Discrete Painlevé equations and special functions

Masatoshi NOUMI (Kobe University, Japan)

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Contents

1 Elliptic difference Painlevé equation

1.1 General idea

$X = \mathbb{C}^N$: affine N-space with coordinates $x = (x_1, \ldots, x_N)$
$\mathcal{K}(X) = \mathbb{C}(x) = \mathbb{C}(x_1, \ldots, x_N)$: field of rational functions on X
$W \curvearrowright X$: birational action of a group W on X
$\rho : W \to \mathrm{Aut}(\mathcal{K}(X))$: group homomorphism

For each $w \in W$ and $\varphi \in \mathcal{K}(X)$, the action $w.\varphi = \rho(w)(\varphi) \in \mathcal{K}(X)$ is defined by

$$(w.\varphi)(x) = \varphi(w^{-1}.x) \quad \text{for generic} \quad x \in X. \tag{1.1}$$

$$w^{-1}. : X \cdots \to X \qquad \begin{cases} w.x_1 = R_1^w(x_1, \ldots, x_N) \\ \quad\vdots \\ w.x_N = R_N^w(x_1, \ldots, x_N) \end{cases} \tag{1.2}$$

The birational action of W on X provides a family of birational mappings which are compatible in the sense

$$R_i^{w_1 w_2}(x) = R_i^{w_2}(R_1^{w_1}(x), \ldots, R_N^{w_1}(x)) \quad (w_1, w_2 \in W;\ i = 1, \ldots, N). \tag{1.3}$$

• *Weyl group*: a group $W = \langle\, s_i \;(i \in I)\,\rangle$ generated by *simple reflections* s_i $(i \in I)$ subject to *fundamental relations* $s_i^2 = 1$ $(i \in I)$ and, for $i, j \in I$, $i \neq j$,

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots \qquad \text{(braid relation)} \tag{1.4}$$

with m_{ij} letters on each side, where $m_{ij} = 2, 3, 4, 6$ or ∞.

• *Affine Weyl group*: Weyl group W isomorphic to the semidirect product $L \rtimes W_0$ of a lattice $L \simeq \mathbb{Z}^l$ and a finite Weyl group W_0 acting linearly on L.

$$W = T_L \rtimes W_0; \qquad T_L = \left\{\, T_\alpha \;\middle|\; \alpha \in L \,\right\} \simeq L, \tag{1.5}$$

Note that $T_0 = 1$, $T_\alpha T_\beta = T_{\alpha+\beta}$ $(\alpha, \beta \in L)$ and $w T_\alpha = T_{w.\alpha} w$ $(\alpha \in L, w \in W_0)$.

When an affine Weyl group $W = T_L \rtimes W_0$ acts birationally on X, the *translation subgroup* T_L defines a commuting family of of birational mapping on X.

$$\begin{cases} T_\alpha(x_1) = R_1^\alpha(x_1, \ldots, x_N) \\ \qquad\vdots \\ T_\alpha(x_N) = R_N^\alpha(x_1, \ldots, x_N) \end{cases} \qquad (\alpha \in L \simeq \mathbb{Z}^l) \tag{1.6}$$

$\Longrightarrow$ *discrete integrable system* of rank N with l discrete time variables

• *Solution*

V: $\mathbb{C}$-vector space, $T_L \curvearrowright V$: affine linear action, $D \subseteq V$: a subset stable by T_L
A T_L-equivariant mapping $\varphi : D \to X$ gives a *solution* of the discrete integrable system specified as above.

1.2 Second order discrete Painlevé equations

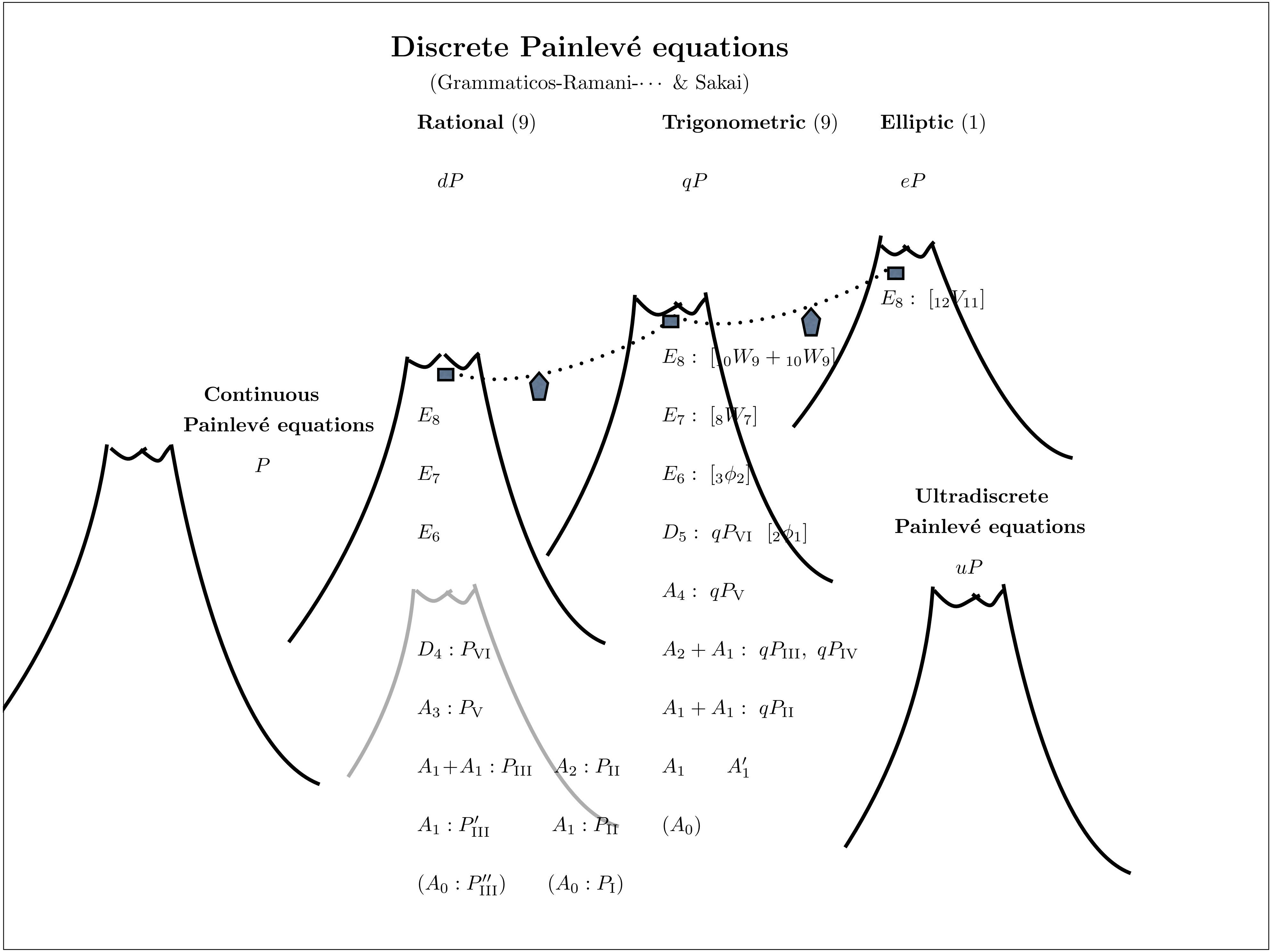
Sakai's table (2001): a standard list of second order discrete Painlevé equations
classification of nine-point blowups of $\mathbb{P}^2$, or eight-point blowups of $\mathbb{P}^1 \times \mathbb{P}^1$, which admit affine Weyl group symmetries.

- **Rational surfaces (anti-canonical divisors)**

$$
\begin{array}{lllllllllll}
(eP): & \boxed{A_0^{(1)}} & & & & & & & & & \\
(qP): & A_0^{(1)} \to & A_1^{(1)} \to & A_2^{(1)} \to & A_3^{(1)} \to & A_4^{(1)} \to & A_5^{(1)} \to & A_6^{(1)} & \to & A_7^{(1)} \to & A_8^{(1)} \\
 & & & & & & & & \searrow & & \\
 & & & & & & & & & A_7'^{(1)} & \\
(dP): & A_0^{(1)} \to & A_1^{(1)} \to & A_2^{(1)} & \to & D_4^{(1)} \to & D_5^{(1)} & \to D_6^{(1)} & \to & D_7^{(1)} \to & D_8^{(1)} \\
 & & & & & & \searrow & & \searrow & & \\
 & & & & & & & E_6^{(1)} & \to & E_7^{(1)} \to & E_8^{(1)}
\end{array}
$$

- **Affine Weyl group symmetry**

$$
\begin{array}{lllllllllll}
(eP): & \boxed{E_8^{(1)}} & & & & & & & & & \\
(qP): & E_8^{(1)} \to & E_7^{(1)} \to & E_6^{(1)} \to & D_5^{(1)} \to & A_4^{(1)} \to & (A_2+A_1)^{(1)} \to & (A_1+A_1')^{(1)} & \to & A_1'^{(1)} \to & A_0^{(1)} \\
 & & & & & & & & \searrow & & \\
 & & & & & & & & & A_1^{(1)} & \\
(dP): & E_8^{(1)} \to & E_7^{(1)} \to & E_6^{(1)} & \to & D_4^{(1)} \to & A_3^{(1)} & \to (2A_1)^{(1)} & \to & A_1'^{(1)} \to & A_0^{(1)} \\
 & & & & & & \searrow & & \searrow & & \\
 & & & & & & & A_2^{(1)} & \to & A_1^{(1)} \to & A_0^{(1)}
\end{array}
$$

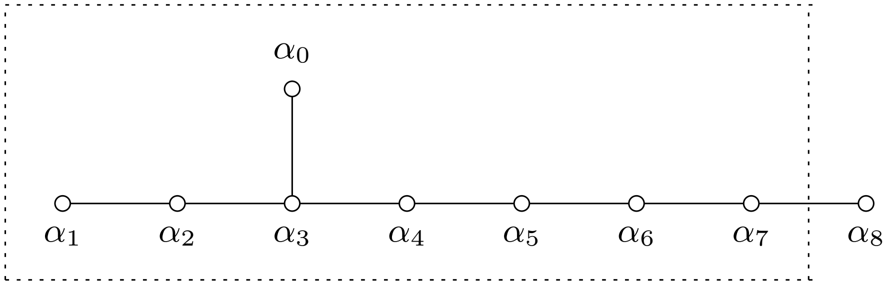

Discrete Painlevé equations
(Grammaticos-Ramani-$\cdots$ & Sakai)
Rational (9)
Trigonometric (9)
Elliptic (1)
dP
qP
eP
$E_8 :\ [{}_{12}V_{11}]$
$E_8 :\ [{}_{10}W_9 + {}_{10}W_9]$
$E_7 :\ [{}_8W_7]$
$E_6 :\ [{}_3\phi_2]$
$D_5 :\ qP_{\mathrm{VI}}\ \ [{}_2\phi_1]$
$A_4 :\ qP_{\mathrm{V}}$
$A_2 + A_1 :\ qP_{\mathrm{III}},\ qP_{\mathrm{IV}}$
$A_1 + A_1 :\ qP_{\mathrm{II}}$
$A_1 \quad A_1'$
(A_0)
Continuous
Painlevé equations
P
E_8
E_7
E_6
$D_4 : P_{\mathrm{VI}}$
$A_3 : P_{\mathrm{V}}$
$A_1{+}A_1 : P_{\mathrm{III}} \quad A_2 : P_{\mathrm{II}}$
$A_1 : P'_{\mathrm{III}} \quad A_1 : P_{\mathrm{II}}$
$(A_0 : P''_{\mathrm{III}}) \quad (A_0 : P_{\mathrm{I}})$
Ultradiscrete
Painlevé equations
uP

1.3 An explicit expression for the Elliptic Painlevé equation

• Affine root system of type $E_8^{(1)}$

Let $\mathfrak{h}$ the Cartan subalgebra of the affine Lie algebra of type $E_8^{(1)}$. We fix a basis of the dual space $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ as follows:

$$\mathfrak{h}^* = \mathbb{C}\kappa_x \oplus \mathbb{C}\kappa_y \oplus \mathbb{C}\varepsilon_1 \oplus \mathbb{C}\varepsilon_2 \oplus \cdots \oplus \mathbb{C}\varepsilon_8. \tag{1.7}$$



$$\alpha_0 = \varepsilon_1 - \varepsilon_2,$$
$$\alpha_1 = \kappa_x - \kappa_y,$$
$$\alpha_2 = \kappa_y - \varepsilon_1 - \varepsilon_2,$$
$$\alpha_j = \varepsilon_{j-1} - \varepsilon_j \quad (j = 3, \ldots, 8)$$

$$\delta = 2\kappa_x + 2\kappa_y - \varepsilon_1 - \varepsilon_2 - \cdots - \varepsilon_8 \qquad (\textit{null root}) \tag{1.8}$$

