Basic hypergeometry and biorthogonal functions related to supersymmetric dualities

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Outline

- Discrete-continuous beta integrals (joint with Ilmar Gahramanov)
- Decoupling phenomenon for elliptic beta integral (after Spiridonov)
- Decoupling phenomenon for discrete-continuous beta integrals

Euler's beta integral

$$\int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

Γ classical gamma function.

Orthogonal polynomials

$$\int_0^1 P_m(t)P_n(t)\,t^{a-1}(1-t)^{b-1}\,dt = C\delta_{mn}$$

are Jacobi polynomials, given by Gauss's hypergeometric function ${}_2F_1$.

More general beta integrals

Many more integral evaluations (discrete and continuous) are called "beta integrals".

Give total mass of measure for hypergeometric orthogonal polynomials, or biorthogonal rational functions.

Spiridonov's beta integral

Top level result for one-variable beta integrals:

$$\oint \frac{\prod_{j=1}^{6} \Gamma(t_j z)\Gamma(t_j/z)}{\Gamma(z^2)\Gamma(z^{-2})} \frac{dz}{2\mathrm{i}\pi z} = \frac{2}{(p;p)_\infty (q;q)_\infty} \prod_{1\leq i<j\leq 6} \Gamma(t_i t_j),$$

$$\Gamma(z) = \prod_{j,k=0}^{\infty} \frac{1-p^{j+1}q^{k+1}/z}{1-p^j q^k z},$$

$(p;p)_\infty = \prod_{j=1}^{\infty}(1-p^j)$ and $t_1 \cdots t_6 = pq$.

Throughout, $\oint$ is over contour separating sequences of poles converging to infinity from sequences converging to zero.

Beta integrals from quantum field theory

Dolan and Osborn (2009) interpreted Spiridonov's integral as index identity for 4D quantum field theories.

3D theories lead to basic (=trigonometric) hypergeometric integrals.
New type of beta integrals appear, with coupled discrete and continuous integration.

Additional motivation from solvable lattice models (talks of Derkachov, Gahramanov, poster of Kels) and proposed relations to three-manifold invariants.

Discrete-continuous beta integrals

Top level result:

$$\sum_{x=-\infty}^{\infty} \oint \frac{(1-q^x z^2)(1-q^x z^{-2})}{q^x z^{6x}} \times \prod_{j=1}^{6} \frac{(q^{1+x/2}/b_j z)_\infty (q^{1-x/2} z/b_j)_\infty}{(q^{N_j+x/2} b_j z)_\infty (q^{N_j - x/2} b_j/z)_\infty} \frac{dz}{2\mathrm{i}\pi z} = \frac{2}{\prod_{j=1}^{6} q^{\binom{N_j}{2}} b_j^{N_j}} \prod_{1\le i<j\le 6} \frac{(q/b_i b_j)_\infty}{(b_i b_j q^{N_i+N_j})_\infty},$$

$(x)_\infty = \prod_{j=0}^{\infty}(1-xq^j)$, $|q|<1$, b_j generic and N_j integer parameters, $b_1 \cdots b_6 = q$ and $N_1 + \cdots + N_6 = 0$.
Integration contour as before. Depends on x.

Scheme of discrete-continuous beta integrals

With Gahramanov, we found "scheme" of four evaluations.
All four evaluations were interpreted in terms of 3D QFT.

In this talk we call them E_1, E_3, E_4, E_6.
E_k contains a product $\prod_{j=1}^{k}$.

$$\begin{tikzcd} & E_6 \arrow[dl] \arrow[dr] & \\ E_4 \arrow[dr] & & E_3 \arrow[dl] \\ & E_1 & \end{tikzcd}$$

Arrows indicate loss of complexity, not rigorous limits.

Remarks on discrete-continuous integrals

Special cases of E_1 are due to Krattenthaler, Spiridonov and Vartanov (2011), Kapustin and Willett (2011) and Yokoyama (2012). The other three evaluations are new.

There are similar integrals at elliptic level, with *finite* discrete component (Kels 2015, Spiridonov 2016).

There are rational ($q = 1$) and hyperbolic ($|q| = 1$) versions (Kels 2014, Gahramanov and Kels 2016).

Several (all?) results can be viewed as star-triangle relation for solvable models, see e.g. Gahramanov and Spiridonov (2015) for E_6.

In an expected relation between 3D QFT and three-manifold invariants (Dimofte, Gaiotto, Gukov 2014), E_3 should correspond to a Pachner move.

