Elliptic hypergeometric combinatorics

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Outline

Notation

The modified Jacobi theta function with argument x and nome p is defined by

$$\theta(x;p) := \prod_{j\geq 0}((1-p^j x)(1-p^{j+1}/x)), \quad \theta(x_1,\ldots,x_m;p) := \prod_{k=1}^{m}\theta(x_k;p),$$

where $x, x_1, \ldots, x_m \neq 0$, $|p| < 1$.

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$$\theta(x;p) = -x\theta(1/x;p),$$
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and Weierstraß' addition formula

$$\theta(xy, x/y, uv, u/v;p) - \theta(xv, x/v, uy, u/y;p) = \frac{u}{y}\theta(yv, y/v, xu, x/u;p).$$

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We call $[n]_q$ the q-number of n,

and $W_q(n)=q^n$ the q-weight of n.

Interpretation of $[n]_q$ as

area generating function

of lattice paths

(with positively directed horizontal and vertical steps of unit length)

from $(0,0)$ to $(1, n-1)$.

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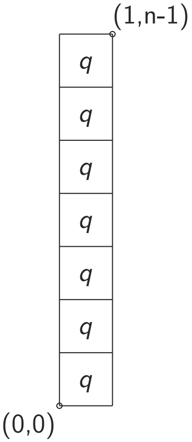

$$W(\mathcal{P}((0,0) \to (1,n-1)))$$
$$= 1 + q + q^2 + \cdots + q^{n-1} = [n]_q.$$

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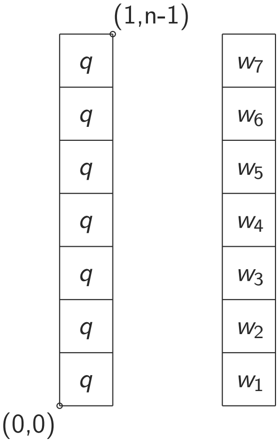

The idea is to generalize this further by suitably modifying the weights.

a; q-analogue: $$[n]_{a;q} + W_{a;q}(n)\,[m-n]_{aq^{2n};q} = [m]_{a;q},$$

where

$$[n]_{a;q} = \frac{(1-q^n)(1-aq^n)}{(1-q)(1-aq)}q^{1-n} \quad \text{and} \quad W_{a;q}(n) = \frac{(1-aq^{1+2n})}{(1-aq)}q^{-n}.$$

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The $b;q$-analogue clearly reduces to the q-analogue when $b \to 0$.

$a, b; q$-analogue (unification of the previous two):

$$[n]_{a,b;q} + W_{a,b;q}(n)\,[m-n]_{aq^{2n},bq^{n};q} = [m]_{a,b;q},$$

where

$$[n]_{a,b;q} = \frac{(1-q^n)(1-aq^n)(1-bq^2)(1-a/b)}{(1-q)(1-aq)(1-bq^{1+n})(1-aq^{n-1}/b)}$$

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The above expressions involve ratios of modified Jacobi theta functions.

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For convenience, we define the theta shifted (or $a, b; q, p$-shifted) factorials:

$$(a; q, p)_k := \prod_{j=0}^{k-1} \theta(aq^j; p),$$

and $$(a_1, \dots, a_m; q, p)_k = (a_1; q, p)_k \dots (a_m; q, p)_k.$$

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Elliptic hypergeometric series: Frenkel & Turaev [1997],

elliptic solutions of the Yang–Baxter equation.

Frenkel and Turaev's ${}_{10}V_9$ summation [1997].

$$\sum_{k=0}^{n} \frac{\theta(aq^{2k};p)}{\theta(a;p)} \frac{(a,b,c,d,e,q^{-n};q,p)_k}{(q,aq/b,aq/c,aq/d,aq/e,aq^{n+1};q,p)_k} q^k = \frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_n}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_n},$$

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The ${}_{10}V_9$ is the most fundamental identity in the theory of elliptic hypergeometric series.