We regard $(\kappa; \varepsilon) = (\kappa_x, \kappa_y; \varepsilon_1, \ldots, \varepsilon_8)$ as coordinates of $\mathfrak{h}$. The translation T_{α_1} with respect to $\alpha_1 = \kappa_x - \kappa_y$ acts on these variables as follows:

$$\begin{aligned} T_{\alpha_1}(\kappa_x) &= \kappa_x - 2\alpha_1 + \delta = \kappa_x + 4\kappa_y - \varepsilon_1 - \cdots - \varepsilon_8 \\ T_{\alpha_1}(\kappa_y) &= \kappa_y - 2\alpha_1 + 3\delta = 4\kappa_x + 9\kappa_y - 3\varepsilon_1 - \cdots - 3\varepsilon_8 \\ T_{\alpha_1}(\varepsilon_j) &= \varepsilon_j - \alpha_1 + \delta \qquad (j = 1, \ldots, 8) \end{aligned} \tag{1.9}$$

- **Parametrization of an elliptic curve in $\mathbb{P}^1 \times \mathbb{P}^1$**

With the notation

$E_\Omega = \mathbb{C}/\Omega$: elliptic curve with the period lattice $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$,
$\sigma(u) = \sigma(u|\Omega)$: Weierstrass sigma function

we use the parameters κ_x, κ_y to define two functions

$$\varphi_a(u) = \sigma(a-u)\sigma(\kappa_x - a - u), \quad \psi_a(u) = \sigma(a-u)\sigma(\kappa_y - a - u) \quad (a, u \in \mathbb{C}). \tag{1.10}$$

Fixing generic constants $a, b \in \mathbb{C}$, we define the *reference curve* $C_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ by

$$C_0: \quad p(u) = (x(u), y(u)); \quad x(u) = \frac{\varphi_b(u)}{\varphi_a(u)}, \quad y(u) = \frac{\psi_b(u)}{\psi_a(u)} \qquad (u \in \mathbb{C}) \tag{1.11}$$

in terms of the inhomogeneous coordinates $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$. This curve can be represented as the zero locus of a polynomial of bidegree $(2, 2)$.

Setting $p_j = p(\varepsilon_j)$ $(j = 1, \ldots, 8)$, we use the parameters $\varepsilon_1, \ldots, \varepsilon_8$ to specify eight points $p_1, \ldots, p_8 \in C_0$. Note that $u = a, b$ corresponds to $(\infty, \infty), (0, 0) \in C_0$. Also, for $t \in \mathbb{C}$, the vertical and horizonal lines

$$\varphi_a(t)x - \varphi_b(t) = 0, \quad \psi_a(t)y - \psi_b(t) = 0 \tag{1.12}$$

of bigree $(1, 0)$ and $(0, 1)$ intersect with C_0 at $p(t) = (x(t), y(t))$.

- **Elliptic Painlevé equation with respect to $\boldsymbol{\alpha_1 = \kappa_x - \kappa_y}$** (KNY 2017, [4])

$$T_{\alpha_1}\left(\frac{\varphi_a(t)\,x - \varphi_b(t)}{\varphi_a(s)\,x - \varphi_b(s)}\right) = \frac{P(x,y;t)}{P(x,y;s)},$$
$$T_{\alpha_1}^{-1}\left(\frac{\psi_a(t)\,y - \psi_b(t)}{\psi_a(s)\,y - \psi_b(s)}\right) = \frac{Q(x,y;t)}{Q(x,y;s)}. \tag{1.13}$$

for any $t, s \in \mathbb{C}$, where $P(x,y;t)$, $Q(x,y;t)$ are characterized as polynomials of bidegree $(1,4)$ and of bidegree $(4,1)$ respectively, having zeros at $p_1,\dots,p_8$ and $p(t) \in C_0$, together with certain normalization conditions.

An explicit representation for $P(x,y;t)$ is given by

$$P(x,y;t) = c_0(t)\big(\varphi_a(t)x - \varphi_b(t)\big)\prod_{j=5}^{8}\big(\psi_a(\varepsilon_j)y - \psi_b(\varepsilon_j)\big)$$
$$+\,(\psi_a(t)y - \psi_b(t))\sum_{k=5}^{8} c_k(t)\,\big(\varphi_a(\varepsilon_k)x - \varphi_b(\varepsilon_k)\big)\prod_{\substack{j=5\\ j\neq k}}^{8}\big(\psi_a(\varepsilon_j)y - \psi_b(\varepsilon_j)\big).$$

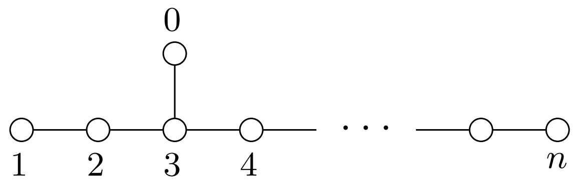
$$c_0(t) = -\frac{\sigma(\kappa_x - \kappa_y - \delta)}{\sigma(\kappa_x - \kappa_y)}\frac{\prod_{i=1}^{4}\sigma(\kappa_y - \varepsilon_j - t)}{\prod_{5\le k\le 8}\sigma(\varepsilon_j - t)}$$
$$c_k(t) = -\frac{\sigma(\kappa_x - \kappa_y - \delta + t - \varepsilon_k)}{\sigma(\kappa_x - \kappa_y)\sigma(t - \varepsilon_k)}\frac{\prod_{i=1}^{4}\sigma(\kappa_y - \varepsilon_j - \varepsilon_k)}{\prod_{5\le j\le 8;\, j\neq k}\sigma(\varepsilon_j - \varepsilon_k)} \quad (k = 5,\dots,8) \tag{1.14}$$

1.4 Weyl group W_n and the Picard lattice L_n

• Weyl group W_n

$W_n = W(T_{2,3,n-2}) = \langle s_0, s_1, \ldots, s_n \rangle$ $(n = 3, 4, \ldots)$

$T_{2,3,n-2}$:



$$s_i^2 = 1$$
$$s_i s_j = s_j s_i \quad (\, i \circ \quad \circ\, j\,)$$
$$s_i s_j s_i = s_j s_i s_j \quad (\, i \circ\!\!-\!\!\!-\!\!\circ\, j\,) \qquad (1.15)$$

n	3	4	5	6	7	8	9	$\cdots$
root system	A_4	D_5	E_6	E_7	E_8	$E_8^{(1)}$	$*$	$\cdots$

($*$: of indefinite type)

W_n: finite group for $n \leq 7$, infinite group for $n \geq 8$.
$W_8 = W(E_8^{(1)})$: affine Weyl group of type $E_8^{(1)}$

The Dynkin diagram $T_{2,3,n-2}$ is also referred to as E_{n+1}: $T_{2,3,5} = E_8$, $T_{2,3,6} = E_9 = E_8^{(1)}$.

This Weyl group W_n is realized as a reflection group on the *Picard lattice*

$$L_n = \mathbb{Z}\mathsf{H}_1 \oplus \mathbb{Z}\mathsf{H}_2 \oplus \mathbb{Z}\mathsf{E}_1 \oplus \mathbb{Z}\mathsf{E}_2 \oplus \cdots \oplus \mathbb{Z}\mathsf{E}_n. \qquad (1.16)$$

We also use the notation $\mathsf{H}_1 = \mathsf{H}_x$, $\mathsf{H}_2 = \mathsf{H}_y$ depending to the situation.

• Picard lattice L_n

$W_n = \langle s_0, s_1, \ldots, s_n \rangle$ is realized as a reflection group on the *Picard lattice*

$$L_n = \mathbb{Z}\mathsf{H}_1 \oplus \mathbb{Z}\mathsf{H}_2 \oplus \mathbb{Z}\mathsf{E}_1 \oplus \mathbb{Z}\mathsf{E}_2 \oplus \cdots \oplus \mathbb{Z}\mathsf{E}_n \tag{1.17}$$

with the symmetric bilinear form $(\ |\ ) : L_n \times L_n \to \mathbb{Z}$ such that

$$\begin{aligned} &(\mathsf{H}_1|\mathsf{H}_1) = (\mathsf{H}_2|\mathsf{H}_2) = 0, \quad (\mathsf{H}_1|\mathsf{H}_2) = -1 \\ &(\mathsf{H}_i|\mathsf{E}_j) = 0 \quad (i = 1, 2; j = 1, \ldots, n), \quad (\mathsf{E}_i|\mathsf{E}_j) = \delta_{ij} \quad (i, j \in \{1, \ldots, n\}). \end{aligned} \tag{1.18}$$

In the geometric terms,

L_n: Picard group attached to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at generic n points $p_1, \ldots, p_n$

H_1 and H_2: divisor classes of lines $x = \text{const.}$ and $y = \text{const.}$, $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$

E_j $(j = 1, \ldots, n)$: exceptional divisors.

$(\Lambda|\Lambda') =$ intersection number of the divisor classes $\Lambda, \Lambda' \in L_n$ multiplied by -1.

$\Lambda = d_1 H_1 + d_2 H_2 - m_1 E_1 - \cdots - m_n E_n \quad (d_1, d_2, m_1, \ldots, m_n \in \mathbb{Z})$

$(\Lambda|H_2) = -d_1, \quad (\Lambda|H_2) = -d_2, \quad (\Lambda|E_j) = -m_j \ (j = 1, \ldots, n)$

... divisors of bidegree (d_1, d_2) intersecting with E_j with multiplicity m_j $(j = 1, \ldots, n)$

In this lattice, the simple roots $\mathsf{a}_0, \mathsf{a}_1, \ldots, \mathsf{a}_n$ of type $T_{2,3,n-2}$ are realized as

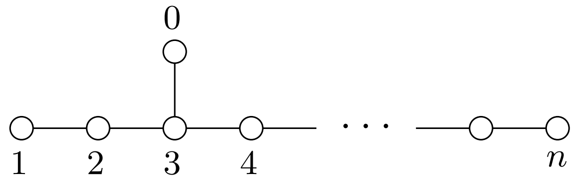
$$\begin{aligned} &\mathsf{a}_0 = \mathsf{E}_1 - \mathsf{E}_2, \quad \mathsf{a}_1 = \mathsf{H}_1 - \mathsf{H}_2, \quad \mathsf{a}_2 = \mathsf{H}_2 - \mathsf{E}_1 - \mathsf{E}_2, \\ &\mathsf{a}_3 = \mathsf{E}_2 - \mathsf{E}_3, \quad \mathsf{a}_4 = \mathsf{E}_3 - \mathsf{E}_4, \quad \ldots, \quad \mathsf{a}_n = \mathsf{E}_{n-1} - \mathsf{E}_n. \end{aligned} \tag{1.19}$$

• Simple roots of the root system of type $T_{2,3,n-2}$

The simple roots $\mathsf{a}_0, \mathsf{a}_1, \ldots, \mathsf{a}_n$ of type $T_{2,3,n-2}$ are realized in the Picard lattice L_n as

$$\mathsf{a}_0 = \mathsf{E}_1 - \mathsf{E}_2,\ \mathsf{a}_1 = \mathsf{H}_1 - \mathsf{H}_2,\ \mathsf{a}_2 = \mathsf{H}_2 - \mathsf{E}_1 - \mathsf{E}_2,\ \mathsf{a}_j = \mathsf{E}_{j-1} - \mathsf{E}_j\ (j = 3, \ldots, n).$$

$T_{2,3,n-2}$:



$$(\mathsf{a}_i|\mathsf{a}_i) = 2$$
$$(\mathsf{a}_i|\mathsf{a}_j) = 0 \qquad (\,i\circ \quad \circ j\,)$$
$$(\mathsf{a}_i|\mathsf{a}_j) = -1 \quad (\,i\circ\!\!-\!\!\!-\!\!\circ j\,) \tag{1.20}$$

• Linear action of W_n on L_n

The complexification of the Picard lattice L_n

$$\mathfrak{h}_n = L_n \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}\mathsf{H}_1 \oplus \mathbb{C}\mathsf{H}_2 \oplus \mathbb{C}\mathsf{E}_1 \oplus \mathbb{C}\mathsf{E}_2 \oplus \cdots \oplus \mathbb{C}\mathsf{E}_8 \tag{1.21}$$

gives a realization of the Cartan subalgebra of the Kac-Moody Lie algeba $\mathfrak{g}(T_{2,3,n-2})$.

