Elliptic hypergeometric sum/integral transformations

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1. Lens elliptic gamma function

Define the multiple Bernoulli polynomial $B_{3,3}(z,\omega_1,\omega_2,\omega_3)$ as [1]

$$B_{3,3} = \frac{z^3}{\omega_1\omega_2\omega_3} - \frac{3z^2\sum_{i=1}^3 \omega_i}{2\omega_1\omega_2\omega_3} + \frac{z(\sum_{i=1}^3 \omega_i^2 + 3\sum_{1\le i<j\le 3}\omega_i\omega_j)}{2\omega_1\omega_2\omega_3} - \frac{(\sum_{i=1}^3 \omega_i)(\sum_{1\le i<j\le 3}\omega_i\omega_j)}{4\omega_1\omega_2\omega_3}, \quad (1)$$

for complex variables $z \in \mathbb{C}$, and $\omega_1,\omega_2,\omega_3 \in \mathbb{C} - \{0\}$.

Introduce two parameters $\sigma,\ \tau \in \mathbb{C}$, where $\mathrm{Im}(\sigma), \mathrm{Im}(\tau) > 0$, and define

$$R(z;\sigma,\tau) = \frac{B_{3,3}(z;\sigma,\tau,-1) + B_{3,3}(z-1;\sigma,\tau,-1)}{12}, \quad (2)$$

$$R_2(z,m;\sigma,\tau) = R(z+m\sigma; r\sigma, \sigma+\tau) + R(z+(r-m)\tau; r\tau, \sigma+\tau), \quad (3)$$

where $z \in \mathbb{C}$, $m \in \mathbb{Z}$, and $r = 1, 2, \ldots$ is an integer parameter.

The **lens elliptic gamma function** [2-4] is defined here as

$$\Gamma(z,m;\sigma,\tau) = \mathrm{e}^{\phi_e(z,m;\sigma,\tau)}\gamma(z,m;\sigma,\tau), \quad (4)$$

where $z \in \mathbb{C}$, $m \in \{0,1,\ldots,r-1\}$,

$$\phi_e = 2\pi i\left(R_2(z,0;\sigma,\tau) + R_2(0,m,1/2,-1/2) - R_2(z,m;\sigma,\tau)\right), \quad (5)$$

$$\gamma(z,m;\sigma,\tau) = \prod_{j,k=0}^{\infty} \frac{1-\mathrm{e}^{-2\pi i z}\mathsf{p}^{-m}(\mathsf{pq})^{j+1}\mathsf{p}^{r(k+1)}}{1-\mathrm{e}^{2\pi i z}\mathsf{p}^{m}(\mathsf{pq})^{j}\mathsf{p}^{rk}} \, \frac{1-\mathrm{e}^{-2\pi i z}\mathsf{q}^{m}(\mathsf{pq})^{j+1}\mathsf{q}^{rk}}{1-\mathrm{e}^{2\pi i z}\mathsf{q}^{-m}(\mathsf{pq})^{j}\mathsf{q}^{r(k+1)}}. \quad (6)$$

The elliptic nomes are defined in terms of σ, τ, as $\mathsf{p} = \mathrm{e}^{2i\pi\sigma}$, $\mathsf{q} = \mathrm{e}^{2i\pi\tau}$.

- For $r = 1$, (4) is the regular elliptic gamma function [5].
- (6) may be expressed as a product of 2 regular elliptic gamma functions.

2. Elliptic beta sum/integral

Theorem 1 [3]

For σ, τ, $t_i \in \mathbb{C}$, and $u_i \in \mathbb{Z}$, $i = 1,2,\ldots,6$, satisfying

$$Im(\sigma), Im(\tau), Im(t_i) > 0, \quad \sum_{i=1}^{6} t_i = \sigma+\tau, \quad \sum_{i=1}^{6} u_i = 0, \quad (7)$$

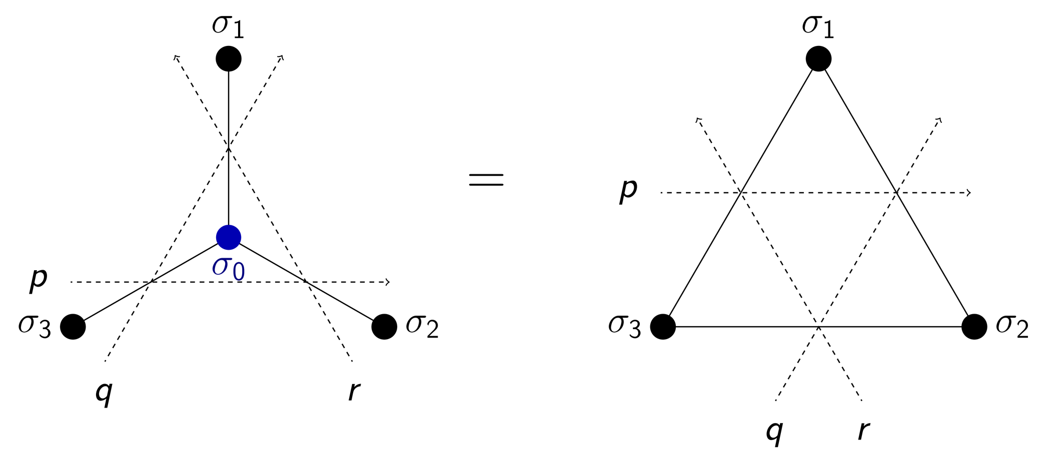
we have (with $\Gamma(z,m) := \Gamma(z,m;\sigma,\tau)$)