Weighted lattice paths

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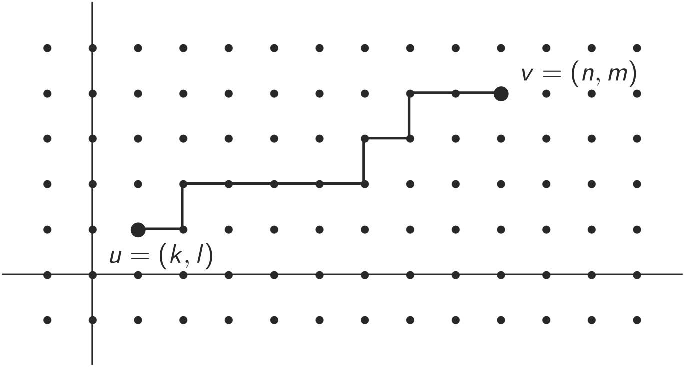

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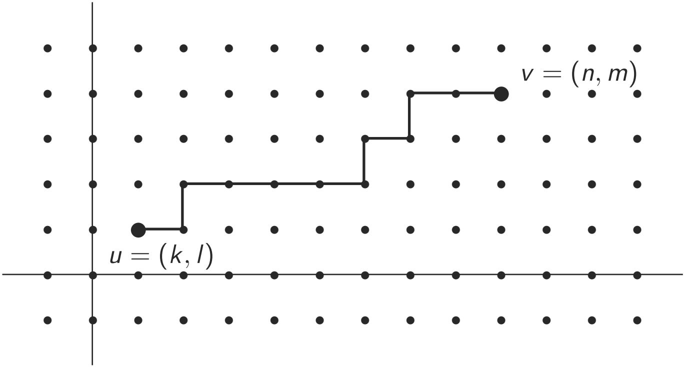
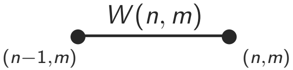


Given $u, v \in \mathbb{Z}^2$, denote the set of all lattice paths from u to v consisting of unit horizontal and vertical steps in the positive direction by $\mathcal{P}(u \to v)$.

To each horizontal edge in $\mathbb{Z}^2$ that connects $(n-1, m)$ and (n, m) we assign the weight $W(n, m)$.



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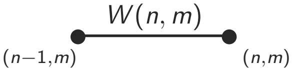

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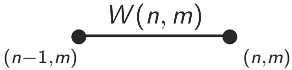

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Most of the subsequent analysis works for general weights.

Here we are interested in employing the elliptic weight

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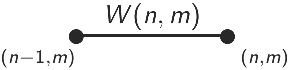

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The weight $W(P)$ of a path P is defined to be the product of the weights of all its horizontal steps.

Denote the weighted generating function of all paths from $(0,0)$ to $(k, n-k)$ by

$$_W\begin{bmatrix} n \\ k \end{bmatrix} := W(\mathcal{P}((0,0) \to (k, n-k))).$$

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and

$$ {}_{W}\begin{bmatrix} n \\ k \end{bmatrix} = 0, \quad \text{whenever} \quad k = -1, -2, \ldots, \quad \text{or} \quad k > n. $$

Furthermore, since the last step of the path is either vertical or horizontal, we have the recursion

$$ {}_{w}\begin{bmatrix} n+1 \\ k \end{bmatrix} = {}_{w}\begin{bmatrix} n \\ k \end{bmatrix} + {}_{w}\begin{bmatrix} n \\ k-1 \end{bmatrix} \, W(k, n+1-k), $$

for nonnegative integers n and k.

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For the elliptic weights $W(s,t) = W_{a,b;q,p}(s,t)$ we have [M.S., 2007]

$$ {}_W\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q, p)_{n-k}}{(q, aq, bq^{1+2k}, aq/b; q, p)_{n-k}}, $$

due to Weierstraß' addition formula for theta functions.

More generally, we have

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Theorem [M.S., 2007].
Let l, k, n, m be four integers with $n-l+m-k \geq 0$.