For each $\alpha \in \mathfrak{h}_n$ with $(\alpha|\alpha) \neq 0$, we define the *reflection* $r_\alpha : \mathfrak{h}_n \to \mathfrak{h}_n$ by

$$r_\alpha(h) = h - (\alpha^\vee|h)\alpha \qquad (h \in \mathfrak{h}_n) \tag{1.22}$$

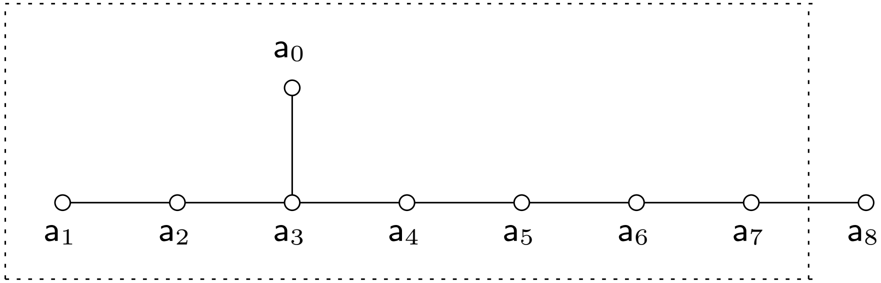
where $\alpha^\vee = 2\alpha/(\alpha|\alpha)$. Then the Weyl group $W_n = \langle s_0, s_1, \ldots, s_n \rangle$ acts linearly on $\mathfrak{h}_n$ through the simple reflections $s_i = r_{\mathsf{a}_i}$ $(i = 0, 1, \ldots, n)$, so that $(\ |\ )$ is W_n-invariant, and that W_n stabilizes $L_n \subset \mathfrak{h}_n$.

• The case $n = 8$: $W_8 = W(E_8^{(1)})$

We denote the root lattices of type E_8 and $E_8^{(1)}$ by $Q(E_8)$ and $Q(E_8^{(1)})$ respectively.

$$Q(E_8) = \mathbb{Z}\mathsf{a}_0 \oplus \mathbb{Z}\mathsf{a}_1 \oplus \cdots \oplus \mathbb{Z}\mathsf{a}_7 \subset Q(E_8^{(1)}) = Q(E_8) \oplus \mathbb{Z}\mathsf{a}_8 \subset L_8 \tag{1.23}$$

• The case $n = 8$: $W_8 = W(E_8^{(1)})$



$$\mathsf{a}_0 = \mathsf{E}_1 - \mathsf{E}_2,$$
$$\mathsf{a}_1 = \mathsf{H}_1 - \mathsf{H}_2,$$
$$\mathsf{a}_2 = \mathsf{H}_2 - \mathsf{E}_1 - \mathsf{E}_2,$$
$$\mathsf{a}_j = \mathsf{E}_{j-1} - \mathsf{E}_j \quad (j = 3, \ldots, 8)$$

$$\begin{aligned} &Q(E_8) \subset \ Q(E_8^{(1)}) = Q(E_8) \oplus \mathbb{Z}\mathsf{c} \subset L_8 = Q(E_8) \oplus \mathbb{Z}\mathsf{c} \oplus \mathbb{Z}\mathsf{E}_8 \\ &\mathsf{c} = 2\mathsf{H}_1 + 2\mathsf{H}_2 - \mathsf{E}_1 - \mathsf{E}_2 - \cdots - \mathsf{E}_8 \\ &\quad = 3\mathsf{a}_0 + 2\mathsf{a}_1 + 4\mathsf{a}_2 + 6\mathsf{a}_3 + 5\mathsf{a}_4 + 4\mathsf{a}_5 + 3\mathsf{a}_6 + 2\mathsf{a}_7 + \mathsf{a}_8 \\ &(\mathsf{c}|\mathsf{a}_j) = 0 \quad (j = 0, 1, \ldots, 8), \quad w.\mathsf{c} = \mathsf{c} \quad (w \in W_8). \end{aligned} \tag{1.24}$$

The *null root* c corresponds to the anti-canonical divisor of the 8-point blowup of $\mathbb{P}^1 \times \mathbb{P}^1$.

For each $\alpha \in Q = Q(E_8)$, the *Kac translation* $T_\alpha : \mathfrak{h}_8 \to \mathfrak{h}_8$ is defined by

$$\begin{aligned} &T_\alpha(\Lambda) = \Lambda + (\mathsf{c}|\Lambda)\alpha - \big(\tfrac{1}{2}(\alpha|\alpha)(\mathsf{c}|\Lambda) + (\alpha|\Lambda)\big)\mathsf{c} \quad (\Lambda \in L_8). \\ &T_0 = 1, \quad T_\alpha T_\beta = T_{\alpha+\beta} \quad (\alpha, \beta \in Q); \quad wT_\alpha = T_{w.\alpha} w \quad (\alpha \in Q,\ w \in W_8). \end{aligned} \tag{1.25}$$

Then the Weyl group $W_8 = W(E_8^{(1)})$ splits into the semi-direct product

$$W_8 = W(E_8^{(1)}) = T_Q \rtimes W(E_8); \quad Q(E_8) \xrightarrow{\sim} T_Q : \ \alpha \mapsto T_\alpha. \tag{1.26}$$

1.5 Birational Weyl group action on the point configuration space

- **Configuration space of generic n points in $\mathbb{P}^1 \times \mathbb{P}^1$**

The Weyl group $W_n = W(T_{2,3,n-2})$ acts birationally on the configuration space of generic n points in $\mathbb{P}^1 \times \mathbb{P}^1$. For two n-tuples $(p_1, \ldots, p_n)$, $(q_1, \ldots, q_n) \in (\mathbb{P}^1 \times \mathbb{P}^1)^n$,

$$\begin{aligned} &(p_1, \ldots, p_n) \sim (q_1, \ldots, q_n) \quad \textit{equivanlent as configurations} \\ &\iff \quad \exists g \in \mathrm{PGL}(2) \times \mathrm{PGL}(2) : \quad g.p_j = q_j \quad (j = 1, 2, \ldots, n). \end{aligned} \tag{1.27}$$

$$\mathbb{X}_n = \left\{ (p_1, \ldots, p_n) \in (\mathbb{P}^1 \times \mathbb{P}^1)^n \;\middle|\; \textit{generic} \right\} / \sim .$$

In terms of the inhomogeneous coordinates, any generic n-tuple of points with $n \geq 3$

$$(p_1, \ldots, p_n) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \ldots & x_n \\ y_1 & y_2 & y_3 & y_4 & \ldots & y_n \end{pmatrix} \in (\mathbb{P}^1 \times \mathbb{P}^1)^n \tag{1.28}$$

is transformed uniquely into

$$(q_1, \ldots, q_n) = \begin{pmatrix} \infty & 0 & 1 & f_4 & \ldots & f_n \\ \infty & 0 & 1 & g_4 & \ldots & g_n \end{pmatrix} \in (\mathbb{P}^1 \times \mathbb{P}^1)^n \tag{1.29}$$

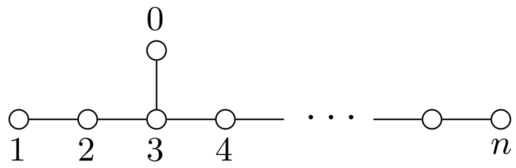
by the pair of fractional linear transformations

$$f = \frac{x - x_2}{x - x_1} \frac{x_3 - x_1}{x_3 - x_2}, \quad g = \frac{y - y_2}{y - y_1} \frac{y_3 - y_1}{y_3 - y_2}. \tag{1.30}$$

• Birational action of W_n on $\mathbb{X}_n$

The Weyl group $W_n = \langle s_0, s_1, \ldots, s_n \rangle$ associated with the Dynkin diagram

$T_{2,3,n-2}$:



$$\begin{aligned} s_i^2 &= 1 \\ s_i s_j &= s_j s_i \qquad (\,i\circ \quad \circ j\,) \\ s_i s_j s_i &= s_j s_i s_j \quad (\,i\circ\!\!-\!\!-\!\!\circ j\,) \end{aligned} \tag{1.31}$$

acts on the field of rational functions $\mathcal{K}(\mathbb{X}_n) = \mathbb{C}(f_4, \ldots, f_n, g_4, \ldots, g_n)$ through the following automorphisms $s_0, s_1, \ldots, s_n$.

$$\begin{aligned} & s_0(f_j) = \frac{1}{f_j}, \qquad s_0(g_j) = \frac{1}{g_j} \qquad s_1(f_j) = g_j, \qquad s_1(g_j) = f_j \\ & s_2(f_j) = \frac{f_j}{g_j}, \qquad s_2(g_j) = \frac{1}{g_j} \qquad s_3(f_j) = 1 - f_j, \qquad s_3(g_j) = 1 - g_j \\ & s_4(f_4) = \frac{1}{f_4}, \quad s_4(g_4) = \frac{1}{g_4}, \qquad s_4(f_j) = \frac{f_j}{f_4}, \quad s_4(g_j) = \frac{g_j}{g_4} \quad (j = 5, \ldots, n) \end{aligned} \tag{1.32}$$

and, for $i = 4, \ldots, n$,

$$\begin{aligned} & s_i(f_{i-1}) = f_i, \quad s_i(f_i) = f_{i-1}, \quad s_i(g_{i-1}) = g_i, \quad s_i(g_i) = g_{i-1} \\ & s_i(f_j) = f_j, \qquad s_i(g_j) = g_j \qquad (j \neq i-1, i). \end{aligned} \tag{1.33}$$

These automorphisms except for s_2 have simple interpretations:

$\mathfrak{S}_n = \langle s_0, s_3, \ldots, s_n \rangle$: permutation of n components in $(p_1, \ldots, p_n)$

s_1: exchanging the two coordinates x, y.

• Linearization of the W_n action in terms of elliptic functions

We identify the $\mathbb{C}$-vector space $\mathfrak{h}_n = \mathbb{C} \otimes_{\mathbb{Z}} L_n$ with the complex affine $(2+n)$-space $\mathbb{C}^{2+n}$ with canonical coordinates $(\kappa; \varepsilon) = (\kappa_1, \kappa_2; \varepsilon_1, \ldots, \varepsilon_n)$ through the expression

$$\begin{aligned} h &= -\kappa_2 \mathsf{H}_1 - \kappa_1 \mathsf{H}_2 + \varepsilon_1 \mathsf{E}_1 + \varepsilon_2 \mathsf{E}_2 + \cdots + \varepsilon_n \mathsf{E}_n \in \mathfrak{h}_n \\ \kappa_i &= (\mathsf{H}_i | \cdot\,) \quad (i = 1, 2), \quad \varepsilon_j = (\mathsf{E}_j | \cdot\,) \quad (j = 1, \ldots, n) \\ \mathfrak{h}_n^* &= \mathrm{Hom}_{\mathbb{C}} = \mathbb{C}\kappa_1 \oplus \mathbb{C}\kappa_2 \oplus \mathbb{C}\varepsilon_1 \oplus \mathbb{C}\varepsilon_2 \oplus \cdots \oplus \mathbb{C}\varepsilon_n. \end{aligned} \tag{1.34}$$

and set $\alpha_j = (\mathsf{a}_j|\cdot)$ $(j = 1, \ldots, n)$ and $\delta = (\mathsf{c}|\cdot) = 2\kappa_1 + 2\kappa_2 - \varepsilon_1 - \cdots - \varepsilon_8$. Setting

$E_\Omega = \mathbb{C}/\Omega$: elliptic curve associated with the period lattice $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

$\sigma(u) = \sigma(u|\Omega)$: Weierstrass sigma function

$\varphi_\lambda(u, v) = \sigma(u - v)\sigma(\lambda - u - v) \quad (\lambda, u, v \in \mathbb{C})$

we consider the reference curve $C_0 : p(u) = (f(u), g(u))$ $(u \in \mathbb{C})$ specified as

$$f(u) = \frac{\varphi_{\kappa_1}(\varepsilon_2, u)}{\varphi_{\kappa_1}(\varepsilon_1, u)} \frac{\varphi_{\kappa_1}(\varepsilon_1, \varepsilon_3)}{\varphi_{\kappa_1}(\varepsilon_2, \varepsilon_3)}, \quad g(u) = \frac{\varphi_{\kappa_2}(\varepsilon_2, u)}{\varphi_{\kappa_2}(\varepsilon_1, u)} \frac{\varphi_{\kappa_2}(\varepsilon_1, \varepsilon_3)}{\varphi_{\kappa_2}(\varepsilon_2, \varepsilon_3)}, \tag{1.35}$$

and define the meromorphic mapping $\Phi: \ \mathfrak{h}_n \otimes_{\mathbb{Z}} E_\Omega = \mathfrak{h}_n/\Omega \otimes_{\mathbb{Z}} L_n \cdots \to \mathbb{X}_n$ by

$$\Phi(\kappa, \varepsilon) = (f(\varepsilon_4), \ldots, f(\varepsilon_n); g(\varepsilon_4), \ldots, g(\varepsilon_n)) \quad (j = 4, \ldots, n). \tag{1.36}$$

in terms of the coordinates $(\kappa; \varepsilon)$ of $\mathfrak{h}_n$ and $(f_4, \ldots, f_n; g_4, \ldots, g_n)$ of $\mathbb{X}_n$. Then it turns out that Φ is a W_n-equivariant mapping, namely, Φ is a *particular solution* of the system of functional equations specified by the birational W_n action on $\mathbb{X}_n$.

