Q-operators for higher spin eight vertex models

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Workshop: Elliptic Hypergeometric Functions in Combinatorics,

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§0. Introduction

- Solving the lattice models

  $\Longleftrightarrow$ diagonalisation of the transfer matrix

- Most well-known diagonalisation: Bethe Ansatz.

- Alternative method: Baxter's Q-operator.

Today's topic

Construction of the Q-operator
for the higher spin eight vertex model
with an even number of sites.

Plan of the talk:

1. The eight vertex model.

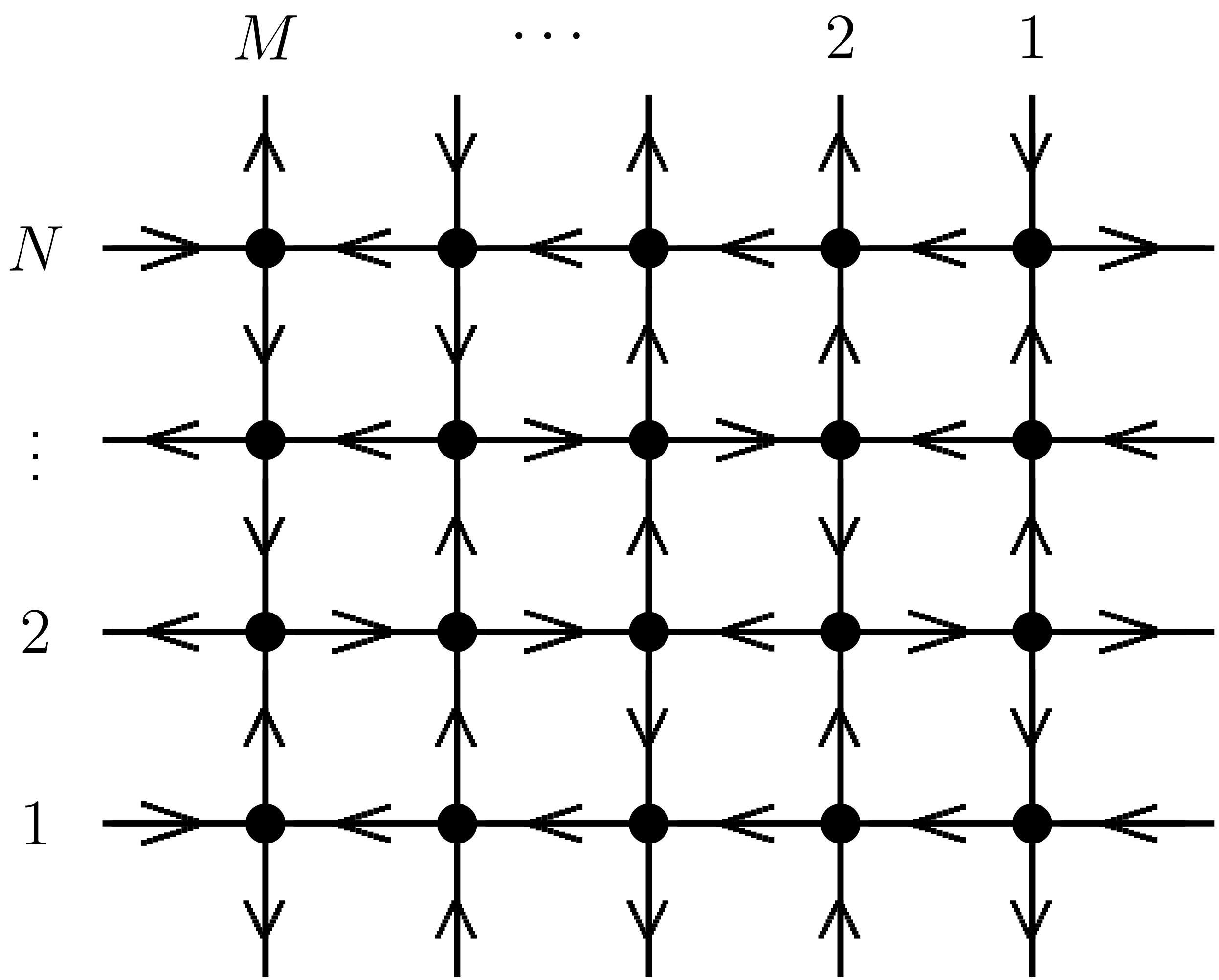
2. Q-operator.