The elliptic generating function of paths running from (l,k) to (n,m) is

$$
\begin{aligned}
&W(\mathcal{P}((l,k)\to(n,m)))\\
&=\frac{(q^{1+n-l}, aq^{1+n+2k}, bq^{1+n+k+l}, aq^{1+k-n}/b; q,p)_{m-k}}{(q, aq^{1+l+2k}, bq^{1+2n+k}, aq^{1+k-l}/b; q,p)_{m-k}}\\
&\quad\times\frac{(aq^{1+l+2k}, aq^{1-n}/b, aq^{-n}/b; q,p)_{n-l}}{(aq^{1+l}, aq^{1+k-n}/b, aq^{k-n}/b; q,p)_{n-l}}\,\frac{(bq^{1+2l}; q,p)_{2n-2l}}{(bq^{1+k+2l}; q,p)_{2n-2l}}\,q^{(n-l)k}.
\end{aligned}
$$

Convolutions

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We have the following convolution of (elliptic) generating functions:

$$W\big(\mathcal{P}((0,0)\to(n,m))\big)$$
$$=\sum_{l=0}^{\min(k,n)} W\big(\mathcal{P}((0,0)\to(l,k-l))\big)\,W\big(\mathcal{P}((l,k-l)\to(n,m))\big).$$

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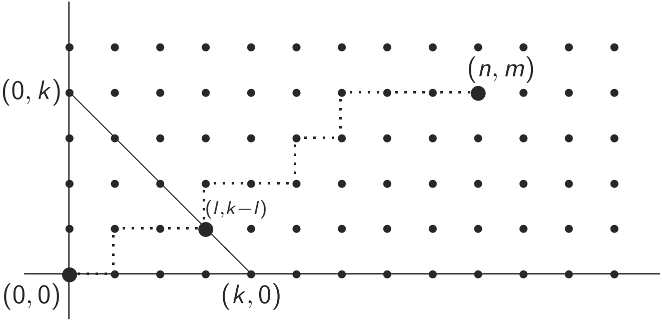

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Elliptic generalization of the q-Chu–Vandermonde identity.
Let n, m, and k be nonnegative integers, let a, b, q, and p be complex numbers with $|p| < 1$. Then there holds:

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{a,b;q,p}$$

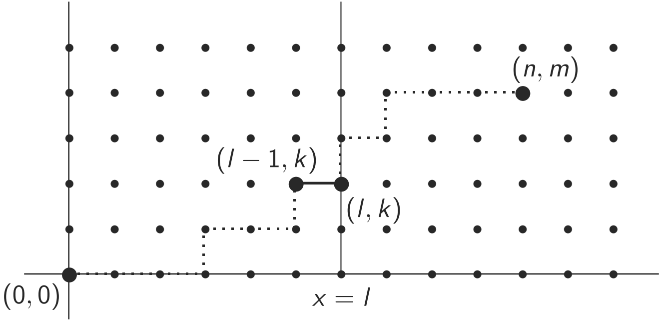
$$= \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} \begin{bmatrix} m \\ k-j \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p} \prod_{i=1}^{k-j} W_{a,b;q,p}(i+j,n-j).$$

We also have the following convolution of (elliptic) generating functions:

$$W\big(\mathcal{P}((0,0)\to(n,m))\big)$$
$$=\sum_{k=0}^{m} W\big(\mathcal{P}((0,0)\to(l-1,k))\big)\,W(l,k)\,W\big(\mathcal{P}((l,k)\to(n,m))\big).$$

We also have the following convolution of (elliptic) generating functions:

$$W\big(\mathcal{P}((0,0)\to(n,m))\big)$$
$$=\sum_{k=0}^{m} W\big(\mathcal{P}((0,0)\to(l-1,k))\big)\,W(l,k)\,W\big(\mathcal{P}((l,k)\to(n,m))\big).$$



Further we have

$$W\big(\mathcal{P}((0,0)\to(n,m))\big)$$
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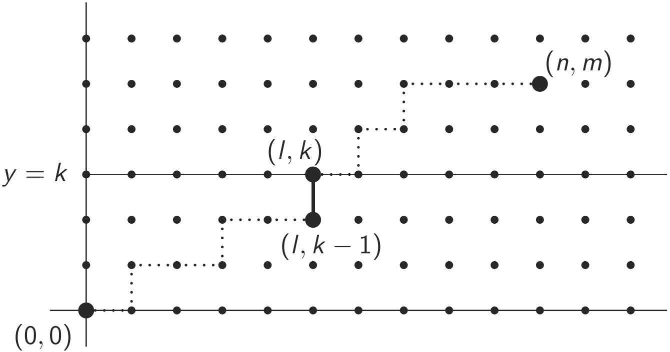
Further we have

$$W\big(\mathcal{P}((0,0)\to(n,m))\big) = \sum_{l=0}^{n} W\big(\mathcal{P}((0,0)\to(l,k-1))\big)\, W\big(\mathcal{P}((l,k)\to(n,m))\big).$$



