Duality domain walls in 5d Supersymmetric gauge theories

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arXiv:1506.03871 with Davide Gaiotto (Perimeter Institute)

Introduction

A large class of BPS domain walls has been studied in 4d maximal supersymmetric (SUSY) gauge theories.

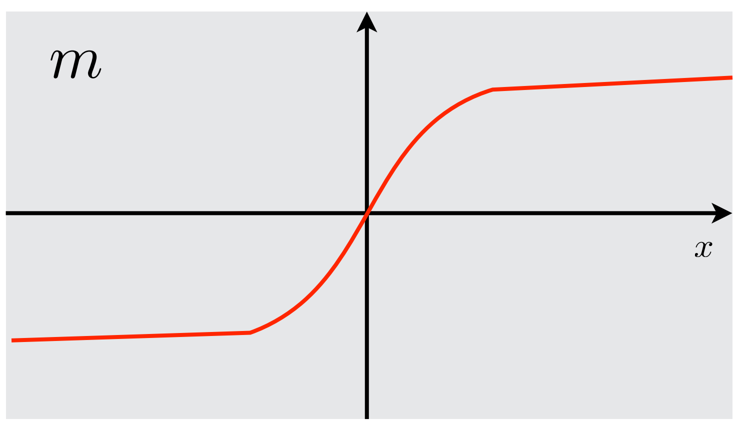
- AdS/CFT, Boundary conditions, S-duality, Branes, …

[Bak, Gutperle, Hirano 03], [Clark, Freedman, Karch, Schnabl 04], [Clark, Karch 05], [D'Hoker, Estes, Gutperle 07], [Bak, Gutperle, Hirano 07], [Gaiotto, Witten 08], …

We are interested in the BPS domain walls in 5d N=1 gauge theories.

We focus on Janus-like domain walls (or interfaces)

- Coupling or mass parameter varies as a function of coordinate.



m : mass parameter

Introduction

I will propose duality domain walls, which involve

- Boundary conditions.
- New 4d degrees of freedom and 4d superpotentials.
- Exact partition functions.

I will show that

- Duality wall is a physical realization of elliptic Fourier transform.
- Instanton partition function is an eigenfunction of elliptic integral equation.
- New dualities motivated by elliptic integral equation.

Outline

1. Introduction.

2. Basics of 5d SUSY gauge theories.

3. Duality walls between SU(N) gauge theories.

4. Partition functions and elliptic integral equations.

5. Duality walls between Sp(N) and SU(N+1) gauge theories.

6. Conclusion

Basics of 5d supersymmetric gauge theories

$\mathcal{N}=1$ gauge theories with gauge group G in five-dimensions

- Vector multiplet $(A_\mu, \phi; \lambda)$
- Hypermultiplet $(q^A; \psi)$
- Preserve 8 SUSY

There is a topological $U(1)_I$ associated to instanton number symmetry

$$J_I = *\mathrm{Tr} F \wedge F \qquad (F = dA)$$

Symmetry of 5d gauge theory is

- $SO(5)$ Lorentz symmetry + $SU(2)_R$ R-symmetry.
- Gauge symmetry $G = SU(N), Sp(N)$.
- Global symmetry $G_F \times U(1)_I$.
  - G_F : Flavor symmetry acting on hypermultiplets $(q^A; \psi)$.

Basics of 5d supersymmetric gauge theories

Large class of 5d SUSY theories flow to interacting conformal field theories (CFT) in UV fixed point (or at high energy).

- QFT analysis
- Branes and string duality
- M-theory on CY3

[Seiberg 96], [Morrison, Seiberg 96], [Douglas, Katz, Vafa 96], [Intriligator, Morrison, Seiberg 97], [Aharony, Hanany 97], [Aharony, Hanany, Kol 97], [DeWolfe, Hanany, Iqbal, Katz 99], …

The gauge theories can be considered as massive deformation of those CFTs at the fixed points.

$$S = \frac{1}{4g^2}\int F^2 + \cdots$$

g : gauge coupling

Many 5d QFTs enjoy global symmetry enhancement at UV CFT.

(Ex : $SU(2)$, $N_f = 5, 6, 7$ have enhanced E_6, E_7, E_8 global symmetries at UV CFT)

Symmetry of UV CFT leads to dualities in IR (or low energy) gauge theories.

$SU(N)$ gauge theory at Chern-Simons level $\kappa = N$

Classical Lagrangian

$$L = \frac{1}{g^2} F \wedge *F + \frac{\kappa}{24\pi^2} A \wedge F \wedge F + \cdots$$

- IR gauge theory has a topological $U(1)_I$ instanton symmetry.
- In UV CFT, global symmetry $U(1)_I$ is enhanced to $SU(2)_I$.

Gauge coupling $1/g^2$ is the mass deformation parameter m of $SU(2)_I$

- Thus, Yang-Mills term breaks $SU(2)_I$ to $U(1)_I$.