3. Sklyanin algebra.

4. Higher spin generalisation of the eight vertex model.

§1. Eight vertex model

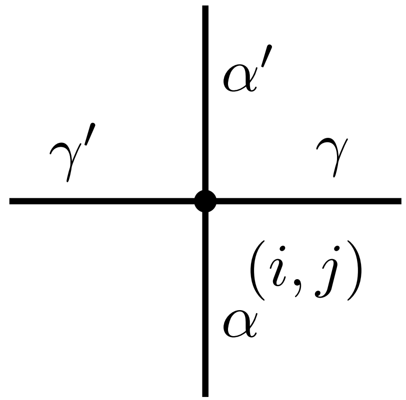
Two-dimensional vertex model in classical statistical mechanics:



$W(\sigma_{i,j})$ = Boltzmann weight at the vertex (i,j)

determined by the arrow-configuration at (i,j).

$\sigma_{i,j} =$

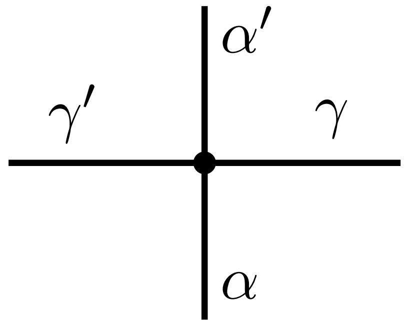


$$\begin{array}{c} \alpha, \alpha' \in \{\uparrow, \downarrow\}, \\ \gamma, \gamma' \in \{\leftarrow, \rightarrow\} \end{array}.$$

- L-matrix:

$L = (L_{\alpha,\gamma}^{\alpha',\gamma'})_{(\alpha,\gamma),(\alpha',\gamma')=(\uparrow \text{ or } \downarrow, \leftarrow \text{ or } \rightarrow)}$ = "table" of Boltzmann weights.

$L_{\alpha,\gamma}^{\alpha',\gamma'} = W\Big($



$\Big).$

L-matrix for Baxter's eight vertex model:

$$L(u) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix} = \sum_{a=0}^{3} W_a(u)\sigma^a \otimes \sigma^a,$$

$$W_a(u) := \frac{\theta_{g_a}(u;\tau)}{\theta_{g_a}(\eta;\tau)}, \quad g_0 = (11),\ g_1 = (10),\ g_2 = (00),\ g_3 = (01).$$

u: spectral parameter, η: unisotropy parameter, τ: elliptic modulus.

σ^a: Pauli matrices. ($\sigma^0 = \mathrm{Id.}$)

Theta functions: ($\theta_{11} = -\vartheta_1$, $\theta_{10} = \vartheta_2$, $\theta_{00} = \vartheta_3$, $\theta_{01} = \vartheta_4$)

$$\theta_{ab}(z;\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i \left(\frac{a}{2} + n\right)^2 \tau + 2\pi i \left(\frac{a}{2} + n\right)\left(\frac{b}{2} + z\right)\right).$$

$V_i = \mathbb{C}\uparrow \oplus \mathbb{C}\downarrow$ $(i = 1, \dots, M)$, $V_0 = \mathbb{C}\leftarrow \oplus \mathbb{C}\rightarrow$.

$\mathcal{H} := V_M \otimes \cdots \otimes V_1$.

- Monodromy matrix:

$$\mathcal{T}(u) := L_{M0}(u) \cdots L_{20}(u) L_{10}(u) : \mathcal{H} \otimes V_0 \to \mathcal{H} \otimes V_0.$$

$(L_{i0} \curvearrowright V_i \otimes V_0)$

- Transfer matrix:

$$T = T(u) := \operatorname{tr}_{V_0} \mathcal{T}(u) : \mathcal{H} \to \mathcal{H}.$$

(Recall: we are assuming the periodic boundary condition.)

Problem

Diagonalise $T(u)$, or *find eivenvalues of* $T(u)$.