… canonical elliptic solution of the W_n-system on $\mathbb{X}_n$.

1.6 Discrete Painlevé equation with $W(E_8^{(1)})$-symmetry

• Birational action of $W_8 = W(E_8^{(1)})$ on $\mathbb{K} = \mathcal{K}(\mathbb{X}_8)$ and $\mathbb{K}(f, g)$

On the configuration space $\mathbb{X}_9$ of generic 9 points in $\mathbb{P}^1 \times \mathbb{P}^1$, we regard

$f_4, \ldots, f_8, g_4, \ldots, g_8$ as parameters for the 8-point configurations, and

$f = f_9,\ g = g_9$ as the coordinates for a generic point $(f, g) \in \mathbb{P}^1 \times \mathbb{P}^1$.

The Weyl group W_8 then acts on $\mathbb{K} = \mathbb{C}(f_4, \ldots, f_8, g_4, \ldots, g_8) = \mathcal{K}(\mathbb{X}_8)$, and also on $\mathbb{K}(f, g) = \mathcal{K}(\mathbb{X}_9)$ through the embedding $W_8 \subset W_9$:

$$\begin{aligned}
&s_0(f) = \frac{1}{f}, \quad s_0(g) = \frac{1}{g}, \qquad s_1(f) = g, \quad s_1(g) = f,\\
&s_2(f) = \frac{f}{g}, \quad s_2(g) = \frac{1}{g}, \qquad s_3(f) = 1 - f, \quad s_3(g) = 1 - g,\\
&s_4(f) = \frac{f}{f_4}, \quad s_4(g) = \frac{g}{g_4}, \qquad s_i(f) = f, \quad s_i(g) = g \quad (i = 5, \ldots, 8).
\end{aligned} \tag{1.37}$$

Having this birational representation of the affine Weyl group $W(E_8^{(1)}) = T_Q \rtimes W(E_8)$, $Q = Q(E_8)$, from the translation part T_Q we obtain the discrete integrable system

$$T_\alpha(f) = R^\alpha(f, g), \quad T_\alpha(g) = S^\alpha(f, g) \quad (\alpha \in Q(E_8)) \tag{1.38}$$

where $R^\alpha(f, g), S^\alpha(f, g) \in \mathbb{K}(f, g)$.

$\Longrightarrow$ *discrete Painlevé equation with* $W(E_8^{(1)})$*-symmetry.*

• Elliptic Painlevé equation $eP(E_8^{(1)})$

From now on, we parametrize the coordinates (f_j, g_j) $(j = 4, \ldots, 8)$ by means of the canonical elliptic solution of the W_8-system:

$$f_j = f(\varepsilon_j) = \frac{\varphi_{\kappa_1}(\varepsilon_2, \varepsilon_j)}{\varphi_{\kappa_1}(\varepsilon_1, \varepsilon_j)} \frac{\varphi_{\kappa_1}(\varepsilon_1, \varepsilon_3)}{\varphi_{\kappa_1}(\varepsilon_2, \varepsilon_3)}, \quad g_j = g(\varepsilon_j) = \frac{\varphi_{\kappa_2}(\varepsilon_2, \varepsilon_j)}{\varphi_{\kappa_2}(\varepsilon_1, \varepsilon_j)} \frac{\varphi_{\kappa_2}(\varepsilon_1, \varepsilon_3)}{\varphi_{\kappa_2}(\varepsilon_2, \varepsilon_3)}. \tag{1.39}$$

Then we obtain a realization of the affine Weyl group $W(E_8^{(1)})$ as an automorphism group of the field of rational functions $\mathcal{M}(\mathfrak{h}_8/\Omega \otimes_{\mathbb{Z}} L_8)(f, g)$. The representation of the translation subgroup $T_Q \subset W(E_8^{(1)})$

$$T_\alpha(f) = R^\alpha(f, g), \quad T_\alpha(g) = S^\alpha(f, g) \quad (\alpha \in Q(E_8)) \tag{1.40}$$

is the *elliptic difference Painlevé equation.*

From the canonical elliptic solution of the W_9-system, we also obtain a one-parameter family of special solutions

$$f = f(u) = \frac{\varphi_{\kappa_1}(\varepsilon_2, u)}{\varphi_{\kappa_1}(\varepsilon_1, u)} \frac{\varphi_{\kappa_1}(\varepsilon_1, \varepsilon_3)}{\varphi_{\kappa_1}(\varepsilon_2, \varepsilon_3)}, \quad g = g(u) = \frac{\varphi_{\kappa_2}(\varepsilon_2, u)}{\varphi_{\kappa_2}(\varepsilon_1, u)} \frac{\varphi_{\kappa_2}(\varepsilon_1, \varepsilon_3)}{\varphi_{\kappa_2}(\varepsilon_2, \varepsilon_3)} \tag{1.41}$$

of the elliptic Painlevé equation (canonical solution). This solution corresponds to the elliptic curve (the curve of bidgree (2,2)) passing through the 8 points specified by

$$(p_1, \ldots, p_8) = \begin{pmatrix} \infty & 0 & 1 & f_4 & \ldots & f_8 \\ \infty & 0 & 1 & g_4 & \ldots & g_8 \end{pmatrix} : \quad \begin{cases} f_j = f(\varepsilon_j) \\ g_j = g(\varepsilon_j) \end{cases} \quad (j = 4, \ldots, 8). \tag{1.42}$$

1.7 $\boldsymbol{\tau}$-functions for $eP(E_8^{(1)})$

We introduce a system of homogeneous coordinates (ξ, η), $\xi = (\xi_1 : \xi_2)$, $\eta = (\eta_1 : \eta_2)$ for $\mathbb{P}^1 \times \mathbb{P}^1$ such that

$$f = \frac{\varphi_{\kappa_1}(\varepsilon_1, \varepsilon_3)}{\varphi_{\kappa_1}(\varepsilon_2, \varepsilon_3)} \frac{\xi_2}{\xi_1}, \quad g = \frac{\varphi_{\kappa_2}(\varepsilon_1, \varepsilon_3)}{\varphi_{\kappa_2}(\varepsilon_2, \varepsilon_3)} \frac{\eta_2}{\eta_1} \tag{1.43}$$

together with new variables $\tau_1, \dots, \tau_8$ corresponding to $p_1, \dots, p_8$. Then the action of W_8 on $\mathcal{K} = \mathbb{K}(f, g)$, $\mathbb{K} = \mathcal{M}(\mathfrak{h}_8)$, can be extended to the field $\mathcal{L} = \mathbb{K}(\xi_1, \xi_2, \eta_1, \eta_2; \tau_1, \dots, \tau_8)$ as follows:

(1.44)

	ξ_1	ξ_2	η_1	η_2	τ_1	τ_2	τ_3	τ_4	$\dots$	τ_8
s_0	ξ_2	ξ_1	η_2	η_1	τ_2	τ_1	τ_3	τ_4	$\dots$	τ_8
s_1	η_1	η_2	ξ_1	ξ_2	τ_1	τ_2	τ_3	τ_4	$\dots$	τ_8
s_2	$\frac{\xi_1\eta_2}{\tau_1\tau_2}$	$\frac{\xi_1\eta_2}{\tau_1\tau_2}$	η_2	η_1	$\frac{\eta_2}{\tau_2}$	$\frac{\eta_1}{\tau_1}$	τ_3	τ_4	$\dots$	τ_8
s_3	ξ_1	ξ_3	η_1	η_3	τ_1	τ_3	τ_2	τ_4	$\dots$	τ_8
s_4 $\vdots$ s_8	ξ_1	ξ_2	η_1	η_2	τ_1	τ_2		$\tau_{(i-1,i)j}$		

$$\xi_j = \frac{\varphi_{\kappa_1}(\varepsilon_j, \varepsilon_2)\xi_1 - \varphi_{\kappa_1}(\varepsilon_j, \varepsilon_1)\xi_2}{\varphi_{\kappa_1}(\varepsilon_1, \varepsilon_2)}, \quad \eta_j = \frac{\varphi_{\kappa_2}(\varepsilon_j, \varepsilon_2)\eta_1 - \varphi_{\kappa_2}(\varepsilon_j, \varepsilon_1)\eta_2}{\varphi_{\kappa_2}(\varepsilon_1, \varepsilon_2)} \tag{1.45}$$

Theorem A: *These automorphisms $s_0, s_1, \dots, s_8$ of $\mathcal{L} = \mathbb{K}(\xi_1, \xi_2, \eta_1, \eta_2; \tau_1, \dots, \tau_8)$ defines a represenation of $W_8 = \langle s_0, s_1, \dots, s_8 \rangle$.*

In this realization we look at the action of s_3 on η_2:

$$s_3(\eta_2) = \eta_3 = \frac{\varphi_{\kappa_2}(\varepsilon_3, \varepsilon_2)\eta_1 - \varphi_{\kappa_2}(\varepsilon_3, \varepsilon_1)\eta_2}{\varphi_{\kappa_2}(\varepsilon_1, \varepsilon_2)} \tag{1.46}$$

By using the relations $\eta_1 = \tau_1 s_2(\tau_2)$, $\eta_2 = \tau_2 s_2(\tau_1)$, this formula can be rewritten as bilinear relations for translates of τ-functions:

$$\varphi_{\kappa_2}(\varepsilon_1, \varepsilon_2)\, s_3(\tau_2)\, s_3 s_2(\tau_1) = \varphi_{\kappa_2}(\varepsilon_3, \varepsilon_2)\, \tau_1\, s_2(\tau_2) - \varphi_{\kappa_2}(\varepsilon_3, \varepsilon_1)\, \tau_2\, s_2(\tau_1) \tag{1.47}$$

1.8 Lattice τ-functions for $eP(E_8^{(1)})$

In order to analyze the action of W_8 on the τ-functions, we consider the W_8-orbit of E_8 in the Picard lattice L_8: $M_8 = W_8\,\mathsf{E}_8 \subset L_8$. This orbit can also be described intrinsically as

$$M_8 = \{\ \Lambda \in L_8 \mid (\Lambda|\Lambda) = 1,\ (\mathsf{c}|\Lambda) = -1\ \}; \quad Q(E_8) \xrightarrow{\sim} M_8 : \quad \alpha \mapsto T_\alpha(\mathsf{E}_8). \tag{1.48}$$

Theorem B: *There exists a unique family of elements $\tau(\Lambda) \in \mathcal{L}$ $(\Lambda \in M_8)$ such that*

$$\tau(\mathsf{E}_j) = \tau_j \quad (j = 1, \ldots, 8); \quad w(\tau(\Lambda)) = \tau(w.\Lambda) \quad (\Lambda \in M_8;\ w \in W_8). \tag{1.49}$$

Furthermore, this family of τ-functions is characterized by the following non-autonomous Hirota equations: For any distinct $i, j, k \in \{1, \ldots, 8\}$ and for $r = 1, 2$,

$$\begin{aligned} &\sigma(\varepsilon_j - \varepsilon_k)\sigma(\kappa_r - \varepsilon_j - \varepsilon_k)\tau(\mathsf{E}_i)\tau(\mathsf{H}_r - \mathsf{E}_i) \\ &+ \sigma(\varepsilon_k - \varepsilon_i)\sigma(\kappa_r - \varepsilon_k - \varepsilon_i)\tau(\mathsf{E}_j)\tau(\mathsf{H}_r - \mathsf{E}_j) \\ &+ \sigma(\varepsilon_i - \varepsilon_j)\sigma(\kappa_r - \varepsilon_i - \varepsilon_j)\tau(\mathsf{E}_k)\tau(\mathsf{H}_r - \mathsf{E}_k) = 0. \end{aligned} \tag{1.50}$$

The homogeneous coordinates $\xi_1, \xi_2, \eta_1, \eta_2$ are recovered from $\tau(\Lambda)$ $(\Lambda \in M_8)$ by

$$\xi_i = \tau(\mathsf{E}_i)\tau(\mathsf{H}_1 - \mathsf{E}_i), \quad \eta_i = \tau(\mathsf{E}_i)\tau(\mathsf{H}_2 - \mathsf{E}_i) \quad (i = 1, 2). \tag{1.51}$$

For each $\Lambda \in M_8$ we define $\tau(\Lambda) = w(\tau_8) \in \mathcal{L}$ by taking a $w \in W_8$ such that $\Lambda = w.\mathsf{E}_8$; this definition does not depend on the choice of w since τ_8 is invariant under the action of the isotropy subgroup W_7 of E_8. With this definition, the bilinear relation

$$\varphi_{\kappa_2}(\varepsilon_1, \varepsilon_2)\, s_3(\tau_2)\, s_3 s_2(\tau_1) = \varphi_{\kappa_2}(\varepsilon_3, \varepsilon_2)\, \tau_1\, s_2(\tau_2) - \varphi_{\kappa_2}(\varepsilon_3, \varepsilon_1)\, \tau_2\, s_2(\tau_1) \tag{1.52}$$

is rewritten in the form

$$\begin{aligned} &\sigma(\varepsilon_1 - \varepsilon_2)\, \sigma(\kappa_2 - \varepsilon_1 - \varepsilon_2)\, \tau(\mathsf{E}_3)\, \tau(\mathsf{H}_2 - \mathsf{E}_3) \\ &= \sigma(\varepsilon_3 - \varepsilon_2)\, \sigma(\kappa_2 - \varepsilon_3 - \varepsilon_2)\, \tau(\mathsf{E}_1)\, \tau(\mathsf{H}_2 - \mathsf{E}_1) \\ &\quad + \sigma(\varepsilon_3 - \varepsilon_1)\, \sigma(\kappa_2 - \varepsilon_3 - \varepsilon_1)\, \tau(\mathsf{E}_2)\, \tau(\mathsf{H}_2 - \mathsf{E}_2). \end{aligned} \tag{1.53}$$

Then by the action of $\mathfrak{S}_8$ and by s_1, we obtain the bilinear equations as described in Theorem B.