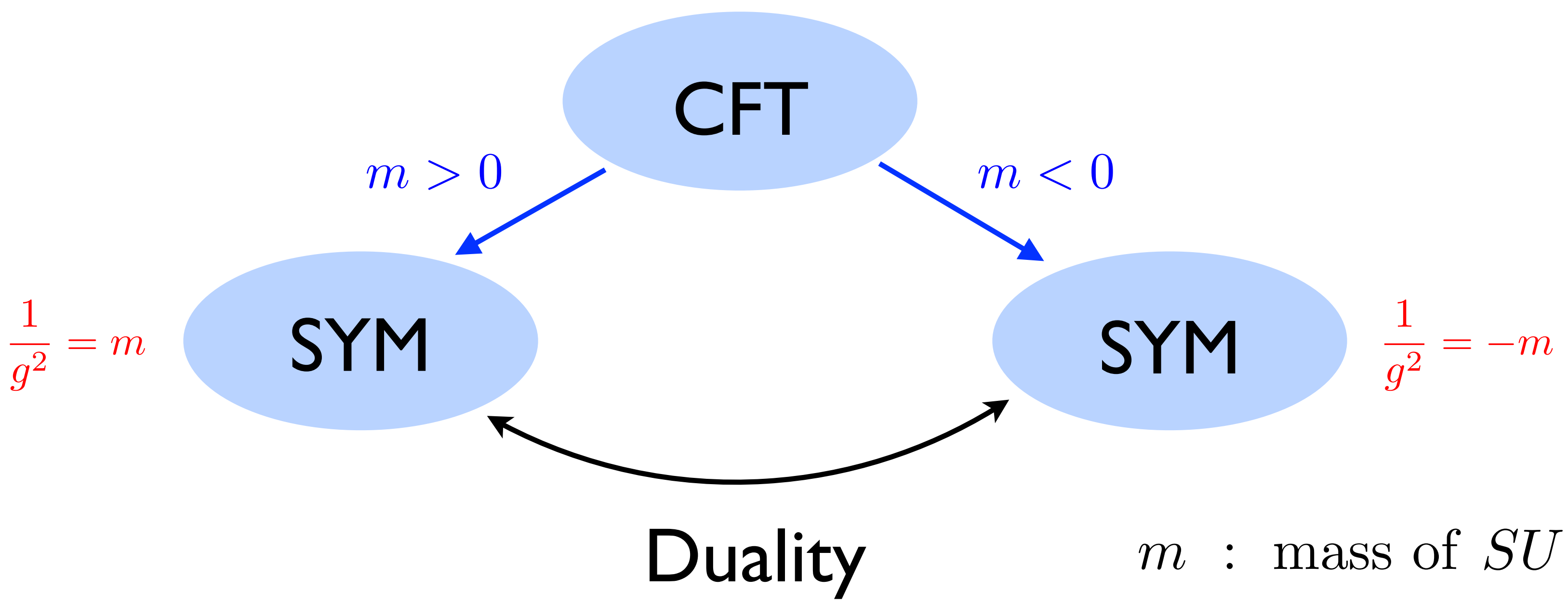
$\mathbb{Z}_2$ Weyl symmetry of $SU(2)_I$ in UV CFT acts as $m \leftrightarrow -m$

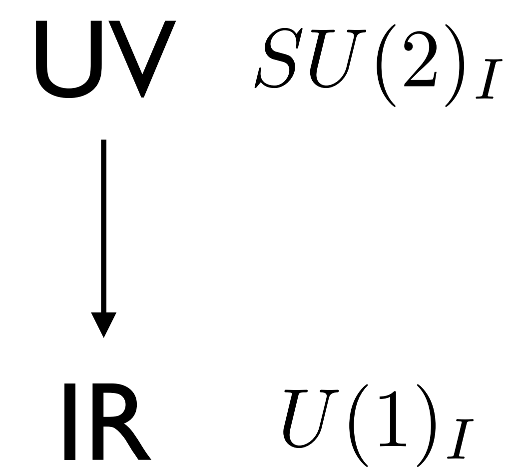
Therefore, the symmetry in UV CFT remains as duality in IR.

$SU(N)_N$ theory with $1/g^2 = m$ $\xleftrightarrow{\text{Duality}}$ $SU(N)_N$ theory with $1/\tilde{g}^2 = -m$

Duality domain wall

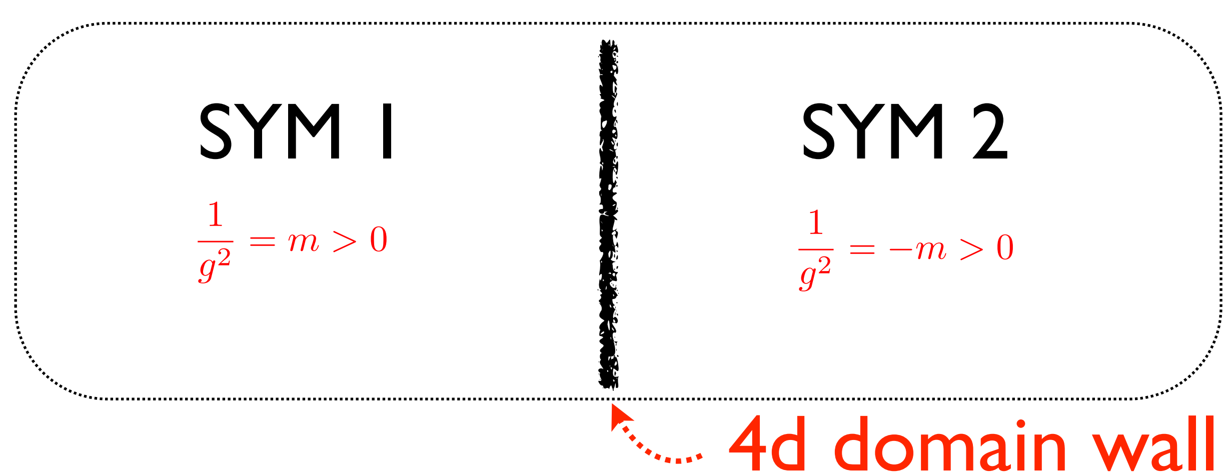
Massive (or gauge coupling) deformation of CFT leads to dualities between low energy supersymmetric Yang-Mills (SYM) gauge theories.