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Clearly,

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

For $q \in \mathbb{C}$, let $\mathbb{C}_q[x, y]$ be the associative algebra over $\mathbb{C}$ with 1 generated by x and y, satisfying the relation

$$yx = qxy.$$

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Note that $\mathbb{C}_q[x, y]$ is a q-deformation of the commutative algebra $\mathbb{C}[x, y]$.

We refer to the variables x, y forming $\mathbb{C}_q[x, y]$ as q-commuting (or quasi-commuting) variables.

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[Harold S.A. Potter (1950); M.P. Schützenberger (1953)]

Binomial theorem for q-commuting variables. The following identity is valid in $\mathbb{C}_q[x, y]$:

$$(x + y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

Writing, as before, $q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$, we denote the space of elliptic functions over $\mathbb{C}$ of the complex variable u, meromorphic in u with the two periods σ^{-1} and $\tau\sigma^{-1}$, by

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More generally, we denote the space of totally elliptic multivariate functions over $\mathbb{C}$ of the complex variables $u_1, \ldots, u_n$, meromorphic in each variable with equal periods, σ^{-1} and $\tau\sigma^{-1}$, of double periodicity, by

$$\mathbb{E}_{q^{u_1},\ldots,q^{u_n};q,p}.$$

Elliptic binomial coefficients

Elliptic binomial coefficients

Let a and b be indeterminates, q and p complex numbers (with $|p| < 1$), n and k nonnegative integers with $n \geq k$.

We recall the definition of the elliptic binomial coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q, p)_{n-k}}{(q, aq, bq^{1+2k}, aq/b; q, p)_{n-k}}.$$

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Note that

$$\lim_{b \to 0} \left(\lim_{a \to 0} \left(\lim_{p \to 0} \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} \right) \right) = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

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This elliptic binomial coefficient is indeed totally elliptic; in particular,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} \in \mathbb{E}_{a,b,q^n,q^k;q,p}.$$

Recall, that using Weierstraß' addition formula, one can verify the following recursion for the elliptic binomial coefficients:

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n \\ n \end{bmatrix}_{a,b;q,p} = 1,$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} + \begin{bmatrix} n \\ k-1 \end{bmatrix}_{a,b;q,p} W_{a,b;q,p}(k, n+1-k),$$

for positive integers n and k with $n \geq k$, where

$$W_{a,b;q,p}(s,t) := \frac{\theta(aq^{s+2t}, bq^{2s}, bq^{2s-1}, aq^{1-s}/b, aq^{-s}/b; p)}{\theta(aq^{s}, bq^{2s+t}, bq^{2s+t-1}, aq^{1+t-s}/b, aq^{t-s}/b; p)} q^{t}.$$

If we let $p \to 0$, $a \to 0$, then $b \to 0$ (in this order), the above relations reduce to

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1,$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{n+1-k},$$

for positive integers n and k with $n \geq k$, which is a well-known recursion for the q-binomial coefficients.

(There exists a second recursion formula for the q-binomial coefficients; that can also be generalized to the elliptic level.)

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For $p, q \in \mathbb{C}$ with $|p| < 1$, and two commuting variables a and b, let $\mathbb{E}_{a,b;q,p}[x, y]$ be the associative algebra over $\mathbb{E}_{a,b;q,p}$ with 1 generated by x and y, satisfying the following relations:

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$\mathbb{E}_{a,b;q,p}[x, y]$ formally reduces to $\mathbb{C}_q[x, y]$ if one lets $p \to 0$, $a \to 0$, then $b \to 0$ (in this order), and drops the conditions of ellipticity.

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Proof.
We proceed by induction on n. For $n=0$ the formula is trivial.