• Why is the eight vertex model "good"?

$\exists$ 4×4-matrix $R(u)(= L(u+\eta))$ (the R-matrix) such that

$$R_{00'}(u-v)L_{i0}(u)L_{i0'}(v) = L_{i0'}(v)L_{i0}(u)R_{00'}(u-v)$$

on $V_i \otimes V_0 \otimes V_{0'}$.

$$\Longrightarrow \quad R_{00'}(u-v)\mathcal{T}_0(u)\mathcal{T}_{0'}(v) = \mathcal{T}_{0'}(v)\mathcal{T}_0(u)R_{00'}(u-v)$$

on $\mathcal{H} \otimes V_0 \otimes V_{0'}$.

Taking $\mathrm{tr}_{V_{0'}\otimes V_0}$ of $R\,\mathcal{T}_0(u)\,\mathcal{T}_{0'}(v)\,R^{-1} = \mathcal{T}_{0'}(v)\,\mathcal{T}_0(u)$,

$$T(u)T(v) = T(v)T(u).$$

$\Longrightarrow$ Eigenvectors do not depend on u.

§2. Q-operator

A tool introduced to find an eigenvalue by Baxter (1972, 1973).

Assume: $\eta \in [-\frac{1}{4}, \frac{1}{4}]$, $\tau = it$ ($t \in \mathbb{R}_{>0}$), $M =$ even as in [Baxter (1973)].

- Q-operator: $Q(u) : \mathcal{H} \to \mathcal{H}$: a linear operator satisfying:
  - TQ-relation: $\left(h_\pm(u) = (2[u \mp \eta])^M,\ [u] = \theta_{11}(u).\right)$

$$T(u)Q(u) = Q(u)T(u) = h_-(u)Q(u-2\eta) + h_+(u)Q(u+2\eta),$$

  - Commutativity: $[Q(u), Q(v)] = 0$.
  - Holomorphicity and quasi-periodicity in u:

$$U_1^{\otimes M} Q(u) = Q(u) U_1^{\otimes M} = e^{-M\pi i/2} Q(u+1), \quad (U_1 = \sigma^1 : \uparrow \mapsto \downarrow,\ \downarrow \mapsto \uparrow)$$

$$U_3^{\otimes M} Q(u) = Q(u) U_3^{\otimes M} = e^{M\pi i(\tau-1)/2 + M\pi i u} Q(u+\tau),$$

$$(U_3 = \sigma^3 : \uparrow \mapsto \uparrow,\ \downarrow \mapsto (-\downarrow)).$$

$T(u)$, $Q(u)$, $U_1^{\otimes M}$, $U_3^{\otimes M}$: all commute with each other for any u.

$\Longrightarrow$ eigenvectors are common for all u and independent of u.

If Ψ is a common eigenvector:

$$
\begin{aligned}
T(u)\Psi &= t(u)\Psi, & Q(u)\Psi &= q(u)\Psi, \\
U_1^{\otimes M}\Psi &= (-1)^{\nu_1}\Psi, & U_3^{\otimes M}\Psi &= (-1)^{\nu_3}\Psi. \qquad (\nu_1, \nu_3 \in \{0, 1\}.)
\end{aligned}
$$

Holomorphicity & quasi-periodicity $\Longrightarrow$ $\exists n_1 \in \mathbb{Z}$, $u_j \in \mathbb{C}$ ($j = 1, \dots, M/2$):

$$
q(u) = C e^{(\nu_1 + 2n_1)\pi i u} \prod_{j=1}^{M/2} [u - u_j].
$$

$u \mapsto u_j$ in the TQ-relation: $0 = h_-(u_j) q(u_j - 2\eta) + h_+(u_j) q(u_j + 2\eta)$.

$\Longleftrightarrow$ equations for u_j's.

$$\left(\frac{[u_j+\eta]}{[u_j-\eta]}\right)^M = e^{4(\nu_1+2n_1)\pi i\eta} \prod_{k=1,k\neq j}^{M/2} \frac{[u_j-u_k+2\eta]}{[u_j-u_k-2\eta]}.$$

(*Bethe equations*; same as those of the Bethe Ansatz.)