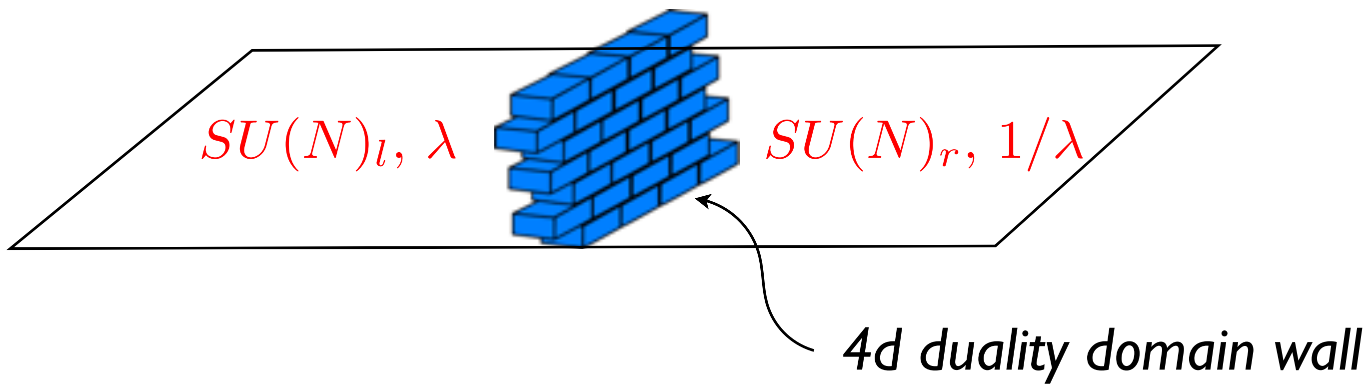
m : mass of $SU(2)_I$

Duality domain wall :



Duality domain wall

Construction of duality domain wall for $SU(N)$ theory with coupling $\lambda \equiv e^{\frac{1}{g^2}}$



1. Boundary condition at the wall :
   - Neumann boundary condition $F_{5i}|_{\partial} = 0$
   - $SU(N)_l \times SU(N)_r$ gauge symmetry survives at the boundary

2. New 4d degrees of freedom
   - 4d $\mathcal{N} = 1$ matter content :

	$SU(N)_l$	$SU(N)_r$	$U(1)_R$	$U(1)_B$
q	N	$\bar{N}$	0	$1/N$
b	1	1	2	-1

   - Superpotential : $W = b\ \det q$
   - +1 (-1) charge of $U(1)_{I_l}$ ($U(1)_{I_r}$) is identified with +1 charge of $U(1)_B$.

5d Nekrasov partition function

Partition function of 5d gauge theory on $S^1\times\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ consists of 1-loop contribution Z_{pert} and non-perturbative instanton contribution Z_{inst}.

$z_i=e^{\alpha_i}$: gauge fugacity

$p=e^{\epsilon_1}, q=e^{\epsilon_2}$: $\Omega-$parameters

$$II(z_i,\lambda;p,q)=Z_{\text{pert}}(z_i;p,q)\cdot Z_{\text{inst}}(z_i,\lambda;p,q)$$

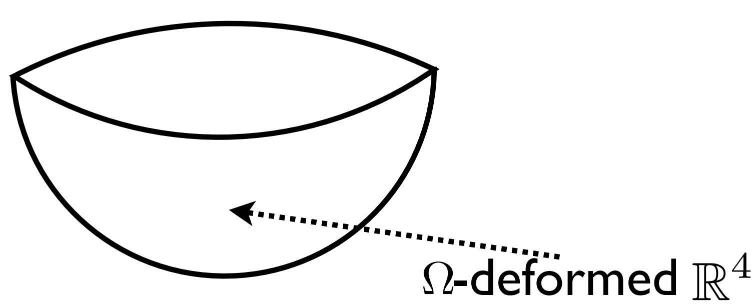
$$Z_{\text{pert}}=(pq;p,q)_\infty^{N-1}\prod_{i\neq j}^{N}(pqz_i/z_j;p,q)_\infty$$

$$Z_{\text{inst}}=\sum_{k=0}^{\infty}\lambda^k\frac{1}{k!}\oint\prod_{I=1}^{k}\frac{d\phi_I}{2\pi i}e^{N\sum_{I=1}^{k}\phi_I}Z_{\text{vec}}(\phi_I,z_i;p,q)$$

$$Z_{\text{vec}}=\frac{\prod_{I\neq J}^{k}2\sinh\frac{\phi_I-\phi_J}{2}\prod_{I,J}^{k}2\sinh\frac{\phi_I-\phi_J+2\epsilon_+}{2}}{\prod_{I,J}^{k}2\sinh\frac{\phi_I-\phi_J+\epsilon_1}{2}2\sinh\frac{\phi_I-\phi_J+\epsilon_2}{2}\prod_{i=1}^{N}\prod_{I=1}^{k}2\sinh\frac{\pm(\phi_I-\alpha_i)+\epsilon_+}{2}}$$