Basic hypergeometric notation

$$(a)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}), \qquad k = 0, 1, 2, \dots, \infty$$

and $(a)_{-k} = 1/(aq^{-k})_k$.
Also $(ax^{\pm})_k = (ax)_k(a/x)_k$.

$${}_r\psi_r\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix}; z\right) = \sum_{k=-\infty}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_r)_k} z^k,$$

$$\begin{aligned} &{}_{r+1}W_r(a; b_1, \dots, b_{r-2}; z) \\ &\qquad = \sum_{k=0}^{\infty} \frac{1-aq^{2k}}{1-a} \frac{(a)_k(b_1)_k \cdots (b_{r-2})_k}{(q)_k(aq/b_1)_k \cdots (aq/b_{r-2})_k} z^k. \end{aligned}$$

Proof of E_6

We sketch proof of E_6.
(E_4 and E_1 are much easier. Our proof of E_3 is different.)

$$\sum_{x=-\infty}^{\infty} \oint \frac{(1-q^x z^2)(1-q^x z^{-2})}{q^x z^{6x}} \times \prod_{j=1}^{6} \frac{(q^{1+x/2}/b_j z)_\infty (q^{1-x/2} z/b_j)_\infty}{(q^{N_j+x/2} b_j z)_\infty (q^{N_j - x/2} b_j/z)_\infty} \frac{dz}{2\mathrm{i}\pi z} = \frac{2}{\prod_{j=1}^{6} q^{\binom{N_j}{2}} b_j^{N_j}} \prod_{1\le i<j\le 6} \frac{(q/b_i b_j)_\infty}{(b_i b_j q^{N_i+N_j})_\infty},$$

$b_1 \cdots b_6 = q$ and $N_1 + \cdots + N_6 = 0$.
Replacing z by $zq^{-x/2}$, the contour of integration can be chosen independently of x.

Proof of E_6, continued

Changing sum and integral then gives

$$\oint (1-z^2)(1-z^{-2}) \prod_{j=1}^{6} \frac{(qz^{\pm}/b_j)_\infty}{(q^{N_j} b_j z^{\pm})_\infty} \, {}_8\psi_8\left(\begin{matrix} q/z, -q/z, b_1/z, \dots, b_6/z \\ 1/z, -1/z, q/b_1 z, \dots, q/b_6 z \end{matrix}; q\right) \frac{dz}{2\mathrm{i}\pi z}.$$

Consider

$$f(z) = \frac{\prod_{j=1}^{6} (qz^{\pm}/b_j)_\infty}{(qz^{\pm 2})_\infty} \times {}_8\psi_8\left(\begin{matrix} q/z, -q/z, b_1/z, \dots, b_6/z \\ 1/z, -1/z, q/b_1 z, \dots, q/b_6 z \end{matrix}; q\right).$$

It is analytic for $z \neq 0$ and satisfies

$$f(z) = f(z^{-1}), \qquad f(qz) = f(z)/qz^2.$$

Proof of E_6, continued

By generalities on theta functions,

$$f(z) = \frac{f(d)}{\theta(cd)\theta(c/d)}\,\theta(cz)\theta(c/z) + \frac{f(c)}{\theta(dc)\theta(d/c)}\,\theta(dz)\theta(d/z),$$

where

$$\theta(z) = \prod_{j=0}^{\infty}(1 - q^j z)(1 - q^{j+1}/z).$$

This is Jackson's ${}_8\psi_8$-transformation,
a special case of Slater's ${}_{2r}\psi_{2r}$-transformation.
Simple proof above due to Ito and Sanada, 2008.

Proof of E_6, concluded

Choose $c = b_5$, $d = b_6$. Then, $f(c)$ and $f(d)$ are ${}_8W_7$-series. We have

$$\oint {}_8\psi_8 = {}_8W_7 \oint + {}_8W_7 \oint .$$

The integrals are Nasrallah–Rahman beta integral ($p = 0$ case of Spiridonov's beta).

Sum of two ${}_8W_7$ is computed by non-terminating Jackson summation. (A different Jackson.)

Two-index biorthogonality

Rahman constructed biorthogonal rational functions with

$$\oint q_k r_l (\cdots) = C\delta_{kl},$$

where $\oint(\cdots)$ is Nasrallah–Rahman beta integral (se below). These functions are ${}_{10}W_9$-series.

Spiridonov found elliptic extensions with *two-index* biorthogonality

$$\oint Q_{k_1} R_{l_1} \tilde{Q}_{k_2} \tilde{R}_{l_2} (\cdots) = C\delta_{k_1 l_1}\delta_{k_2 l_2},$$

where $\sim$ means $p \leftrightarrow q$ and $\oint(\cdots)$ is Spiridonov beta.

We will explain this as a consequence of "decoupling phenomenon".

Shifting parameters

In the integral

$$\oint \frac{\prod_{j=1}^6 \Gamma(t_j z)\Gamma(t_j/z)}{\Gamma(z^2)\Gamma(z^{-2})}\frac{dz}{2\mathrm{i}\pi z} = \frac{2}{(p;p)_\infty (q;q)_\infty} \prod_{1\leq i<j\leq 6} \Gamma(t_i t_j)$$

replace t_j by $t_j p^{k_j} q^{l_j}$, where $k_1 + \cdots + k_6 = l_1 + \cdots + l_6 = 0$.