$$\frac{\lambda}{2}\sum_{y=0}^{\lfloor r/2\rfloor}\varepsilon(y)\int_0^1 dz\,\frac{\prod_{i=1}^{6}\Gamma(t_i\pm z, u_i\pm y)}{\Gamma(\pm 2z,\pm 2y)} = \prod_{1\le i<j\le 6}\Gamma(t_i+t_j, u_i+u_j), \quad (8)$$

where

$$\lambda = (\mathsf{p}^r;\mathsf{p}^r)_\infty(\mathsf{q}^r;\mathsf{q}^r)_\infty, \qquad \varepsilon(y) = \begin{cases} 1 & y = 0 \text{ or } r/2, \\ 2 & \text{otherwise}, \end{cases} \quad (9)$$

- For $r > 1$, (8) is a sum/integral generalisation of Spiridonov's elliptic beta integral [6], and (8) is equivalent to Spiridonov's case when $r = 1$.
- Eq. (8) is also equivalent to a **star-triangle relation** [3,4].



This is a fundamental identity (**Yang-Baxter eqn.**) for integrability of 2-d lattice models of statistical mechanics, *e.g.* Ising model, Chiral Potts model.

3. $A_n \leftrightarrow A_m$ sum/integral transformation

Introduce t_i, $s_i \in \mathbb{C}$, and a_i, $b_i \in \mathbb{Z}$, for $i = 0,1,\ldots,m+n+1$, satsifying

$$\mathrm{Im}(t_i), \mathrm{Im}(s_i) > 0, \quad \sum_{i=0}^{m+n+1}\frac{t_i+s_i}{m+1} = \sigma+\tau, \quad \sum_{i=0}^{m+n+1} a_i = \sum_{i=0}^{m+n+1} b_i = 0. \quad (10)$$

Define $I^m_{A_n}(\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b})$ as the following elliptic hypergeometric sum/integral

$$I^m_{A_n}(\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b}) = \frac{\lambda^n}{(n+1)!}\sum_{\substack{y_i=0\\ \sum_{i=0}^n y_i=0}}^{r-1}\int_{\substack{0\\ \sum_{i=0}^n z_i=0}}^1 \Delta^m_{A_n}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b})\prod_{i=0}^{n-1}dz_i, \quad (11)$$

where

$$\Delta^m_{A_n} = \frac{\prod_{i=0}^{n}\prod_{j=0}^{m+n+1}\Gamma(t_j+z_i, a_j+y_i)\Gamma(s_j-z_i, b_j-y_i)}{\prod_{0\le i<j\le n}\Gamma(z_i-z_j, y_i-y_j)\Gamma(z_j-z_i, y_j-y_i)}, \quad (12)$$

and $m,n = 0,1,\ldots$.

$\mathbf{A_n \leftrightarrow A_m}$ elliptic hypergeometric sum/integral transformation:

Theorem 2 [7]