TQ-relation gives the eigenvalue of $T(u)$:

$$t(u) = h_-(u)\frac{q(u-2\eta)}{q(u)} + h_+(u)\frac{q(u+2\eta)}{q(u)}.$$

Quasi-periodicity says more: the sum rule of Bethe roots.

$$\sum_{j=1}^{M/2} u_j \equiv -\frac{\nu_1\tau}{2} + \frac{\nu_3}{2} \pmod{\mathbb{Z}+\tau\mathbb{Z}}.$$

(The Bethe Ansatz does not give the sum rule.)

§3. Sklyanin algebra

$$[T(u), T(v)] = 0 \Longleftarrow R\mathcal{T}\mathcal{T} = \mathcal{T}\mathcal{T}R \Longleftarrow RLL = LLR.$$

Generalise $L(u) = \sum_{a=0}^{3} W_a(u)\sigma^a \otimes \sigma^a$

$\longrightarrow L(u) = \sum_{a=0}^{3} W_a(u)S^a \otimes \sigma^a$, so that $RLL = LLR$ still holds.

[Sklyanin (1982)]

$RLL = LLR$ gives the commutation relation of S^a's:

$$[S^\alpha, S^0]_- = -iJ_{\alpha,\beta}[S^\beta, S^\gamma]_+, \qquad [S^\alpha, S^\beta]_- = i[S^0, S^\gamma]_+,$$

($[A, B]_\pm = AB \pm BA$, $(\alpha, \beta, \gamma) =$ cyclic permutation of $(1, 2, 3)$.)
The structure constants $J_{\alpha,\beta} = \frac{(W_\alpha)^2-(W_\beta)^2}{(W_\gamma)^2-(W_0)^2}$ do not depend on u.

$U_{\tau,\eta} := \langle S^0, S^1, S^2, S^3 \rangle$: Sklyanin algebra. (the "first" quantum *algebra*)

[Sklyanin (1983)]

spin l representation: $\rho^l : U_{\tau,\eta} \to \mathrm{End}_{\mathbb{C}}(\Theta_{00}^{4l+})$,

$\Theta_{00}^{4l+} := \{f(z) \mid f(z+1) = f(-z) = f(z), f(z+\tau) = e^{-4l\pi i(2z+\tau)} f(z)\}$,

$$(\rho^l(S^a)f)(z) = \frac{s_a(z - l\eta)f(z+\eta) - s_a(-z - l\eta)f(z-\eta)}{\theta_{11}(2z,\tau)},$$

$$s_0(z) = \theta_{11}(\eta,\tau)\theta_{11}(2z,\tau), \qquad s_1(z) = \theta_{10}(\eta,\tau)\theta_{10}(2z,\tau),$$

$$s_2(z) = i\theta_{00}(\eta,\tau)\theta_{00}(2z,\tau), \qquad s_3(z) = \theta_{01}(\eta,\tau)\theta_{01}(2z,\tau).$$

- Deformation of spin l representation of $sl_2(\mathbb{C})$.
- $l \in \frac{1}{2}\mathbb{Z}_{\geq 0} \Longrightarrow \dim \Theta_{00}^{4l+} = 2l + 1$.
- $l = 1/2$: $\rho^{1/2}(S^a) \propto \sigma^a \Longrightarrow$ eight vertex model.

Assume: $\eta \in [-\frac{1}{2(2l+1)}, \frac{1}{2(2l+1)}]$, $l \in \frac{1}{2}\mathbb{Z}_{>0}$. Recall $\tau = it$, $t > 0$.