Conversely, suppose that the family $\tau(\Lambda)$ $(\Lambda \in M_8)$ satisfies the property as stated in Theorem B. Then the variables ξ_i, η_i $(i = 1, 2)$ are recovered by

$$\xi_i = \tau(\mathsf{E}_i)\tau(\mathsf{H}_1 - \mathsf{E}_i), \quad \eta_i = \tau(\mathsf{E}_i)\tau(\mathsf{H}_2 - \mathsf{E}_i). \tag{1.54}$$

The non-autonomous Hirota equations mentioned above guarantee the validity of relations to be satisfied under the action of s_3.

1.9 Linear systems $\mathcal{L}(\Lambda)$

In the homogeneous coordinates $(\xi,\eta)\in\mathbb{P}^1\times\mathbb{P}^1$, $\xi=(\xi_1:\xi_2)$, $\eta=(\eta_1;\eta_2)$, we specify the parametrization of the reference curve C_0 by $p(u)=(\xi(u),\eta(u))$ $(u\in\mathbb{C})$ where

$$\begin{aligned}\xi_i(u)&=\varphi_{\kappa_1}(\varepsilon_i,u)=\sigma(\varepsilon_i-u)\sigma(\kappa_1-\varepsilon_i-u)\\ \eta_i(u)&=\varphi_{\kappa_2}(\varepsilon_i,u)=\sigma(\varepsilon_i-u)\sigma(\kappa_2-\varepsilon_i-u)\end{aligned}\tag{1.55}$$

and the eight reference points $p_1,\ldots,p_8$ by $p_j=p(\varepsilon_j)$.

For each element $\Lambda=d_1\mathsf{H}_1+d_2\mathsf{H}_2-m_1\mathsf{E}_1-m_2\mathsf{E}_2-\cdots-m_8\mathsf{E}_8\in L_8$ $(d_i,m_j\in\mathbb{Z})$ of the Picard lattice, we denote by $\mathcal{L}(\Lambda)$ the $\mathbb{K}$-vector space of functions of the form $f(\xi,\eta)\tau_1^{-m_1}\cdots\tau_8^{-m_8}$ such that

(1) $f(\xi,\eta)\in\mathbb{K}[\xi,\eta]$: homogeneous of bidegree (d_1,d_2), and

(2) $f(\xi,\eta)$ has a zero of multiplicity $\geq m_j$ at $p_j=p(\varepsilon_j)$ for $j=1,\ldots,8$.

Note that $\mathcal{L}(\mathsf{H}_1)=\mathbb{K}\xi_1\oplus\mathbb{K}\xi_2$, $\mathcal{L}(\mathsf{H}_2)=\mathbb{K}\eta_1\oplus\mathbb{K}\eta_2$ and that, for each $j=1,\ldots,8$,

$$\begin{gathered}\mathcal{L}(\mathsf{E}_j)=\mathbb{K}\,\tau_j=\tau(\mathsf{E}_j);\quad \mathcal{L}(\mathsf{H}_1-\mathsf{E}_j)=\mathbb{K}\,\xi_j\tau_j^{-1},\quad \mathcal{L}(\mathsf{H}_2-\mathsf{E}_j)=\mathbb{K}\,\eta_j\tau_j^{-1}\\ \xi_j=\frac{\varphi_{\kappa_1}(\varepsilon_j,\varepsilon_2)\xi_1-\varphi_{\kappa_1}(\varepsilon_j,\varepsilon_1)\xi_2}{\varphi_{\kappa_1}(\varepsilon_1,\varepsilon_2)},\quad \eta_j=\frac{\varphi_{\kappa_2}(\varepsilon_j,\varepsilon_2)\eta_1-\varphi_{\kappa_2}(\varepsilon_j,\varepsilon_1)\eta_2}{\varphi_{\kappa_2}(\varepsilon_1,\varepsilon_2)}.\end{gathered}\tag{1.56}$$

Also, each $w\in W_8$ induces a $\mathbb{C}$-isomorphism $w.\ :\ \mathcal{L}(\Lambda)\xrightarrow{\sim}\mathcal{L}(w.\Lambda)$ for all $\Lambda\in L_8$. In particular, for each $\Lambda\in M_8=W_8\{\mathsf{E}_1,\ldots,\mathsf{E}_8\}$, we have $\mathcal{L}(\Lambda)=\mathbb{K}\,\tau(\Lambda)$.

• $\boldsymbol{\tau}$-Cocycles $\boldsymbol{\phi_\Lambda(\xi,\eta)}$

Suppose that $\Lambda \in M_8$ and $\Lambda = d_1\mathsf{H}_1 + d_2\mathsf{H}_2 - m_1\mathsf{E}_1 - \cdots - m_8\mathsf{E}_8$. Then the τ-function $\tau(\Lambda)$ is expressed as $\tau(\Lambda) = \phi_\Lambda(\xi,\eta)\,\tau_1^{-m_1}\cdots\tau_8^{-m_8}$ with a homogeneous polynomial $\phi_\Lambda(\xi,\eta)$ of bidegree (d_1,d_2) such that $\mathrm{ord}_{p_j}\phi_\Lambda = m_j$ $(j=1,\ldots,8)$. Furthemore, $\phi_\Lambda(\xi,\eta)$ is normalized so that its restriction of to the reference curve C_0 is given by

$$\phi_\Lambda(\xi(u),\eta(u)) = \sigma(\lambda-u)\prod_{j=1}^{8}\sigma(\varepsilon_j-u), \quad \lambda = d_1\kappa_1 + d_2\kappa_2 - m_1\varepsilon_1 - \cdots - m_8\varepsilon_8. \tag{1.57}$$

We now consider the inhomogeneous coordinates (x,y) of $\mathbb{P}^1\times\mathbb{P}^1$ defined by

$$x = \frac{\xi_2}{\xi_1} = \frac{\tau(\mathsf{E}_2)\tau(\mathsf{H}_1-\mathsf{E}_2)}{\tau(\mathsf{E}_1)\tau(\mathsf{H}_1-\mathsf{E}_1)}, \quad y = \frac{\eta_2}{\eta_1} = \frac{\tau(\mathsf{E}_2)\tau(\mathsf{H}_2-\mathsf{E}_2)}{\tau(\mathsf{E}_1)\tau(\mathsf{H}_2-\mathsf{E}_1)} \tag{1.58}$$

Then the action of each $w \in W_8$ is expressed as

$$w.x = \frac{\tau(w.\mathsf{E}_2)\tau(w.(\mathsf{H}_1-\mathsf{E}_2))}{\tau(w.\mathsf{E}_1)\tau(w.(\mathsf{H}_1-\mathsf{E}_1))} = \frac{\phi_{w.\mathsf{E}_2}(\xi,\eta)\phi_{w.(\mathsf{H}_1-\mathsf{E}_2)}(\xi,\eta)}{\phi_{w.\mathsf{E}_1}(\xi,\eta)\phi_{w.(\mathsf{H}_1-\mathsf{E}_1)}(\xi,\eta)}. \tag{1.59}$$

Hence, in terms of inhomogenous polynomials $P_\Lambda(x,y) = \phi_\Lambda(\xi,\eta)\xi_1^{-d_1}\eta_1^{-d_2}$, we have

$$w.x = \frac{P_{w.\mathsf{E}_2}(x,y)P_{w.(\mathsf{H}_1-\mathsf{E}_2)}(x,y)}{P_{w.\mathsf{E}_1}(x,y)P_{w.(\mathsf{H}_1-\mathsf{E}_1)}(x,y)}, \quad w.y = \frac{P_{w.\mathsf{E}_2}(x,y)P_{w.(\mathsf{H}_2-\mathsf{E}_2)}(x,y)}{P_{w.\mathsf{E}_1}(x,y)P_{w.(\mathsf{H}_2-\mathsf{E}_1)}(x,y)}. \tag{1.60}$$

1.10 From the lattice τ-functions to the ORG τ-functions

Among the τ-functions $\tau(\Lambda)$ $(\Lambda \in M_8)$, $\tau_8 = \tau(\mathsf{E}_8)$ is a distinguished τ-function. It is $W(E_8)$-invariant, and all the τ-functions $\tau(\Lambda)$ $(\Lambda \in M_8)$ are expressible as the translates

$$\tau(\Lambda) = T_{\mathsf{E}_8-\Lambda}(\tau_8) \quad (\Lambda \in M_8); \qquad M_8 = T_Q(\mathsf{E}_8). \tag{1.61}$$

The system of non-autonomous Hirota equations for $\{\tau(\Lambda)\}_{\Lambda \in M_8}$ is then translated into a $W(E_8)$-invariant system of *difference equations* for a single τ-function $\tau = \tau_8$, which we formulate in terms of *ORG τ-functions* in the final section.

In working with difference equations, it is more convenient to use the root lattice $Q(E_8)$ of type E_8 and the associated complex vector space

$$V = \mathfrak{h}(E_8) = Q(E_8) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}\mathsf{a}_0 \oplus \mathbb{C}\mathsf{a}_1 \oplus \cdots \oplus \mathbb{C}\mathsf{a}_7 \subset \mathfrak{h}_8 \tag{1.62}$$

rather than the complexification $\mathfrak{h}_8 = \mathfrak{h}(E_8^{(1)}) = L_8 \otimes_{\mathbb{Z}} \mathbb{C}$ of the Picard lattice.

• Orthonormal basis of V

In view of

$$V = \mathfrak{h}(E_8) = \left\{\, h \in \mathfrak{h}_8 \;\middle|\; (\mathsf{c}|h) = (\mathsf{E}_8|h) = 0 \,\right\} \subset \mathfrak{h}_8 = \mathfrak{h}(E_8^{(1)}), \tag{1.63}$$

we take the orthonormal basis $v_0, v_1, \ldots, v_7$ of V defined by

$$\begin{aligned} v_1 &= \mathsf{H}_1 - \mathsf{E}_1 - \tfrac{1}{2}(\mathsf{H}_1 + \mathsf{H}_2 - \mathsf{E}_1 - \mathsf{E}_8) + \tfrac{1}{2}\mathsf{c} \\ v_2 &= \mathsf{H}_2 - \mathsf{E}_1 - \tfrac{1}{2}(\mathsf{H}_1 + \mathsf{H}_2 - \mathsf{E}_1 - \mathsf{E}_8) + \tfrac{1}{2}\mathsf{c} \\ v_j &= \mathsf{E}_{j-1} - \tfrac{1}{2}(\mathsf{H}_1 + \mathsf{H}_2 - \mathsf{E}_1 - \mathsf{E}_8) + \tfrac{1}{2}\mathsf{c} \quad (j = 3, \ldots, 8), \quad v_0 = -v_8 \end{aligned} \tag{1.64}$$

In terms of the orthonormal basis $v_0, v_1, \ldots, v_7$ of V, the simple roots $\mathsf{a}_0, \mathsf{a}_1, \ldots, \mathsf{a}_7$ of type E_8 are expressed as

$$\mathsf{a}_0 = \phi - v_0 - v_1 - v_2 - v_3, \quad \mathsf{a}_j = v_j - v_{j+1} \quad (j = 1, \ldots, 6), \quad \mathsf{a}_7 = v_7 + v_0. \tag{1.65}$$

where $\phi = \frac{1}{2}(v_0 + v_1 + \cdots + v_7)$.