Now let $n>0$ (n being fixed) and assume that we have already shown the formula for all nonnegative integers $\leq n$. We need to show

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{a,b;q,p} x^k y^{n+1-k}.$$

By the recursion formula for the elliptic binomial coefficients, the right-hand side is

$$\begin{aligned}
&\sum_{k=0}^{n+1} \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} x^k y^{n+1-k} \\
&+ \sum_{k=0}^{n+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{a,b;q,p} W_{a,b;q,p}(k, n+1-k)\, x^k y^{n+1-k} \\
= &\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} x^k y^{n-k} y \\
&+ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} W_{a,b;q,p}(k+1, n-k)\, x^{k+1} y^{n-k},
\end{aligned}$$

with $W_{a,b;q,p}(k+1, n-k)$ defined earlier.

It remains to be shown that

$$W_{a,b;q,p}(k+1,n-k)\,x^{k+1}y^{n-k}=x^k y^{n-k}x.$$

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However, using the defining relations of the algebra $\mathbb{E}_{a,b;q,p}[x,y]$ it is straightforward to show by induction with respect to k and l (we omit the details here) that

$$x^k W_{a,b;q,p}(s,t)=W_{a,b;q,p}(s+k,t)x^k$$
$$y^l W_{a,b;q,p}(s,t)=\frac{W_{a,b;q,p}(s,t+l)}{W_{a,b;q,p}(s,l)}y^l,$$

from which, together with the first defining relation of the algebra $\mathbb{E}_{a,b;q,p}[x,y]$ that can be written in the form

$$yx=W_{a,b;q,p}(1,1)\,xy,$$

one readily establishes the formula, as stated. $\square$

This elliptic binomial theorem can be used to recover Frenkel and Turaev's ${}_{10}V_9$ summation in the following form (where the requirement of n and m being nonnegative integers can be removed by analytic continuation):

Frenkel and Turaev's ${}_{10}V_9$ summation. Let n, m, and k be nonnegative integers, let a, b, q, and p be complex numbers with $|p| < 1$. Then there holds the following convolution formula:

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{a,b;q,p} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} \begin{bmatrix} m \\ k-j \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p} \prod_{i=1}^{k-j} W_{a,b;q,p}(i+j,n-j).$$

Proof. (Working in $\mathbb{E}_{a,b;q,p}[x,y]$) we expand $(x+y)^{n+m}$ in two different ways and then suitably extract coefficients. On the one hand, we have

$$(x+y)^{n+m} = \sum_{k=0}^{n+m} \begin{bmatrix} n+m \\ k \end{bmatrix}_{a,b;q,p} x^k y^{n+m-k}.$$

On the other hand, we have

$$\begin{aligned}(x+y)^{n+m} &= (x+y)^n (x+y)^m \\ &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} x^j y^{n-j} \sum_{l=0}^{m} \begin{bmatrix} m \\ l \end{bmatrix}_{a,b;q,p} x^l y^{m-l} \\ &= \sum_{j=0}^{n} \sum_{l=0}^{m} \begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} \begin{bmatrix} m \\ l \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p} x^j y^{n-j} x^l y^{m-l}.\end{aligned}$$

We now apply

$$\begin{aligned} x^j y^{n-j} x^l y^{m-l} &= x^j \left(\prod_{i=1}^{l} W_{a,b;q,p}(i, n-j) \right) x^l y^{n+m-j-l} \\ &= \left(\prod_{i=1}^{l} W_{a,b;q,p}(i+j, n-j) \right) x^{j+l} y^{n+m-j-l} \end{aligned}$$

and after extracting and equating (left) coefficients of $x^k y^{n+m-k}$ on the two right-hand sides of the equations on the previous page, we immediately obtain the convolution formula as stated. □

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Elliptic commuting variables can also be used to prove other combinatorial identities.

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A.M. Navon (1973) showed that the normal order coefficients of a word in the Weyl algebra are rook numbers.

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Special combinatorial numbers

Many special combinatorial numbers closely related to hypergeometric series.