Sklyanin form

$\langle , \rangle : \Theta_{00}^{4l+} \times \Theta_{00}^{4l+} \to \mathbb{C}$: sesquilinear form, such that

$$\langle \rho^{(l)}(S^a) f, g \rangle = \langle f, \rho^{(l)}(S^a) g \rangle.$$

Explicit definition:

$$\langle f(z), g(z) \rangle := \int_0^1 dx \int_0^t dy \, \overline{f(x+iy)} \, g(x+iy) \, \mu(x+iy, x-iy),$$

$$\mu(z,w) := \frac{\theta_{11}(2z,\tau)\theta_{11}(2w,\tau)}{\prod_{j=0}^{2l+1} \theta_{00}(z+w+(2j-2l-1)\eta,\tau)\,\theta_{00}(z-w+(2j-2l-1)\eta,\tau)}.$$

§4. Higher spin generalisation of the eight vertex model

Idea: Use the spin l representation to define the L-matrix!

$$L(u) := \sum_{a=0}^{3} W_a(u)\rho^{(l)}(S^a) \otimes \sigma^a : \Theta_{00}^{4l+} \otimes V_0 \to \Theta_{00}^{4l+} \otimes V_0.$$

$V_0 \cong \mathbb{C}^2$ as before. Hereafter $l \in \frac{1}{2}\mathbb{Z}_{>0}$.

- Monodromy matrix:

$$\mathcal{T}(u) := L_{M0}(u) \cdots L_{20}(u) L_{10}(u) : \mathcal{H} \otimes V_0 \to \mathcal{H} \otimes V_0,$$

where $\mathcal{H} := V_M \otimes \cdots \otimes V_1$, $V_i \cong \Theta_{00}^{4l+}$.

- Transfer matrix: $T(u) = \operatorname{tr}_{V_0} \mathcal{T}(u)$.

Goal: Construct the Q-operator for this $T(u)$.

Strategy: (same as that in [Baxter (1973)], [Baxter's book])

1. Twist the L-operator, so that $(2,1)$-component becomes degenerate.

2. The tensor product of the null vectors of the degenerate component $=$ a column vector of Q_R-operator, satisfying

$$T(u)Q_R(u) = h_-(u)Q_R(u-2\eta)+h_+(u)Q_R(u+2\eta), \quad h_\pm(u) = (2[u\mp 2l\eta])^M.$$

3. $Q_L(u) := (Q_R(-\bar{u}))^*$ (hermitian conjugate with respect to $\langle , \rangle$):

$$Q_L(u)T(u) = h_-(u)Q_L(u-2\eta) + h_+(u)Q_L(u+2\eta),$$

4. $Q(u) = Q_R(u)Q_R(u_0)^{-1} = Q_L(u_0)^{-1}Q_L(u)$ is the Q-operator, where u_0 is a suitably fixed parameter.

• Twisting the L-matrix:

Gauge transformation matrix:

$$M_\lambda(v) := \begin{pmatrix} -\theta_{00}\left((\lambda - v)/2, \tau/2\right) & -\theta_{00}\left((\lambda + v)/2, \tau/2\right) \\ \theta_{01}\left((\lambda - v)/2, \tau/2\right) & \theta_{01}\left((\lambda + v)/2, \tau/2\right) \end{pmatrix}.$$

Twisted L-matrix:

$$L_{\lambda,\lambda'}(u;v) = \begin{pmatrix} \alpha_{\lambda,\lambda'}(u;v) & \beta_{\lambda,\lambda'}(u;v) \\ \gamma_{\lambda,\lambda'}(u;v) & \delta_{\lambda,\lambda'}(u;v) \end{pmatrix} := M_\lambda(v)^{-1} L(u) M_{\lambda'}(v),$$

$\gamma_{\lambda+4l\eta,\lambda}(u;v)$ is degenerate. Its null vector is:

$$\omega_\lambda(u;v) := \left[z + \tfrac{\lambda+u-v}{2} + (-l+1)\eta\right]_{2l} \left[-z + \tfrac{\lambda+u-v}{2} + (-l+1)\eta\right]_{2l},$$

where $[z]_k := \prod_{j=0}^{k-1}[z + 2j\eta] = [z][z+2\eta]\cdots[z+2(k-1)\eta]$.

We can construct column vectors of $Q_R(u)$ by $\omega_\lambda(u;v)$:

Take parameters

- $\vec{\sigma} := (\sigma_M, \ldots, \sigma_2, \sigma_1)$: $\sigma_j = \pm 1$, $\sum_{k=1}^{M} \sigma_k = 0$. (Recall: M is even!)
- $\lambda_j = \lambda_j(\vec{\sigma}) := \lambda + 4l\eta(\sigma_1 + \cdots + \sigma_{j-1})$. (In particular, $\lambda_{M+1} = \lambda_1$.)