• Hirota equations in the coordinates of $\boldsymbol{V}$

Setting $x_i = (v_i|\cdot) \in \mathfrak{h}_8^*$ $(i = 0, 1, \ldots, 7)$, we use the coordinates $x = (x_0, x_1, \ldots, x_7)$ for $V = \mathbb{C}^8$ defined by

$$\begin{aligned} x_1 &= \kappa_1 - \varepsilon_1 - \tfrac{1}{2}(\kappa_1 + \kappa_2 - \varepsilon_1 - \varepsilon_8) + \tfrac{1}{2}\delta \\ x_2 &= \kappa_2 - \varepsilon_1 - \tfrac{1}{2}(\kappa_1 + \kappa_2 - \varepsilon_1 - \varepsilon_8) + \tfrac{1}{2}\delta \\ x_j &= \varepsilon_{j-1} - \tfrac{1}{2}(\kappa_1 + \kappa_2 - \varepsilon_1 - \varepsilon_8) + \tfrac{1}{2}\delta \quad (j = 3, \ldots, 8), \quad x_0 = -x_8 \end{aligned} \tag{1.66}$$

instead of the coordinates $(\kappa, \varepsilon) = (\kappa_1, \kappa_2; \varepsilon_1, \ldots, \varepsilon_8)$ for $\mathfrak{h}_8$. On these variables the Kac translations T_{v_i} $(i = 0, 1, \ldots, 7)$ act as *shift operators* such that

$$T_{v_i}(x_i) = x_i - \delta, \quad T_{v_i}(x_j) = x_j \quad (j \in \{0, 1, \ldots, 7\}\,;\ j \neq i). \tag{1.67}$$

Then the $\tau = \tau_8$ is characterized as a $W(E_8)$-invariant τ function satisfying the non-autonomous Hirota equations

$$\sigma(x_j \pm x_k)T_{v_i}(\tau)T_{v_i}^{-1}(\tau) + \sigma(x_k \pm x_i)T_{v_j}(\tau)T_{v_j}^{-1}(\tau) + \sigma(x_i \pm x_j)T_{v_k}(\tau)T_{v_k}^{-1}(\tau) = 0 \tag{1.68}$$

for any triple $i, j, k \in \{0, 1, \ldots, 7\}$. In the final section, we will introduce the notion of an *ORG τ-function* as a function in these coordinates satisfying the $W(E_8)$-invariant system of difference equations including these Hirota equations, and use them to construct special solutions to the elliptic Painlevé equation.

2 Elliptic hypergeometric functions

2.1 Theta function and elliptic gamma function

Assuming that $\Omega = \mathbb{Z}1 \oplus \mathbb{Z}\tau$, $\mathrm{Im}\,\tau > 0$, we set $p = e(\tau) = e^{2\pi\sqrt{-1}\tau}$, $|p| < 1$. We also use the multiplicative notation

$$\theta(u;p) = (u;p)_\infty (p/u;p)_\infty; \quad \theta(p/u;p) = \theta(u;p), \quad \theta(pz;p) = -u^{-1}\theta(u;p) \tag{2.1}$$

for theta functions. Then $[z] = u^{-\frac{1}{2}}\,\theta(u;p)$, $u = e(z)$, satisfies the functional equation

$$[z \pm a][b \pm c] + [z \pm b][c \pm a] + [z \pm c][a \pm b] = 0, \tag{2.2}$$

where $[a \pm b] = [a+b][a-b]$. *Ruijsenaars' elliptic gamma function* is defined by

$$\begin{aligned} &\Gamma(u;p,q) = \frac{(pq/u;p,q)_\infty}{(u;p,q)_\infty}, \quad (u;p,q)_\infty = \prod_{i,j=0}^{\infty} (1 - p^i q^j u) \quad (|q| < 1), \\ &\Gamma(pq/u;p,q) = \frac{1}{\Gamma(u;p,q)}, \quad \frac{\Gamma(qu;p,q)}{\Gamma(u;p,q)} = \theta(u;p), \end{aligned} \tag{2.3}$$

and the *triple elliptic gamma function* by

$$\begin{aligned} &\Gamma(u;p,q,r) = (u;p,q,r)_\infty (pqr/u;p,q,r)_\infty, \quad (u;p,q,r)_\infty = \prod_{i,j,k=0}^{\infty} (1 - p^i q^j r^k u) \quad (|r| < 1), \\ &\Gamma(pqr/u;p,q,r) = \Gamma(u;p,q,r), \quad \frac{\Gamma(ru;p,q,r)}{\Gamma(u;p,q,r)} = \Gamma(u;p,q). \end{aligned}$$

2.2 Elliptic hypergeometric integrals (van Diejen, Spiridonov, Rains)

$$I(u_0, u_1, \ldots, u_{m-1}; p, q) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{\prod_{i=0}^{m-1} \Gamma(u_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z} \tag{2.4}$$

• Elliptic beta integral ($m = 6$)

Under the balancing condition $u_0 u_1 \cdots u_5 = pq$,

$$I(u_0, u_1, \ldots, u_5; p, q) = \prod_{0 \le i < j \le 5} \Gamma(u_i u_j; p, q) \tag{2.5}$$

• Two transformation formulas ($m = 8$)

Under the balancing condition $u_0 u_1 \cdots u_7 = p^2 q^2$,

$$\begin{aligned} &I(u_0, u_1, \ldots, u_7; p, q) \\ &= I(\widetilde{u}_0, \widetilde{u}_1, \ldots, \widetilde{u}_7; p, q) \prod_{0 \le i < j \le 3} \Gamma(u_i u_j; p, q) \prod_{4 \le i < j \le 7} \Gamma(u_i u_j; p, q) \\ &\widetilde{u}_i = u_i \sqrt{pq/u_0 u_1 u_2 u_3} \quad (i = 0, 1, 2, 3), \quad u_i \sqrt{pq/u_4 u_5 u_6 u_7} \quad (i = 4, 5, 6, 7) \end{aligned} \tag{2.6}$$

$$\begin{aligned} &I(u_0, u_1 \ldots, u_7; p, q) \\ &= I(\sqrt{pq}/u_0, \sqrt{pq}/u_1, \ldots, \sqrt{pq}/u_7; p, q) \prod_{0 \le i < j \le 7} \Gamma(u_i u_j; p, q) \end{aligned}$$

- **Three term relations**

$$T_{q,u_i}\Gamma(u_i z^{\pm\pm1}) = \Gamma(qu_i z^{\pm\pm1}) = \Gamma(u_i z^{\pm1}; p, q)\theta(u_i z^{\pm1}; p) \tag{2.7}$$

From the functional equation

$$\begin{aligned} u_k\theta(u_j u_k^{\pm1}; p)\theta(u_i z^{\pm1}; p) + u_i\theta(u_k u_i^{\pm1}; p)\theta(u_j z^{\pm1}; p) \\ + u_j\theta(u_i u_j^{\pm1}; p)\theta(u_k z^{\pm1}; p) = 0, \end{aligned} \tag{2.8}$$

we obtain the three term relations for $I(u) = I(u_0, \ldots, u_7; p, q)$:

$$u_k\theta(u_j u_k^{\pm1}; p)T_{q,u_i}I(u) + u_i\theta(u_k u_i^{\pm1}; p)T_{q,u_j}I(u) + u_j\theta(u_i u_j^{\pm1}; p)T_{q,u_k}I(u) = 0. \tag{2.9}$$

In additive variables $x = (x_0, x_1, \ldots, x_7)$ with $u_i = e(x_i)$ $(i = 0, 1, \ldots, 7)$ and δ with $\mathrm{Im}\delta > 0$, $q = e(\delta)$,

$$J(x) = e(-Q(x))I(u), \quad Q(x) = \tfrac{1}{2\delta}(x|x) = \tfrac{1}{2\delta}(x_0^2 + \cdots + x_7^2). \tag{2.10}$$

satisfies

$$[x_j \pm x_k]T_{x_i}^{\delta}J(x) + [x_k \pm x_i]T_{x_j}^{\delta}J(x) + [x_i \pm x_j]T_{x_k}^{\delta}J(x) = 0. \tag{2.11}$$

Three term relations + Bailey type transformations

$\Longrightarrow$ System of elliptic hypergeometric difference equations

2.3 Elliptic hypergeometric integrals of type BC_n

$$I^{(n)}(u_0,u_1,\ldots,u_{m-1};p,q,t)$$
$$= \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n!(2\pi\sqrt{-1})^n} \int_{C^n} \prod_{k=1}^{n} \frac{\prod_{i=0}^{m-1}\Gamma(u_i z_k^{\pm 1};p,q)}{\Gamma(z_k^{\pm 2};p,q)} \prod_{1\le k<l\le n} \frac{\Gamma(t z_k^{\pm 1} z_l^{\pm 1};p,q)}{\Gamma(z_k^{\pm 1} z_l^{\pm 1};p,q)} \frac{dz_1\cdots dz_n}{z_1\cdots z_n}. \tag{2.12}$$

• **Evaluation formula ($m=6$)** (van Diejen-Spiridonov 2001, Rains)
Under the balancing condition $u_0 u_1 \cdots u_5 t^{2n-2} = pq$,

$$I^{(n)}(u_0,u_1,\ldots,u_5;p,q,t) = \prod_{i=1}^{n} \left(\frac{\Gamma(t^i;p,q)}{\Gamma(t;p,q)} \prod_{0\le k<l\le 5} \Gamma(t^{i-1}u_k u_l;p,q) \right) \tag{2.13}$$

(Elliptic extension of Gustafson's q-Selberg integral)

• **BC_n elliptic hypergeometric integral ($m=8$)** (Rains)
When $t=q$, the sequence of integrals $I^{(n)}(u_0,\ldots,u_7;p,q,q)$ $(n=0,1,2,\ldots)$ provides with a *hypergeometric τ-function* of the E_8 elliptic Painlevé equation (Rains 2005, Noumi: arXiv:1604.06869). In this case, $I^{(n)}(u_0,\ldots,u_7;p,q,q)$ can also be expressed as an $n\times n$ Casorati deteminant whose entries are elliptic hypergeometric integrals in one variable.

3 $eP(E_8^{(1)})$ as a system of non-autonomous Hirota equations

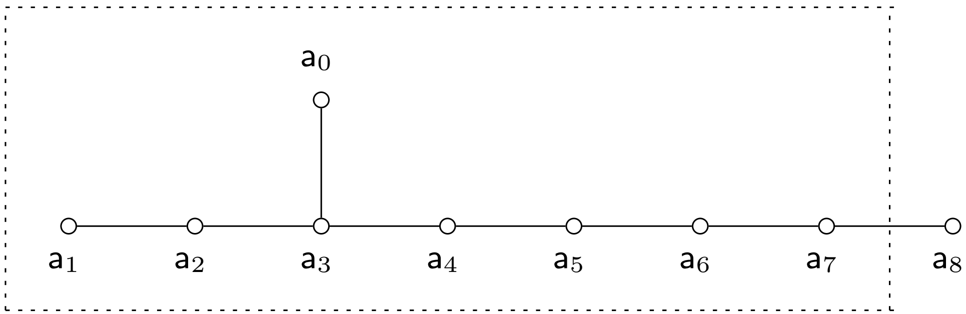
3.1 A standard realization of the root lattice $P = Q(E_8)$

$$V = \mathbb{C}^8 = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_7; \quad (v_i|v_j) = \delta_{ij} \quad (i, j \in \{0, 1, \ldots, 7\}). \tag{3.1}$$

$$\begin{gathered} P = \left\{ a \in \mathbb{Z}^8 \cup (\phi + \mathbb{Z}^8) \mid (\phi|a) \in \mathbb{Z} \right\} \\ \phi = \tfrac{1}{2}(1,1,1,1,1,1,1,1) = \tfrac{1}{2}(v_0 + v_1 + \cdots + v_7) \end{gathered} \tag{3.2}$$

$$\begin{aligned} &\Delta(E_8) = \{ \alpha \in P \mid (\alpha|\alpha) = 2 \}, \quad |\Delta(E_8)| = 240. \\ &(1): \ \pm v_i \pm v_j \quad (0 \le i < j \le 7) \qquad \cdots \ \tbinom{8}{2} \cdot 4 = 112 \\ &(2): \ \tfrac{1}{2}(\pm v_0 \pm \cdots \pm v_7) \quad (\text{even number of } - \text{ signs}) \quad \cdots \ 2^7 = 128 \end{aligned} \tag{3.3}$$

$$\sum_{a \in P} q^{(a|a)} = 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + \cdots \tag{3.4}$$



$$\begin{aligned} \mathsf{a}_0 &= \phi - v_0 - v_1 - v_2 - v_3, \\ \mathsf{a}_j &= v_j - v_{j+1} \quad (j = 1, \ldots, 6) \\ \mathsf{a}_7 &= v_7 + v_0 \\ \mathsf{a}_8 &= \mathsf{c} - \phi \end{aligned}$$