- This function counts number of BPS states on Ω-deformed $\mathbb{R}^4$.

$$II(z_i,\lambda)=\text{spectrum in}$$



Duality wall partition function

Contribution from 4d boundary degrees of freedom

	$SU(N)_l$	$SU(N)_r$	$U(1)_R$	$U(1)_\lambda$
q	N	$\bar{N}$	0	$1/N$
b	1	1	2	-1

$$\Rightarrow \quad I^{4d} = \frac{\prod_{i,j=1}^{N} \Gamma(\lambda^{1/N} z_i / z'_j)}{\Gamma(\lambda)}$$

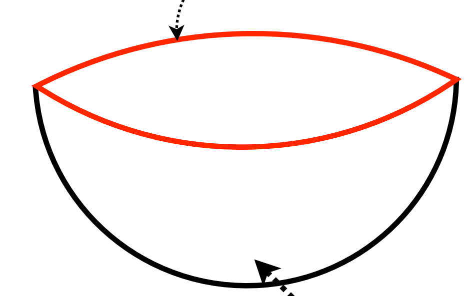
$z_i : SU(N)_l$
$z'_i : SU(N)_r$
$\lambda : U(1)_\lambda$
$\prod_i z_i = \prod_i z'_i = 1$

($\Gamma(x) \equiv \Gamma(x; p, q)$: Elliptic gamma function)

Duality domain wall partition function :

$$\hat{D}II^N(z,\lambda) \equiv \oint \prod_{i=1}^{N-1} \frac{dz'_i}{2\pi i z'_i} \frac{I^{4d}(z,z',\lambda)}{\prod_{i,j}^{N} \Gamma(z'_i/z'_j)} II^N(z'_i,\lambda)$$

4d SU(N) vectormultiplet



Since duality wall is conjectured to flip $U(1)_I$ charge, we claim that

$$\hat{D}II^N(z_i,\lambda) = II^N(z_i,\lambda^{-1})$$

Elliptic integral equation

Therefore we claim that duality domain wall partition function gives the following elliptic integral equation :

$$II(z_i,\lambda^{-1};p,q)=\oint\prod_{i=1}^{N-1}\frac{dz'_i}{2\pi i z'_i}\frac{\prod_{i,j=1}^{N}\Gamma(\lambda^{1/N}z_i/z'_k)}{\Gamma(\lambda)\prod_{i,j=1}^{N}\Gamma(z'_i/z'_j)}II(z'_i,\lambda;p,q)$$

[Gaiotto, H.-C. Kim 15]

- 5d Nekrasov partition function

$$II(z_i,\lambda;p,q)=Z_{\rm pert}(z_i;p,q)\cdot Z_{\rm inst}(z_i,\lambda;p,q)$$

$$Z_{\rm pert}=(pq;p,q)_\infty^{N-1}\prod_{i\neq j}^{N}(pqz_i/z_j;p,q)_\infty$$

$$Z_{\rm inst}=\sum_{k=0}^{\infty}\lambda^k\frac{1}{k!}\oint\prod_{I=1}^{k}\frac{d\phi_I}{2\pi i}e^{N\sum_{I=1}^{k}\phi_I}Z_{\rm vec}(\phi_I,z_i;p,q)$$

$$Z_{\rm vec}=\frac{\prod_{I\neq J}^{k}2\sinh\frac{\phi_I-\phi_J}{2}\prod_{I,J}^{k}2\sinh\frac{\phi_I-\phi_J+2\epsilon_+}{2}}{\prod_{I,J}^{k}2\sinh\frac{\phi_I-\phi_J+\epsilon_1}{2}2\sinh\frac{\phi_I-\phi_J+\epsilon_2}{2}\prod_{i=1}^{N}\prod_{I=1}^{k}2\sinh\frac{\pm(\phi_I-\alpha_i)+\epsilon_+}{2}}$$

is an eigenfunction of the elliptic integral equation.