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$$F_0 = F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \qquad \text{for } n \geq 2,$$

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I. Schur's q-analogue of the Fibonacci numbers, $F_n(q)$,

$$F_0(q) = F_1(q) = 1, \quad F_n(q) = F_{n-1}(q) + q^{a+n-2} F_{n-2}(q), \quad \text{for } n \geq 2,$$

satisfy

$$F_{n+1}(q) = \sum_{k=0}^{n} \begin{bmatrix} n-k \\ k \end{bmatrix}_q q^{k(k-1)+ak},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=1}^{k} \frac{[n-j+1]_q}{[j]_q}$ is the q-binomial coefficient.

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Explict formula:

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

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Explicit formula:

$$S_q(n,k) = \frac{1}{[k]_q!} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [j]_q^n,$$

where $[k]_q! = \prod_{j=1}^{k} [j]_q$.

Weighted Stirling numbers of the second kind

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Consider a partition of $[n+1] = \{1, 2, \ldots, n+1\}$ into k nonempty blocks, ordered by their minima from left-to-right.

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If the element $n+1$ forms a separate block,
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This immediately yields the recurrence

$$_V S_W(n+1, k) = V_k \, _V S_W(n, k-1) + (W_1 + \cdots + W_k) \, _V S_W(n, k).$$

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Consider a partition of $[n+1] = \{1, 2, \ldots, n+1\}$ into k nonempty blocks, ordered by their minima from left-to-right.

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that block must be the k-th one and $n+1$ the minimum of that block.
$\longrightarrow$ Assign weight V_k to the element $n+1$.

Otherwise, $n+1$ is in the j-th block for some $1 \le j \le k$,
but not a minimal element of any block
$\longrightarrow$ Assign weight W_j to the element $n+1$.

This immediately yields the recurrence

$$ {}_V S_W(n+1, k) = V_k \, {}_V S_W(n, k-1) + (W_1 + \cdots + W_k) \, {}_V S_W(n, k). $$

Carlitz' q-case is obtained when $V_j = W_j = q^{j-1}$ for all j.

Taking $V_j = W_j = W_{a,b;q,p}(j-1)$, where

$$W_{a,b;q,p}(j-1) = \frac{\theta(aq^{-1+2j}, bq, bq^2, a/b, a/bq; p)}{\theta(aq, bq^j, bq^{1+j}, aq^{j-2}/b, aq^{j-1}/b; p)} q^{j-1},$$

and using

$$\sum_{j=1}^{k} W_{a,b;q,p}(j-1) = [k]_{a,b;q,p} = \frac{\theta(q^k, aq^k, bq^2, a/b; p)}{\theta(q, aq, bq^{1+k}, aq^{k-1}/b; p)},$$

which telescopes due to the $n = k-1$ case of

$$[n]_{a,b;q,p} + W_{a,b;q,p}(n)\,[k-n]_{aq^{2n},bq^n;q,p} = [k]_{a,b;q,p},$$

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$$[n]_{a,b;q,p} + W_{a,b;q,p}(n)\,[k-n]_{aq^{2n},bq^n;q,p} = [k]_{a,b;q,p},$$

we obtain the following elliptic extension of the Stirling numbers of the second kind:

$$S_{a,b;q,p}(n+1,k) = W_{a,b;q,p}(k-1)S_{a,b;q,p}(n,k-1) + [k]_{a,b;q,p}S_{a,b;q,p}(n,k).$$

Weighted unsigned Stirling numbers of the first kind
[de Médicis & Leroux, 1995; Kereskényiné Balogh & M.S.]

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Consider a permutation of $[n+1]$, deomposed into to exactly k cycles, ordered by their minima from left-to-right.

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$\longrightarrow$ Assign weight v_{n+1} to the element $n+1$.

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A relevant q-case is obtained when $v_j = w_j = q^{1-j}$ for all j.
Likewise, we can assign elliptic weights to give an
elliptic extension of the unsigned Stirling numbers of the first kind.