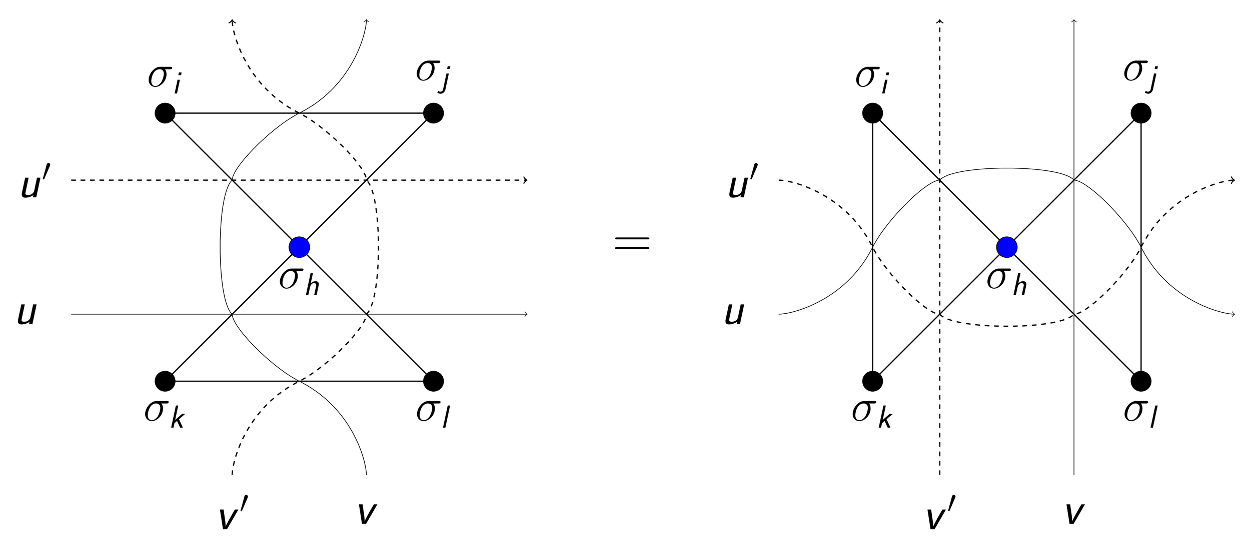
The sum/integral (11) *satisfies*

$$I^m_{A_n}(\boldsymbol{t},\boldsymbol{a};\boldsymbol{s},\boldsymbol{b}) = I^n_{A_m}(\tilde{\boldsymbol{t}},\tilde{\boldsymbol{a}};\tilde{\boldsymbol{s}},\tilde{\boldsymbol{b}})\prod_{i,j=0}^{m+n+1}\Gamma(t_i+s_j, a_i+b_j), \quad (13)$$

where

$$\tilde{\boldsymbol{t}} = \sum_{j=0}^{m+n+1}\frac{t_j}{m+1} - \boldsymbol{t}, \quad \tilde{\boldsymbol{s}} = \sum_{j=0}^{m+n+1}\frac{s_j}{m+1} - \boldsymbol{s}, \quad \tilde{\boldsymbol{a}} = -\boldsymbol{a}, \quad \tilde{\boldsymbol{b}} = -\boldsymbol{b}. \quad (14)$$

- The case $r = 1$ of Theorem 3 is equivalent to the $A_n \leftrightarrow A_m$ elliptic hypergeometric integral transformations proven by Rains [8].
- The $m = 0, n = 1$ case of (13) is the elliptic beta/sum integral (8).
- Eq. (13) (for $m = n$) is also equivalent to a **star-star relation** [9].



This is another condition of integrability for 2-d lattice models [10].

4. $BC_n \leftrightarrow BC_m$ sum/integral transformation

Introduce σ, τ, $t_i \in \mathbb{C}$, and $a_i \in \mathbb{Z}$, for $i = 0,1,\ldots,2m+2n+3$, satisfying

$$\mathrm{Im}(t_i) > 0, \quad \sum_{i=0}^{2m+2n+3} t_i = (m+1)(\sigma+\tau), \quad \sum_{i=0}^{2m+2n+3} a_i = 0. \quad (15)$$

Define $I^m_{BC_n}(\boldsymbol{t},\boldsymbol{a})$ as the following elliptic hypergeometric sum/integral

$$I^m_{BC_n}(\boldsymbol{t},\boldsymbol{a}) = \frac{\lambda^n}{2^n\, n!}\sum_{y_i=0}^{r-1}\int_0^1 \Delta^m_{BC_n}(\boldsymbol{z},\boldsymbol{y};\boldsymbol{t},\boldsymbol{a})\prod_{i=1}^{n}dz_i, \quad (16)$$

where

$$\Delta^m_{BC_n} = \frac{\prod_{i=1}^{n}\prod_{j=0}^{2m+2n+3}\Gamma(t_j+z_i, a_j+y_i)\Gamma(t_j-z_i, a_j-y_i)}{\prod_{i=1}^{n}\Gamma(\pm 2z_i,\pm 2y_i)\prod_{1\le i<j\le n}\Gamma(\pm z_i\pm z_j, \pm y_i\pm y_j)}. \quad (17)$$

$\mathbf{BC_n \leftrightarrow BC_m}$ elliptic hypergeometric sum/integral transformation:

Theorem 3 [7]

The sum/integral (16) *satisfies*

$$I^m_{BC_n}(\boldsymbol{t},\boldsymbol{a}) = I^n_{BC_m}(\tilde{\boldsymbol{t}},\tilde{\boldsymbol{a}})\prod_{0\le i<j\le 2m+2n+3}\Gamma(t_i+t_j, a_i+a_j), \quad (18)$$

where

$$\tilde{\boldsymbol{t}} = \frac{\sigma+\tau}{2} - \boldsymbol{t}, \quad \tilde{\boldsymbol{a}} = -\boldsymbol{a}. \quad (19)$$

- The case $r = 1$ of Theorem 3 is equivalent to the $BC_n \leftrightarrow BC_m$ elliptic hypergeometric integral transformations proven by Rains [8].
- The $m = 0$ case of Theorem 3, is equivalent to a elliptic hypergeometric sum/integral (rarefied) identity proven by Spiridonov [11].
- The $m = 0, n = 1$ case of (18) is the elliptic beta/sum integral (8).

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