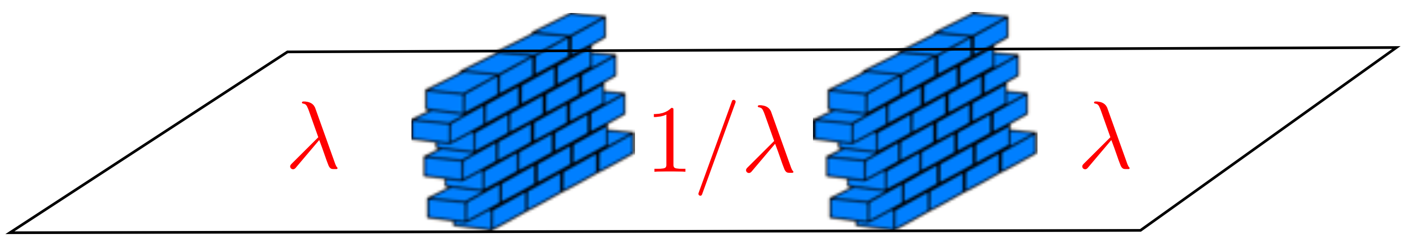
- Duality wall acts as an elliptic integral equation

$$\hat{D} II(z,\lambda;p,q) = II(z,\lambda^{-1};p,q)$$

- Perturbative check of this integral equation can be done by expanding both sides in terms of energy fugacity $x \equiv (pq)^{1/2}$.

- Analytic proof of $\hat{D} II(\lambda) = II(\lambda^{-1})$??

- Physics of duality wall implies $\hat{D}^2 = 1$



- Analytic proof of $\hat{D}^2 = 1$ is given by an elliptic Fourier transform

$$\oint d\mu_{z'} \frac{\prod_{i,j}^N \Gamma(\lambda^{1/N} z_i/z'_j)}{\Gamma(\lambda)\prod_{i,j}^N \Gamma(z'_i/z'_j)} \oint d\mu_{z''} \frac{\prod_{i,j}^N \Gamma(\lambda^{-1/N} z'_i/z''_j)}{\Gamma(\lambda^{-1})\prod_{i,j}^N \Gamma(z''_i/z''_j)} f(z'') \sim f(z)$$

[Spiridonov, Warnaar 04]

Duality domain wall with matter hypermultiplets

Consider $SU(N)$ gauge theory at level $\kappa = N - N_f/2$ with N_f fund. matters.

- Symmetry enhancement at UV CFT fixed point.

$$SU(N_f)\times U(1)_f \times U(1)_I \to SU(N_f)\times SU(2)_+ \times U(1)_-$$

- There is an IR Duality associated to $\mathbb{Z}_2$ Weyl symmetry in $SU(2)_+$.

We claim that duality domain wall for this theory with matters is given by

1. Boundary conditions
   - Vector multiplet : $F_{5i}|_\partial = 0$
   - Matter hypermultiplet $\Phi = (X,Y)$: $X|_\partial = 0\ ,\ \partial_5 Y|_\partial = 0$

2. Couple to a 4d boundary degrees of freedom
   - 4d matter content :

	$SU(N)_l$	$SU(N)_r$	$U(1)_R$	$U(1)_B$
q	N	$\bar{N}$	0	$1/N$
b	1	1	2	-1

   - Superpotential : $W = b\,\mathrm{det}q + YqX'$

X, Y : matters in LHS

X', Y' : matters in RHS

Elliptic integral equation

- Partition function with matter hypermultiplets

$$II(z_i, w_a, \lambda; p, q) = Z_{\text{pert}}(z_i, w_a; p, q) \cdot Z_{\text{inst}}(z_i, w_a, \lambda; p, q)$$

$$Z_{\text{pert}} = \frac{(pq;p,q)_\infty^{N-1} \prod_{i\neq j}^{N} (pqz_i/z_j;p,q)_\infty}{\prod_{i=1}^{N} \prod_{a=1}^{N_f} (\sqrt{pq} z_i/w_a;p,q)_\infty}$$

$$Z_{\text{inst}} = \sum_{k=0}^{\infty} \lambda^k \frac{1}{k!} \oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} e^{(N-N_f/2)\sum_{I=1}^{k} \phi_I} Z_{\text{vec}}(\phi_I, z_i; p, q) \cdot Z_{\text{matter}}(\phi_I, w_a)$$

$$Z_{\text{vec}} = \frac{\prod_{I\neq J}^{k} 2\sinh\frac{\phi_I-\phi_J}{2} \prod_{I,J}^{k} 2\sinh\frac{\phi_I-\phi_J+2\epsilon_+}{2}}{\prod_{I,J}^{k} 2\sinh\frac{\phi_I-\phi_J+\epsilon_1}{2} 2\sinh\frac{\phi_I-\phi_J+\epsilon_2}{2} \prod_{i=1}^{N}\prod_{I=1}^{k} 2\sinh\frac{\pm(\phi_I-\alpha_i)+\epsilon_+}{2}}$$

$$Z_{\text{matter}} = \prod_{I=1}^{k} \prod_{a=1}^{N_f} 2\sinh\frac{\phi_I - m_a}{2}$$