Then

$$\phi(u; v, \lambda, \vec{\sigma}) := \omega_{\sigma_M \lambda_M}(u; \sigma_M v) \otimes \cdots \otimes \omega_{\sigma_1 \lambda_1}(u; \sigma_1 v) \in \mathcal{H}$$

satisfies

$$T(u)\phi(u; v, \lambda, \vec{\sigma}) = h_-(u)\phi(u - 2\eta; v, \lambda, \vec{\sigma}) + h_+(u)\phi(u + 2\eta; v, \lambda, \vec{\sigma}).$$

<u>Assumption</u>: $\exists$ a set of parameters $\{(v_k, \lambda_k, \vec{\sigma}_k)\}_{k=1,\ldots,\dim \mathcal{H}}$ such that generically $\{\phi_k(u) := \phi(u; v_k, \lambda_k, \vec{\sigma}_k)\}_{k=1,\ldots,\dim \mathcal{H}}$ spans $\mathcal{H}$.

$$Q_R(u) : \mathbb{C}^{\dim \mathcal{H}} = \bigoplus_{k=1}^{\dim \mathcal{H}} \mathbb{C} e_k \ni e_k \mapsto \phi_k(u) \in \mathcal{H}$$

satisfies

$$T(u)Q_R(u) = h_-(u)Q_R(u-2\eta) + h_+(u)Q_R(u+2\eta).$$

Assumption $\Longrightarrow \exists u_0 \in \mathbb{C}$: $Q_R(u_0)$ is invertible.

$Q(u) := Q_R(u)Q_R(u_0)^{-1} : \mathcal{H} \to \mathcal{H}$ satisfies

$$T(u)Q(u) = h_-(u)Q(u-2\eta) + h_+(u)Q(u+2\eta).$$

Eventually, this $Q(u)$ is what we want.

Need to show: $T(u)Q(u) = Q(u)T(u)$, $Q(u)Q(v) = Q(v)Q(u)$.

• Hermitian conjugate of $T(u)$ and definition of $Q_L(u)$

$(\cdot)^*$: Hermitian conjuagate with respect to the Sklyanin form.

$$(S^a)^* = S^a \implies (T(u))^* = T(-\bar{u})$$

$\implies Q_L(u) := (Q_R(-\bar{u}))^* : \mathcal{H} \to \mathbb{C}^{\dim \mathcal{H}}$ satisfies

$$Q_L(u)T(u) = h_-(u)Q_L(u-2\eta) + h_+(u)Q_L(u+2\eta).$$

The commutation relation $Q_L(u)Q_R(u') = Q_L(u')Q_R(u)$ holds, which implies:

Theorem

$Q(u) := Q_R(u)Q_R(u_0)^{-1} = Q_L(u_0)^{-1}Q_L(u)$ satisfies

- $T(u)Q(u) = Q(u)T(u) = h_-(u)Q(u-2\eta) + h_+(u)Q(u+2\eta)$.
- $[Q(u), Q(u')] = 0$.

Idea of the proof of $Q_L(u)Q_R(u') = Q_L(u')Q_R(u)$:

$$(i,j)\text{-element of } Q_L(u)\,Q_R(u') = \langle \phi_i(-\bar{u}), \phi_j(u')\rangle.$$

$\Longrightarrow$ Enough to show that

$$\begin{aligned}\Phi(u,u') &:= \langle \phi(-\bar{u}; v, \lambda, \vec{\sigma}), \phi(u'; v', \lambda', \vec{\sigma}')\rangle \\ &= \prod_{k=1}^{M} \langle \omega_{\sigma_k \lambda_k}(-\bar{u}; \sigma_k v), \omega_{\sigma'_k \lambda'_k}(u'; \sigma'_k v')\rangle\end{aligned}$$

is a symmetric function in (u,u') for any λ, λ', $\vec{\sigma}$ and $\vec{\sigma}'$.

• One can compute the product $\langle \omega_{\sigma\lambda}(-\bar{u}; \sigma v), \omega_{\sigma'\lambda'}(u'; \sigma' v')\rangle$ explicitly thanks to the results of [Rosengren (2004, 2007)].