Basis transitions

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We denote the falling factorials by

$$z^{\underline{n}} := \begin{cases} z(z-1)\dots(z-n+1) & \text{if } n = 1, 2, \dots, \\ 1 & \text{if } n = 0, \end{cases}$$

and denote the raising factorials by

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$$\begin{aligned} S(n,0) &= \delta_{n,0}, \\ S(n,k) &= 0 \qquad\qquad \text{for } k > n, \\ S(n+1,k) &= S(n,k-1) + k\, S(n,k). \end{aligned}$$

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$$\begin{aligned} s(n,0) &= \delta_{n,0}, \\ s(n,k) &= 0 \qquad\qquad \text{for } k>n, \\ s(n+1,k) &= s(n,k-1) - n\, s(n,k). \end{aligned}$$

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The Lah numbers count the number of placements of $1, 2, \ldots, n$ into exactly k nonempty tubes with linear order on its elements.

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  These are defined, for $c \in \mathbb{N}$, as the following connection coefficients:

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  The Abel numbers $A(n,k)$ count the number of forests of n labelled vertices composed of k rooted trees where each of the vertices can have one of c colors but the k roots must all have the first color.

We denote the elliptic falling factorials by

$$[z]^{\underline{n}}_{a,b;q,p} := \begin{cases} [z]_{a,b;q,p}[z-1]_{aq^2,bq;q,p}\dots[z-n+1]_{aq^{2n-2},bq^{n-1};q,p} & \text{if } n=1,2,\dots, \\ 1 & \text{if } n=0. \end{cases}$$

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Similarly, we denote the elliptic raising factorials by

$$[z]^{\overline{n}}_{a,b;q,p} := \begin{cases} [z]_{a,b;q,p}[z+1]_{aq^{-2},bq^{-1};q,p}\dots[z+n-1]_{aq^{2-2n},bq^{1-n};q,p} & \text{if } n=1,2,\dots, \\ 1 & \text{if } n=0. \end{cases}$$

The elliptic Stirling numbers of the second kind $S_{a,b;q,p}(n,k)$ satisfy the following connection identity.

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Using $[z]_{a,b;q,p} = W_{a,b;q,p}(k)[z-k]_{aq^{2k},bq^k;q,p} + [k]_{a,b;q,p}$,
this can be easily deduced from the recurrence relation

$$S_{a,b;q,p}(n+1,k) = W_{a,b;q,p}(k-1) S_{a,b;q,p}(n,k-1) + [k]_{a,b;q,p}\, S_{a,b;q,p}(n,k).$$

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On the contrary, the above connection identity can be used to define the sequence $S_{a,b;q,p}(n,k)$.

The elliptic Stirling numbers of the first kind $s_{a,b;q,p}(n,k)$ are defined as the following connection coefficients:

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As one can verify, the $s_{a,b;q,p}(n,k)$ satisfy the following recursion:

$$\begin{aligned} s_{a,b;q,p}(n,0) &= \delta_{n,0}, \\ s_{a,b;q,p}(n,k) &= 0 \qquad\qquad \text{for } k > n, \\ s_{a,b;q,p}(n+1,k) &= W^{-1}_{a,b;q,p}(n)\big(s_{a,b;q,p}(n,k-1) - [n]_{a,b;q,p}\, s_{a,b;q,p}(n,k)\big). \end{aligned}$$

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Clearly,

$$\sum_{k=l}^{n} S_{a,b;q,p}(n,k)\, s_{a,b;q,p}(k,l) = \delta_{n,l},$$

or

$$\big(S_{a,b;q,p}(n,k)\big)^{-1}_{n,k\in\mathbb{N}_0} = \big(s_{a,b;q,p}(k,l)\big)_{k,l\in\mathbb{N}_0}.$$

Summary

Various elliptic extensions of combinatorial special numbers
(a lot of these have been obtained in joint work with Meesue Yoo,
and with Zsófia R. Kereskényiné Balogh):

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Various elliptic extensions of combinatorial special numbers
(a lot of these have been obtained in joint work with Meesue Yoo,
and with Zsófia R. Kereskényiné Balogh):

- Elliptic binomial coefficients
- Elliptic Fibonacci and Lucas numbers
- Elliptic Stirling numbers of the second kind,
- Elliptic Stirling numbers of the first kind,
- Elliptic Lah numbers,
- Elliptic Abel numbers,
- r-Restricted versions and other generalizations of the above,
- Elliptic rook numbers (in different models: see talk of Meesue Yoo).