3.2 ORG τ-function (Ohta-Ramani-Grammaticos)

Definition A set of $2l$ vectors $\{\pm a_1, \dots, \pm a_l\}$ in V is called a C_l-*frame* if

$$\begin{aligned} &(1) \quad (a_i|a_j) = \delta_{ij} \quad (i, j \in \{1, \dots, l\}), \\ &(2) \quad \{\, \pm a_i \pm a_j \mid 1 \le i < j \le l \,\} \cup \{\, \pm 2a_i \mid 1 \le i \le l \,\} \subset P. \end{aligned} \tag{3.5}$$

There are 2160 vectors $a \in \frac{1}{2}P$ with $(a|a) = 1$. Let $\mathcal{C}_l$ be the set of all C_l frames in P:

$$(\tfrac{1}{2}P)_1 = \bigsqcup_{A \in \mathcal{C}_8} A; \quad |\mathcal{C}_8| = 135, \quad |\mathcal{C}_3| = 135 \cdot \binom{8}{3} = 7560 \tag{3.6}$$

Hereafter we use the notation $[u] = \sigma(u|\Omega)$ or $[u] = z^{-\frac{1}{2}}\theta(z;p)$, $z = e^{2\pi\sqrt{-1}u}$ so that

$$[\beta \pm \gamma][u \pm \alpha] + [\gamma \pm \alpha][u \pm \beta] + [\alpha \pm \beta][u \pm \gamma] = 0. \tag{3.7}$$

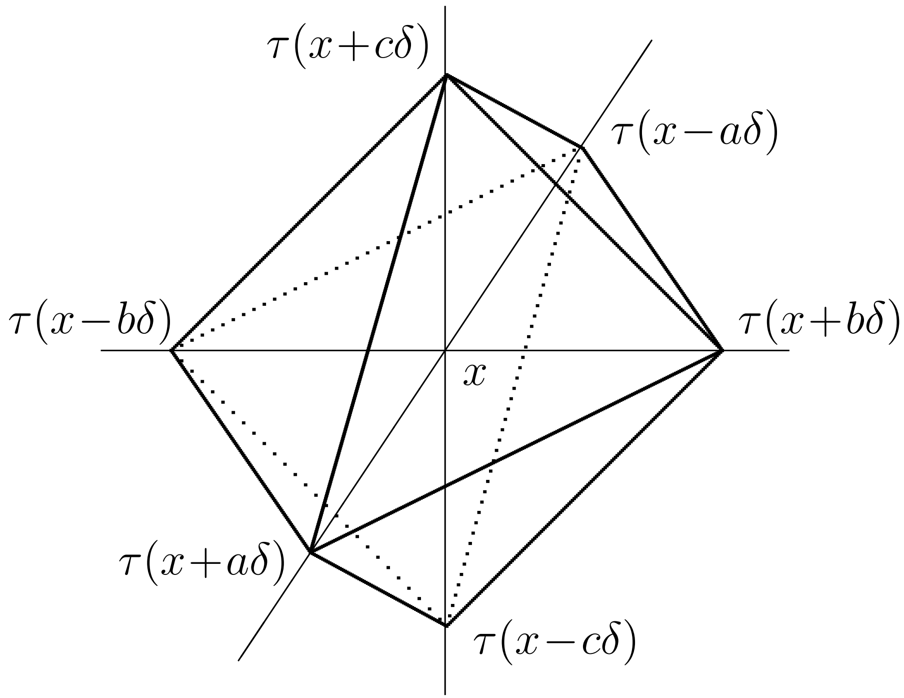
Fix a nonzero constant δ. Let D be a subset of $V = \mathbb{C}^8$ such that $D + P\delta = D$.

Definition A function $\tau(x)$ defined over D is called an *ORG τ-function* if it satisfies the non-autonomous Hirota equation

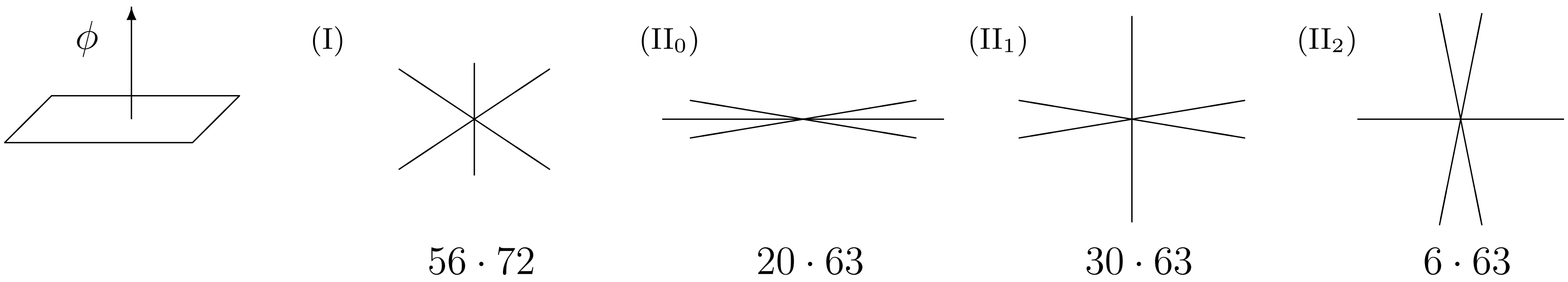
$$[(b \pm c|x)]\,\tau(x \pm a\delta) + [(c \pm a|x)]\,\tau(x \pm b\delta) + [(a \pm b|x)]\,\tau(x \pm c\delta) = 0 \tag{3.8}$$

for any C_3-frame $\{\pm a, \pm b, \pm c\}$ in $P = Q(E_8)$.

Each of the six points $x \pm a\delta$, $x \pm b\delta$, $x \pm c\delta$ belongs to D if and only if the others do. In this formulation $eP(E_8)$ is a $W(E_8)$-invariant system of 7560 non-autonomous Hirota equations.



$$[(b \pm c|x)]\tau(x \pm a\delta) + [(c \pm a|x)]\tau(x \pm b\delta) + [(a \pm b|x)]\tau(x \pm c\delta) = 0$$



Four types of 7560 C_3-frames

3.3 $eP(E_8)$ τ-function as an infinite chain of $eP(E_7)$ τ-functions

In the E_8 root lattice $P = Q(E_8)$, the E_7 root lattice is realized as

$$Q(E_7) = \{a \in P \mid (\phi|a) = 0\ \} \subset P = Q(E_8); \quad \Delta(E_7) = \Delta(E_8)^{\perp\phi}. \tag{3.9}$$

Fixing a constant $c \in \mathbb{C}$, we consider the union of parallel hyperplanes

$$D_c = \bigsqcup_{n\in\mathbb{Z}} H_{c+n\delta}; \quad H_{c+n\delta} = \{x \in V \mid (\phi|x) = c + n\delta\} \quad (n \in \mathbb{Z}). \tag{3.10}$$

Then an ORG τ-function $\tau(x)$ on D_c can be regarded as a chain $\{\tau^{(n)}(x)\}_{n\in\mathbb{Z}}$ of $eP(E_7)$ τ-functions on parallel hyperplanes by setting $\tau^{(n)} = \tau|_{H_{c+n\delta}}$ $(n \in \mathbb{Z})$.

Four types of bilinear equations corresponding to the types I, II_0, II_1, II_2 of C_3-frames:

$$\begin{aligned}
(\mathrm{I})_{n+\frac{1}{2}}:\ & [(a_1 \pm a_2|x)]\tau^{(n)}(x - a_0\delta)\tau^{(n+1)}(x + a_0\delta) + \cdots = 0 \\
(\mathrm{II}_0)_n:\ & [(a_1 \pm a_2|x)]\tau^{(n)}(x - a_0\delta)\tau^{(n)}(x + a_0\delta) + \cdots = 0 \\
(\mathrm{II}_1)_n:\ & [(a_1 \pm a_2|x)]\tau^{(n-1)}(x - a_0\delta)\tau^{(n+1)}(x + a_0\delta) \\
& = [(a_0 \pm a_2|x)]\tau^{(n)}(x \pm a_1\delta) - [(a_0 \pm a_1|x)]\tau^{(n)}(x \pm a_2\delta) \\
(\mathrm{II}_2)_n:\ & [(a_1 \pm a_2|x)]\tau^{(n)}(x \pm a_0\delta) \\
& = [(a_0 \pm a_2|x)]\tau^{(n-1)}(x - a_1\delta)\tau^{(n+1)}(x + a_1\delta) - \cdots
\end{aligned} \tag{3.11}$$

Definition A meromorphic ORG τ function $\tau(x)$ on $D_c = \bigsqcup_{n\in\mathbb{Z}} H_{c+n\delta}$ is called a *hypergeometric τ-function* if

$$\tau^{(n)}(x) = 0 \quad (n < 0), \qquad \tau^{(0)}(x) \neq 0. \tag{3.12}$$

Theorem C: *Let $\tau^{(0)}(x)$, $\tau^{(1)}(x)$ be nonzero meromorphic functions on H_c, $H_{c+\delta}$ respectively. Suppose that they satisfy*

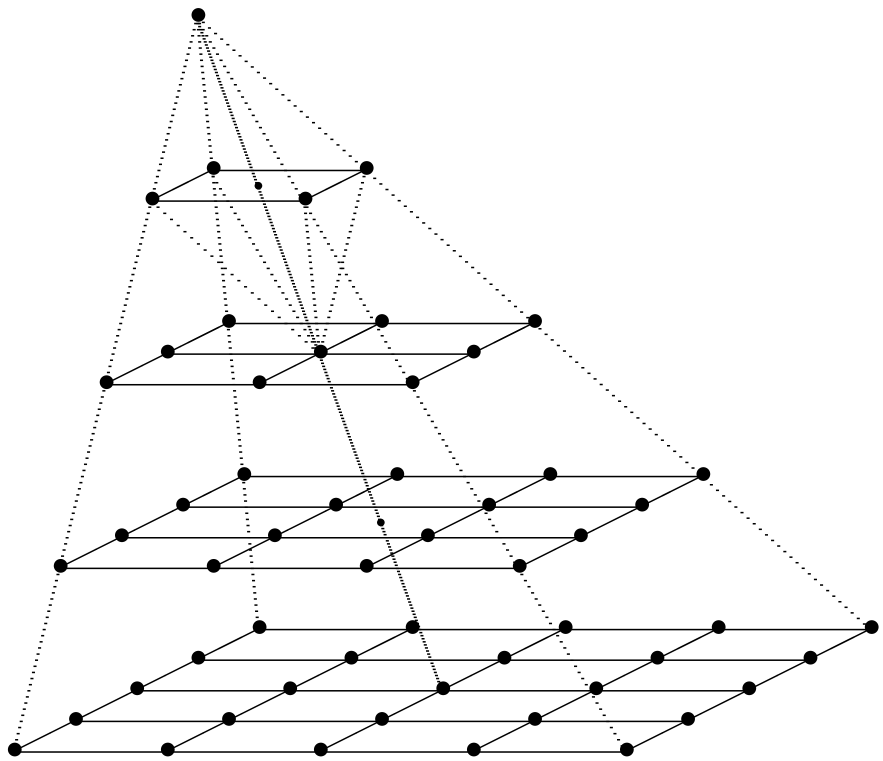
$$[(a_0 \pm a_2|x)]\tau^{(0)}(x \pm a_1\delta) = [(a_0 \pm a_1|x)]\tau^{(0)}(x \pm a_2\delta) \tag{3.13}$$

for any C_3-frame of type II_1*, and*

$$[(a_1 \pm a_2|x)]\tau^{(0)}(x - a_0\delta)\tau^{(1)}(x + a_0\delta) + \cdots = 0 \tag{3.14}$$

for any C_3-frame of type I*. Then these exists a unique a hypergeometric τ-function $\tau(x)$ on D_c such that $\tau^{(0)} = \tau|_{H_c}$ and $\tau^{(1)} = \tau|_{H_{c+\delta}}$.*

Toda equations produce 2-directional Casorati determinants



$$(\mathrm{II}_1)_n: \ [(a_1 \pm a_2|x)]\tau^{(n-1)}(x - a_0\delta)\tau^{(n+1)}(x + a_0\delta)$$
$$= [(a_0 \pm a_2|x)]\tau^{(n)}(x \pm a_1\delta) - [(a_0 \pm a_1|x)]\tau^{(n)}(x \pm a_2\delta)$$

3.4 Determinant representation of hypergeometric $\boldsymbol{\tau}$-functions

Theorem D: *Under the assumption of Theorem C, suppose that* $\tau^{(1)}(x)$ *is expressed as* $\tau^{(1)}(x) = \gamma^{(1)}(x)\,\varphi(x)$ *with a nonzero meromorphic function* $\gamma^{(1)}(x)$ *satisfying*

$$[(a_0 + a_2|x)]\gamma^{(1)}(x \pm a_1\delta) = [(a_0 + a_1|x)]\gamma^{(1)}(x \pm a_2\delta) \tag{3.15}$$

for a C_3*-frame of type* II_1 *with* $(\phi|a_0) = 1$, $(\phi|a_1) = (\phi|a_2) = 0$*. Then the components* $\tau^{(n)}(x)$ *of the hypergeometric* τ*-function* $\tau(x)$ *are expressed as follows in terms of 2-directional Casorati determinants:*

$$\begin{aligned}
&\tau^{(n)}(x) = \gamma^{(n)}(x)K^{(n)}(x) \quad (x \in H_{c+n\delta};\ n = 0, 1, 2, \ldots) \\
&K^{(n)}(x) = \det\left(\varphi_{ij}^{(n)}(x)\right)_{i,j=1}^{n} \\
&\varphi_{ij}(x) = \varphi^{(n)}(x-(n-1)a_0\delta+(n+1-i-j)a_1\delta+(j-i)a_1\delta) \quad (1 \le i, j \le n).
\end{aligned} \tag{3.16}$$

The gauge factors $\gamma^{(n)}(x)$ *are determined inductively from* $\gamma^{(0)}(x) = \tau^{(0)}(x)$, $\gamma^{(1)}(x)$ *by*

$$[(a_0 \pm a_2|x)]\gamma^{(n-1)}(x - a_0\delta)\gamma^{(n+1)}(x + a_0\delta) = [(a_1 \pm a_2|x)]\gamma^{(n)}(x \pm a_1\delta). \tag{3.17}$$

The Toda equation $(\mathrm{II}_1)_n$ corresponds to the *Lewis-Carroll formula* for determinants.