$\Longrightarrow \Phi(u,u')$ is factorised as

$$\Phi(u,u') = (\text{const.}) \prod_{k=1}^{M} F\big(\tfrac{\lambda'_k - \bar{\lambda}_k}{2} + \tfrac{\sigma'_k u' + \sigma_k u}{2} + (\sigma'_k - \sigma_k) l\eta + \tfrac{-v' + \bar{v}}{2}\big) \times$$
$$\times \prod_{k=1}^{M} G\big(\tfrac{\lambda'_k + \bar{\lambda}_k}{2} + \tfrac{\sigma'_k u' - \sigma_k u}{2} + (\sigma'_k + \sigma_k) l\eta + \tfrac{-v' - \bar{v}}{2}\big).$$

- [Baxter (book)]: A function factorised as above is symmetric in (u,u').

  (Induction on the sequence $\{\sigma_k\}$.)

$\Longrightarrow Q_L(u) Q_R(u') = Q_L(u') Q_R(u).$ $\square$

• <u>Quasi-periodicity</u>:

$$U_1^{\otimes M} Q(u) = Q(u) U_1^{\otimes M} = e^{-Ml\pi i} Q(u+1),$$
$$U_3^{\otimes M} Q(u) = Q(u) U_3^{\otimes M} = e^{Ml\pi i(\tau-1)+2Ml\pi i u} Q(u+\tau),$$

where $U_1,\ U_3 \in \mathrm{End}_{\mathbb{C}}(\Theta_{00}^{4l+})$:

$$(U_1 f)(z) = e^{\pi i \ell} f\left(z + \frac{1}{2}\right), \quad (U_3 f)(z) = e^{\pi i \ell} e^{\pi i \ell (4z+\tau)} f\left(z + \frac{\tau}{2}\right).$$

(When $l = 1/2$, $U_1 \sim \sigma^1$, $U_3 \sim \sigma^3$.)

$\Longrightarrow$ <u>*the Bethe Ansatz equations*</u>:

$$\left(\frac{[u_j + 2l\eta]}{[u_j - 2l\eta]}\right)^M = e^{4(\nu_1 + 2n_1)\pi i \eta} \prod_{k=1, k\neq j}^{Ml} \frac{[u_j - u_k + 2\eta]}{[u_j - u_k - 2\eta]}.$$

(as in the case of the eight vertex model.)

The eigenvalue of $T(u)$:

$$\Lambda(u) = (2[u+2l\eta])^M e^{-2\tilde{\nu}_1 \pi i \eta} \prod_{j=1}^{Ml} \frac{[u-u_j-2\eta]}{[u-u_j]}$$

$$+(2[u-2l\eta])^M e^{2\tilde{\nu}_1 \pi i \eta} \prod_{j=1}^{Ml} \frac{[u-u_j+2\eta]}{[u-u_j]},$$

The sum rule:

$$\sum_{j=1}^{Ml} u_j \equiv -\frac{\nu_1 \tau}{2} + \frac{\nu_3}{2} \pmod{\mathbb{Z} + \tau\mathbb{Z}}.$$

where $\nu_1, \nu_3 \in \{0, 1\}$, $\tilde{\nu}_1 = \nu_1 +$ (even number).

[T (1992), (1995)]: the modified Bethe Ansatz for this model.

- the Bethe equations and $\Lambda(u)$ were obtained.
- The sum rule was not proved.

Conclusion

- The higher spin eight vertex model ($M =$ even) has the Q-operator.
  - the Bethe equations.
  - the sum rule of Bethe roots.
- Baxter's construction: *seemingly* too technical.
  (heavily dependent on the explicit structure of the transfer matrix)
  However, it can be generalised to higher spin cases.
  $\rightsquigarrow$ existence of mathematical background?

Problems

- The case $M =$ odd. (On-going work; $\eta \in \mathbb{Q}$ (cf. [Baxter (1972)]).)
- [Zabrodin (2000)], [Chicherin, Derkachov, Karakhanyan, Kirschner (2013)]: ∞-dimensional Q-operators. Relations? Reduction?