$w_a = e^{m_a}$: fugacity for flavor symmetry

- Duality domain wall action on this partition function yields an elliptic integral equation

$$II(z_i, w_a, \lambda^{-1}; p, q) = \oint \prod_{i=1}^{N-1} \frac{dz_i'}{2\pi i z_i'} \frac{\prod_{i,j=1}^{N} \Gamma(\lambda^{1/N} z_i/z_k')}{\Gamma(\lambda) \prod_{i,j=1}^{N} \Gamma(z_i'/z_j')} II(z_i', w_a', \lambda; p, q)$$

$$w_a = \lambda^{-1/N} w_a'$$

Elliptic integral equation and instanton partition function

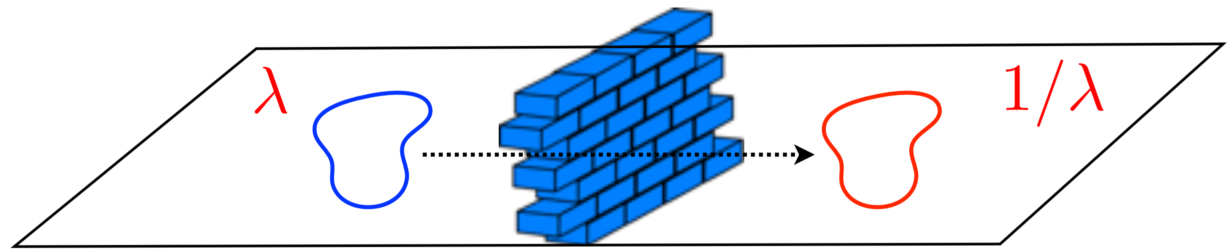
Note that for a given 1-loop partition function $Z_{\rm pert}$ we can uniquely determine the instanton partition function $Z_{\rm inst}$ by solving the elliptic integral equation with an assumption $Z_{\rm inst} = 1 + \cdots$.

$$Z_{\rm pert}(z_i, w_a; p, q) \cdot (1 + \mathcal{O}(\lambda^{<0})) = \oint \prod_{i=1}^{N-1} \frac{dz'_i}{2\pi i z'_i} \frac{\prod_{i,j=1}^{N} \Gamma(\lambda^{1/N} z_i / z'_k)}{\Gamma(\lambda) \prod_{i,j=1}^{N} \Gamma(z'_i / z'_j)} Z_{\rm pert}(z'_i, w_a; p, q) \cdot (1 + \mathcal{O}(\lambda^{>0}))$$

→ $Z_{\rm inst}(z_i, w_a, \lambda; p, q) = 1 + \mathcal{O}(\lambda^{>0})$ is uniquely determined !!

Wilson loop operators and duality domain wall

We can consider duality domain wall system in the presence of Wilson loop operators.



Partition function with a Wilson loop operator in representation R is

$$\mathcal{W}_R(z_i,\lambda;p,q) = Z_{\text{pert}}(z_i;p,q)\cdot \mathcal{W}_{R,\text{inst}}(z_i,\lambda;p,q)$$

$$Z_{\text{pert}} = (pq;p,q)_\infty^{N-1}\prod_{i\neq j}^{N}(pqz_i/z_j;p,q)_\infty$$

$$\mathcal{W}_{R,\text{inst}} = \sum_{k=0}^{\infty}\lambda^k\frac{1}{k!}\oint\prod_{I=1}^{k}\frac{d\phi_I}{2\pi i}\,\underline{Ch_R(z,\phi_I;p,q)}\cdot e^{N\sum_{I=1}^{k}\phi_I}Z_{\text{vec}}(\phi_I,z_i;p,q)$$

$$Z_{\text{vec}} = \frac{\prod_{I\neq J}^{k}2\sinh\frac{\phi_I-\phi_J}{2}\prod_{I,J}^{k}2\sinh\frac{\phi_I-\phi_J+2\epsilon_+}{2}}{\prod_{I,J}^{k}2\sinh\frac{\phi_I-\phi_J+\epsilon_1}{2}2\sinh\frac{\phi_I-\phi_J+\epsilon_2}{2}\prod_{i=1}^{N}\prod_{I=1}^{k}2\sinh\frac{\pm(\phi_I-\alpha_i)+\epsilon_+}{2}}$$

$Ch_R(z,\phi;p,q)$: equivariant Chern character of universal bundle in representation R

Elliptic integral equation for Wilson loop operator

Domain wall partition function with a Wilson loop operator gives the following elliptic integral equation.

$$\underline{\lambda^{k(R)/N}} \cdot \mathcal{W}_R(z_i, \lambda^{-1}; p, q) = \oint \prod_{i=1}^{N-1} \frac{dz'_i}{2\pi i z'_i} \frac{\prod_{i,j=1}^{N} \Gamma(\lambda^{1/N} z_i / z'_k)}{\Gamma(\lambda) \prod_{i,j=1}^{N} \Gamma(z'_i / z'_j)} \mathcal{W}_R(z'_i, \lambda; p, q)$$

- Note that there is an extra weight $\lambda^{k(R)/N}$.
- $k(R) = n$ for rank n symmetric or anti-symmetry representation.

- Wilson loop partition function

$$\mathcal{W}_R(z_i, \lambda; p, q) = Z_{\text{pert}}(z_i; p, q) \cdot \mathcal{W}_{R,\text{inst}}(z_i, \lambda; p, q)$$

$$Z_{\text{pert}} = (pq; p, q)_\infty^{N-1} \prod_{i \neq j}^{N} (pq z_i / z_j; p, q)_\infty$$

$$\mathcal{W}_{R,\text{inst}} = \sum_{k=0}^{\infty} \lambda^k \frac{1}{k!} \oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \, Ch_R(z, \phi_I; p, q) \cdot e^{N \sum_{I=1}^{k} \phi_I} Z_{\text{vec}}(\phi_I, z_i; p, q)$$

$$Z_{\text{vec}} = \frac{\prod_{I \neq J}^{k} 2 \sinh \frac{\phi_I - \phi_J}{2} \prod_{I,J}^{k} 2 \sinh \frac{\phi_I - \phi_J + 2\epsilon_+}{2}}{\prod_{I,J}^{k} 2 \sinh \frac{\phi_I - \phi_J + \epsilon_1}{2} 2 \sinh \frac{\phi_I - \phi_J + \epsilon_2}{2} \prod_{i=1}^{N} \prod_{I=1}^{k} 2 \sinh \frac{\pm(\phi_I - \alpha_i) + \epsilon_+}{2}}$$

is an eigenfunction of the elliptic integral equation.