3.5 $W(E_7)$-invariant hypergeometric τ-function

We consider the case $[\zeta] = z^{-\frac{1}{2}}\theta(z;p)$, $z = e(\zeta) = e^{2\pi\sqrt{-1}\zeta}$. An example of hypergeometric τ-function for $eP(E_8)$ is given by the multiple elliptic hypergeometric integrals:

$$\begin{aligned} &I^{(n)}(u;p,q,q) = I^{(n)}(u_0,\ldots,u_7;p,q,q) \\ &= \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n!(2\pi\sqrt{-1})^n} \int_{C^n} \prod_{k=1}^{n} \frac{\prod_{i=0}^{7}\Gamma(u_i z_k^{\pm1};p,q)}{\Gamma(z_k^{\pm2};p,q)} \prod_{1\le k<l\le n} \theta(z_k^{\pm1} z_l^{\pm1};p) \frac{dz_1\cdots dz_n}{z_1\cdots z_n}. \end{aligned} \tag{3.18}$$

We consider to construct a hypergeometric τ-function on

$$D_\tau = \bigsqcup_{n\in\mathbb{Z}} H_{\tau+n\delta} \quad \text{with} \quad p = e(\tau), \quad q = e(\delta). \tag{3.19}$$

• $\boldsymbol{\tau^{(0)}(x)}$ The system of first order difference equations for $\tau^{(0)}(x)$ $(x \in H_\tau)$ is solved by a product of triple elliptic gamma functions:

$$\tau^{(0)}(x) = \prod_{0\le i<j\le 7} \Gamma(qu_iu_j;p,q,q) \quad (x \in H_\tau) \tag{3.20}$$

in the multiplicative variables $u_i = e(x_i)$ $(i = 0,1,\ldots,7)$, where

$$\begin{aligned} &\Gamma(u;p,q,r) = (u;p,q,r)_\infty (pqr/u;p,q,r)_\infty, \\ &(u;p,q,r)_\infty = \prod_{i,j,k=0}^{\infty} (1 - p^i q^j r^k u) \quad (|p|,|q|,|r| < 1). \end{aligned} \tag{3.21}$$

• $\boldsymbol{\tau^{(1)}(x)}$ Then, the system of Hirota equations between $\tau^{(0)}(x)$ and $\tau^{(1)}(x)$ is solved by the elliptic hypergeometric integral:

$$\begin{aligned} &\tau^{(1)}(x) = \prod_{0\le i<j\le 7} \Gamma(u_i u_j; p, q, q)\, e(-Q(x)) I(u; p, q) \quad (x \in H_{\tau+\delta}), \\ &Q(x) = \tfrac{1}{2\delta}(x|x), \\ &I(u; p, q) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{\prod_{i=0}^{7} \Gamma(u_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}. \end{aligned} \tag{3.22}$$

Note that the condition $x \in H_{\tau+\delta}$ corresponds to the balancing condition $u_0 u_1 \cdots u_7 = p^2 q^2$ in multiplicative variables. In fact, the system of linear difference equations for $\tau^{(1)}(x)$ reduces to the three term relations

$$[x_j \pm x_k] T^{\delta}_{x_i} J(x) + [x_k \pm x_i] T^{\delta}_{x_j} J(x) + [x_i \pm x_j] T^{\delta}_{x_k} J(x) = 0. \tag{3.23}$$

for $J(x) = e(-Q(x)) I(u; p, q)$.

- **Determinant formula for $\boldsymbol{\tau^{(n)}(x)}$**

Using the decomposition $\tau^{(1)}(x) = \gamma^{(1)}(x)\varphi(x)$ with $\varphi(x) = J(x)$, by Theorem D we know that $\tau^{(n)}(x)$ has the determinant formula

$$\begin{aligned} &\tau^{(n)}(x) = \gamma^{(n)}(x) \det\left(\varphi_{ij}^{(n)}(x)\right)_{i,j=1}^{n} \\ &\varphi_{ij}^{(n)}(x) = \varphi(x - (n-1)a_0\delta + (n+1-i-j)a_1\delta + (j-i)a_2\delta) \end{aligned} \tag{3.24}$$

for any C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ of type II_1 with $(\phi|a_0) = 1$.

- **$\boldsymbol{\tau^{(n)}(x)}$ as a multiple elliptic hypergeometric integral**

This 2-directional Casorati determinant can be rewritten into multiple integrals. By Warnaar's elliptic extension of the Krattenthaler determinant, we finally obtain the expression of $\tau^{(n)}(x)$ in terms of the multiple elliptic hypergeometric integral of Rains:

$$\begin{aligned} &\tau^{(n)}(x) = p^{\binom{n}{2}} \prod_{0\le i<j\le 7} \Gamma(q^{1-n}u_i u_j; p, q, q)\ e(-nQ(x))\ I^{(n)}(q^{\frac{1}{2}(1-n)}u; p, q, q), \\ &I^{(n)}(u; p, q, q) \\ &= \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n! (2\pi\sqrt{-1})^n} \int_{C^n} \prod_{k=1}^{n} \frac{\prod_{i=0}^{7} \Gamma(u_i z_k^{\pm 1}; p, q)}{\Gamma(z_k^{\pm 2}; p, q)} \prod_{1\le k<l\le n} \theta(z_k^{\pm 1} z_l^{\pm 1}; p) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}. \end{aligned} \tag{3.25}$$

The sequence $\tau^{(n)}(x)$ $(n = 0, 1, 2, \ldots)$ determined as above provides a $W(E_7)$-invariant hypergeometric τ-function. This fact follows from the $W(E_7)$-invariance of $\tau^{(0)}(x)$, $\tau^{(1)}(x)$ and the uniqueness of extension to $\tau^{(n)}(x)$.

3.6 From the determinant representation to the multiple integral

We compute the determinant

$$K^{(n)}(x) = \det\left(\varphi_{ij}^{(n)}(x)\right)_{i,j=1}^{n},$$
$$\varphi_{ij}^{(n)}(x) = I(q^{n-i}t_0, q^{n-j}t_1, q^{j-1}t_2, q^{i-1}t_3, t_4, t_5, t_6, t_7; p, q). \tag{3.26}$$
$$t_i = u_i\sqrt{pq/u_0u_1u_2u_3} \ \ (i = 0, 1, 2, 3), \quad t_i = u_i\sqrt{pq/u_4u_5u_6u_7} \ \ (i = 4, 5, 6, 7).$$

Hence $\varphi_{ij}^{(n)}(x)$ is expressed as

$$\varphi_{ij}^{(n)}(x) = \kappa \int_C h(z) f_i(z) g_j(z) \frac{dz}{z}, \quad \kappa = \frac{(p;p)_\infty (q;q)_\infty}{4\pi\sqrt{-1}},$$
$$h(z) = \frac{\prod_{k=0}^{7} \Gamma(t_k z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)}, \tag{3.27}$$
$$f_i(z) = \theta(t_0 z^{\pm 1}; p; q)_{n-i}\, \theta(t_3 z^{\pm 1}; p; q)_{i-1},$$
$$g_j(z) = \theta(t_1 z^{\pm 1}; p; q)_{n-j}\, \theta(t_2 z^{\pm 1}; p; q)_{j-1},$$

for $i, j = 1, 2, \ldots, n$, where $\theta(z; p; q)_k = \theta(z; p)\theta(qz; p) \cdots \theta(q^{k-1}z; p)$ $(k = 0, 1, 2, \ldots)$.

We now rewrite the determinant $K^{(n)}(x)=\det\left(\varphi_{ij}^{(n)}(x)\right)_{i,j=1}^{n}$ as

$$\begin{aligned}K^{(n)}(x)&=\frac{1}{n!}\sum_{\sigma,\tau\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_{k=1}^{n}\varphi_{\sigma(k),\tau(k)}^{(n)}(x)\\&=\frac{\kappa^n}{n!}\sum_{\sigma,\tau\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\int_{C^n}\prod_{k=1}^{n}h(z_k)f_{\sigma(k)}(z_k)g_{\sigma(k)}(z_k)\frac{dz_1\cdots dz_n}{z_1\cdots z_n}\\&=\frac{\kappa^n}{n!}\int_{C^n}h(z_1)\cdots h(z_n)\det(f_j(z_i))_{i,j=1}^{n}\det(g_j(z_i))_{i,j=1}^{n}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}.\end{aligned}\tag{3.28}$$

$$\begin{aligned}f_j(z)&=\theta(t_0z^{\pm1};p;q)_{n-j}\,\theta(t_3z^{\pm1};p;q)_{j-1},\\g_j(z)&=\theta(t_1z^{\pm1};p;q)_{n-j}\,\theta(t_2z^{\pm1};p;q)_{j-1},\end{aligned}$$

Then the determinants $\det(f_j(z_i))_{i,j=1}^{n}$, $\det(g_j(z_i))_{i,j=1}^{n}$ can be evaluated by means of Warnaar's elliptic extension of the *Krattenthaler determinant*.

Lemma [Warnaar 2002] *For a set of complex variables* $(z_1,\ldots,z_n)$ *and two parameters* a,b, *one has*

$$\begin{aligned}&\det\left(\theta(az_i^{\pm1};p;q)_{j-1}\theta(bz_i^{\pm1};p;q)_{n-j}\right)_{i,j=1}^{n}\\&=q^{\binom{n}{3}}a^{\binom{n}{2}}\prod_{k=1}^{n}\theta(b(q^{k-1}a)^{\pm1};p;q)_{n-k}\prod_{1\le i<j\le n}z_i^{-1}\theta(z_iz_j^{\pm};p).\end{aligned}\tag{3.29}$$

Hence we obtain

$$
\begin{aligned}
K^{(n)}(x) &= \det\left(\varphi_{ij}^{(n)}(x)\right)_{i,j=1}^{n} &(3.30)\\
&= \frac{\kappa^n}{n!}\int_{C^n} h(z_1)\cdots h(z_n)\det(f_j(z_i))_{i,j=1}^{n}\det(g_j(z_i))_{i,j=1}^{n}\frac{dz_1\cdots dz_n}{z_1\cdots z_n} &(3.31)\\
&= d^{(n)}(x)\, I^{(n)}(t;p,q,q)
\end{aligned}
$$

where

$$
I^{(n)}(t;p,q,q) = \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n!(2\pi\sqrt{-1})^n}\int_{C^n}\prod_{i=1}^{n} h(z_i)\prod_{1\le i<j\le n}\theta(z_i^{\pm 1}z_j^{\pm 1};p)\frac{dz_1\cdots dz_n}{z_1\cdots z_n}, \tag{3.32}
$$

and

$$
\begin{aligned}
d^{(n)}(x) &= q^{2\binom{n}{3}}(t_2t_3)^{\binom{n}{2}}\prod_{k=1}^{n}\theta(t_0(q^{k-1}t_3)^{\pm 1};p;q)_{n-k}\theta(t_1(q^{k-1}t_2)^{\pm 1};p;q)_{n-k}\\
&= q^{2\binom{n}{3}}(pq/u_0u_1)^{\binom{n}{2}}\prod_{(i,j)=(0,3),(1,2)}\prod_{k=1}^{n}\theta(q^{1-n}u_iu_j;p;q)_{k-1}\theta(q^{k-n}u_i/u_k;p;q)_{k-1}.
\end{aligned}
\tag{3.33}
$$

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