi_B(\boldsymbol{p},\boldsymbol{u}) \,\middle|\, \Psi_A(\boldsymbol{x}) \right\rangle$$

Substituting the expressions for $\boldsymbol{\mu}_N^{(B)}(\boldsymbol{p},\boldsymbol{u})$ and all scalar products one gets after some algebra the analog of the first Gustafson's integral

$$\frac{1}{(N-1)!}\int \mathcal{D}_{N-1}\boldsymbol{u}\, \frac{\prod_{k=1}^{N}\prod_{j=1}^{N-1} a(1+i(x_k-u_j))\, a(1+i(u_j-x_k'))}{\prod_{m<j} a(1+i(u_j-u_m))a(1+i(u_m-u_j))} = \frac{\prod_{k,j=1}^{N} a(1+i(x_k-x_j'))}{a(1+i(X-X'))}$$

the integration variables: $u_k = -in_k/2+\nu_k\,, \bar{u}_k = in_k/2+\nu_k\,, n_k\in\mathbb{Z}\,, \nu_k\in\mathbb{R}$
the external variables:

$$x_k = -\frac{im_k}{2}+\mu_k, \quad \bar{x}_k = \frac{im_k}{2}+\mu_k, \quad x_k' = -\frac{im_k'}{2}+\mu_k', \quad \bar{x}_k = \frac{im_k'}{2}+\mu_k',$$

where m_k, m_k' are integers and μ_k and μ_k' are complex numbers such that $\operatorname{Im}\mu_k > 0$ and $\operatorname{Im}\mu_k' < 0$. For such a prescription the ν-poles of the functions $a(1+i(x_k-u_j))$ and $a(1+i(u_j-x_k'))$ are separated by the integration contour.

Open problems

- The clear proof of the completeness.
- Generalization to the another type of representations.
  - $SL(2,\mathbb{R})$ principal unitary series
  - $SL(2,\mathbb{R})$ complementary series
- New matrix elements $\longrightarrow$ new integrals ?
- Generalization up to the elliptic level.
  - Faddeev modular double
  - elliptic modular double