Duality between $Sp(N)$ and $SU(N+1)$ theories

Duality between 1 . $Sp(N)$ gauge theory with N_f fundamental hypers.

2 . $SU(N+1)$ gauge theory with N_f fundamental hypers at CS-level $\kappa = N + 3 - N_f/2$. $(N_f < 2N + 6)$

[Gaiotto, H.-C Kim 15]

- Same dimension of Coulomb branch : $\dim \mathcal{M}_{\text{Coulomb}} = N$
- Classical global symmetries are different. However two theories flow the same UV CFT fixed point with same global symmetry $SO(2N_f) \times U(1)_I$.

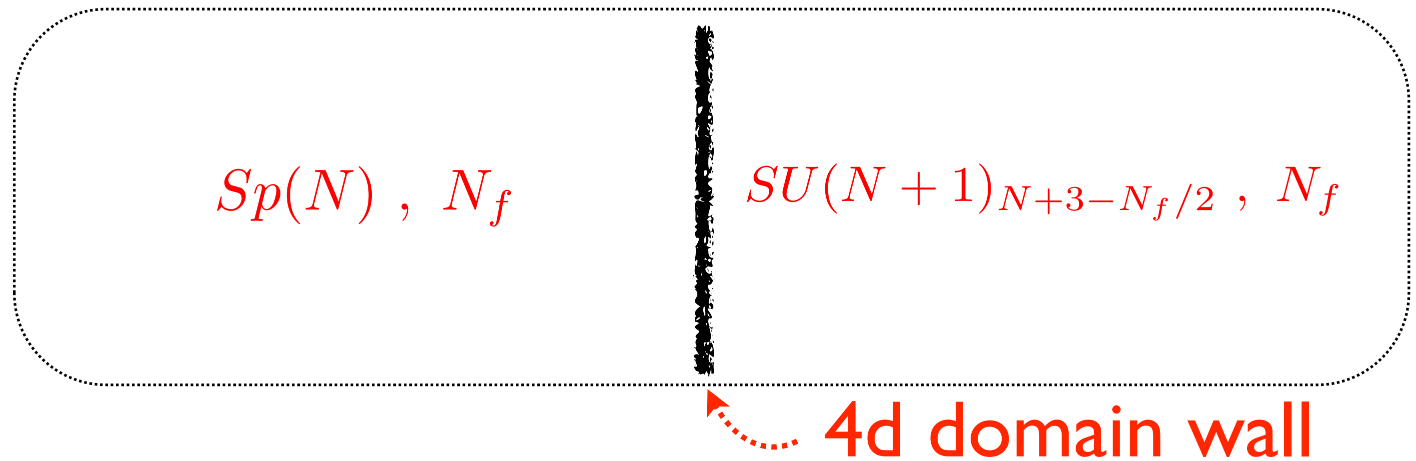
This duality was tested by 1-instanton analysis and also comparing superconformal indices. [Gaiotto, H.-C Kim 15]

Duality can also been seen from (p,q) 5-Branes web construction.

[Bergman, Zafrir 14,15], [Hayashi, S.-S Kim, K. Lee, Taki, Yagi 15]

Duality domain wall between $Sp(N)$ and $SU(N+1)$ theories

We propose a duality wall :



We use a similar boundary conditions $F_{5i}|_\partial = 0\ ,\ \ X|_\partial = 0\ ,\ \ Y|_\partial \neq 0$

And couple it to 4d degrees of freedom at the interface

- 4d $\mathcal{N}=1$ matter content

	$Sp(N)$	$SU(N+1)$	$U(1)_R$	$U(1)_\lambda$
q	N	$N+1$	0	$1/2$
M	1	$N(N+1)/2$	2	-1

- Superpotential $\quad W = \mathrm{Tr}\ qMq^T w + XqX'$

w : symplectic form of $Sp(N)$
X : chiral half of hypermultiplet in $SU(N+1)$
X' : chiral half of hypermultiplet in $Sp(N)$

- When N=1, it reduces to duality wall in previous SU(2) theory

We propose that this is the duality wall that interpolates Sp(N) and SU(N+1) gauge theories.

Partition function and elliptic integral equations

Partition function of $Sp(N)$ gauge theory with N_f fundamental matters :

$$II^{N,N_f}_{Sp(N)}(z_i, w_a, \lambda_{Sp}; p, q) = Z^{N,N_f}_{Sp,\text{pert}}(z_i, w_a; p, q) \cdot Z^{N,N_f}_{Sp,\text{inst}}(z_i, w_a, \lambda_{Sp}; p, q)$$

$$Z^{N,N_f}_{Sp,\text{pert}} = \frac{\prod_{i>j}^{N}(pqz_i^{\pm} z_j^{\pm})_\infty \prod_{i=1}^{N}(pqz_i^{\pm 2}; p, q)_\infty}{\prod_{i=1}^{N}\prod_{a=1}^{N_f}(\sqrt{pq} z_i^{\pm 1}/w_a; p, q)_\infty}$$

Duality domain wall connecting $Sp(N)$ gauge theory and $SU(N+1)$ gauge theory gives rise to two new elliptic integral equations.

$$II^{N+1,N_f}_{SU}(z'_i, w'_a, \lambda_{SU}; p, q) = \oint \frac{dz_i}{2\pi i z_i} \Delta^{(C)}(z, z', \lambda)\ II^{N,N_f}_{Sp}(z_i, w_a, \lambda_{Sp}; p, q)$$

$$II^{N,N_f}_{Sp}(z_i, w_a, \lambda_{Sp}; p, q) = \oint \frac{dz'_i}{2\pi i z'_i} \Delta^{(A)}(z', z, \lambda)\ II^{N+1,N_f}_{SU}(z'_i, w'_a, \lambda_{SU}; p, q)$$

$$\Delta^{(C)} = \frac{\prod_{i=1}^{N+1}\prod_{j=1}^{N}\Gamma(\sqrt{\lambda} z'_i z_j^{\pm 1})}{\prod_{i>j}^{N+1}(\lambda z'_i z'_j)\prod_{i>j}^{N}\Gamma(z_i^{\pm} z_j^{\pm})\prod_{i=1}^{N}\Gamma(z_i^{\pm 2})}$$

$$\Delta^{(A)} = \frac{\prod_{i=1}^{N}\prod_{j=1}^{N+1}\Gamma(\sqrt{\lambda}^{-1} z'_j z_i^{\pm 1})}{\prod_{i\neq j}^{N+1}\Gamma(z'_i/z'_j)\prod_{i>j}^{N+1}(\lambda^{-1}(z'_i z'_j))^{-1}}$$

$$w_a = \lambda^{1/2} w'_a\ , \quad \mathfrak{q}_{Sp} = \lambda^{(N+1)/2}\prod_{a=1}^{N_f}(w_a)^{-1/2}\ , \quad \mathfrak{q}_{SU} = \lambda^{-1}\prod_{a=1}^{N_f}(w'_a)^{-1/2}$$

Duality domain wall converts $Sp(N)$ gauge theory into $SU(N+1)$ gauge theory and vice versa.

$$II_{SU}^{N+1,N_f}(z_i',w_a',\lambda_{SU};p,q)=\oint\frac{dz_i}{2\pi i z_i}\Delta^{(C)}(z,z',\lambda)\ II_{Sp}^{N,N_f}(z_i,w_a,\lambda_{Sp};p,q)$$

$$II_{Sp}^{N,N_f}(z_i,w_a,\lambda_{Sp};p,q)=\oint\frac{dz_i'}{2\pi i z_i'}\Delta^{(A)}(z',z,\lambda)\ II_{SU}^{N+1,N_f}(z_i',w_a',\lambda_{SU};p,q)$$

$$\Delta^{(C)}=\frac{\prod_{i=1}^{N+1}\prod_{j=1}^{N}\Gamma(\sqrt{\lambda}z_i'z_j^{\pm1})}{\prod_{i>j}^{N+1}(\lambda z_i'z_j')\prod_{i>j}^{N}\Gamma(z_i^{\pm}z_j^{\pm})\prod_{i=1}^{N}\Gamma(z_i^{\pm2})}$$

$$\Delta^{(A)}=\frac{\prod_{i=1}^{N}\prod_{j=1}^{N+1}\Gamma(\sqrt{\lambda}^{-1}z_j'z_i^{\pm1})}{\prod_{i\neq j}^{N+1}\Gamma(z_i'/z_j')\prod_{i>j}^{N+1}(\lambda^{-1}(z_i'z_j'))^{-1}}$$

CA- and AC-type inversion formula in [Spiridonov, Warnaar 04]

$$\oint d\mu_{z'}\Delta^{(A)}(z',x,\lambda)\oint d\mu_z\Delta^{(C)}(z,z',\lambda)f(z)=f(x)\ ,$$

$$\oint d\mu_{z}\Delta^{(C)}(z,x,\lambda)\oint d\mu_{z'}\Delta^{(A)}(z',z,\lambda)f(z')=f(x)\ .$$

guarantee that

$$\hat{D}^2 II_{SU}(z,w,\lambda_{SU})=II_{SU}(z,w,\lambda_{SU})\ ,\quad \hat{D}^2 II_{Sp}(z,w,\lambda_{Sp})=II_{Sp}(z,w,\lambda_{Sp})$$

Conclusion

- 5d CFTs can have enhanced global symmetries by strong dynamics which lead to dualities between gauge theories at low energy.
- We propose duality domain wall connecting two dual SU(N) gauge theories.
- Partition function of duality domain wall is a physical realization of elliptic integral equations.
- We also propose a new duality between Sp(N) and SU(N+1) gauge theories, and construct a duality domain wall connecting two dual theories.