Note that

$$\frac{\Gamma(p^k q^l z)}{\Gamma(z)} = \frac{(-1)^{kl}}{z^{kl} p^{l\binom{k}{2}} q^{k\binom{l}{2}}} (z)_k [z]_l,$$

where

$$(z)_k = (z;q,p)_k = \theta(z;p)\theta(qz;p)\cdots\theta(q^{k-1}z;p),$$

$$[z]_k = (z;p,q)_k = \theta(z;q)\theta(pz;q)\cdots\theta(p^{k-1}z;q).$$

Parameter shifts give decoupling

After simplification we get

$$\oint \frac{\prod_{j=1}^{6} [t_j z^{\pm}]_{k_j} (t_j z^{\pm})_{l_j} \Gamma(t_j z^{\pm})}{\Gamma(z^{\pm 2})} \frac{dz}{2\mathrm{i}\pi z} = \frac{2}{(p;p)_\infty (q;q)_\infty} \prod_{1\leq i<j\leq 6} \Gamma(t_i t_j) \, [t_i t_j]_{k_i+k_j} (t_i t_j)_{l_i+l_j}.$$

On the right, k_j and l_j are decoupled!

Decoupling phenomenon

If

$$\mu(f) = \frac{(p;p)_\infty (q;q)_\infty}{2\prod_{1\leq i<j\leq 6}\Gamma(t_i t_j)} \oint f(z) \frac{\prod_{j=1}^{6}\Gamma(t_j z^{\pm})}{\Gamma(z^{\pm 2})} \frac{dz}{2\mathrm{i}\pi z}$$

we find that

$$\mu(fg) = \mu(f)\mu(g)$$

for f, g in the linear span of

$$\prod_{j=1}^{6} [t_j z^{\pm}]_{k_j} \quad \text{resp.} \quad \prod_{j=1}^{6} (t_j z^{\pm})_{k_j}, \qquad \sum k_j = 0.$$

Concretely, f and g are elliptic with nome q resp. p, inversion symmetric, with poles only at specified points.

Calculus student's integration formula

The identity

$$\mu(fg) = \mu(f)\mu(g)$$

is like the calculus student's useful rule

$$\int e^x x \, dx = \int e^x \, dx \int x \, dx.$$

Two-index biorthogonality

In $\mu(fg) = \mu(f)\mu(g)$, let $f = Q_{k_1} R_{l_1}$, $g = \tilde{Q}_{k_2} \tilde{R}_{l_2}$ and recover

$$\mu\left(Q_{k_1} \tilde{Q}_{k_2} R_{l_1} \tilde{R}_{l_2}\right) = \delta_{k_1 l_1} \delta_{k_2 l_2}.$$

Rahman integral

Symmetric extension of Nasrallah–Rahman integral:

$$\oint \frac{(z^{\pm 2})_\infty \theta(\lambda z^{\pm})}{\prod_{j=1}^6 (b_j z^{\pm})_\infty} \frac{dz}{2\mathrm{i}\pi z} = \frac{2\left(\prod_{j=1}^6 \theta(\lambda b_j) - q\lambda^{-2}\prod_{j=1}^6 \theta(\lambda/b_j)\right)}{(q)_\infty \theta(\lambda^2)\prod_{1\leq i<j\leq 6}(b_i b_j)_\infty},$$

$b_1 \cdots b_6 = q$.
Replace b_j by $b_j q^{k_j}$, where $k_1 + \cdots + k_6 = 0$. Gives

$$\oint \frac{(z^{\pm 2})_\infty \theta(\lambda z^{\pm})}{\prod_{j=1}^6 (b_j z^{\pm})_\infty} \prod_{j=1}^{6} (b_j z^{\pm})_{k_j} \frac{dz}{2\mathrm{i}\pi z}$$
$$= \frac{2\left(\prod_{j=1}^6 \theta(\lambda b_j) - q\lambda^{-2}\prod_{j=1}^6 \theta(\lambda/b_j)\right)\prod_{1\leq i<j\leq 6}(b_i b_j)_{k_i+k_j}}{(q)_\infty \theta(\lambda^2)\prod_{1\leq i<j\leq 6}(b_i b_j)_\infty \prod_{j=1}^6 q^{\binom{k_j}{2}} b_j^{k_j}}.$$

Rahman functional

The functions $\prod_{j=1}^6 (b_j z^{\pm})_{k_j}$, $\sum_j k_j = 0$, span the space of rational functions in $(z+z^{-1})/2$ which are regular everywhere (including at infinity) except at $z^{\pm} \in b_j q^{\mathbb{Z}_{<0}}$.

It follows that there exists functional $\mathbf{J}$ on that space with

$$\mathbf{J}\left(\prod_{j=1}^{6} (b_j z^{\pm})_{k_j}\right) = \frac{\prod_{1\leq i<j\leq 6}(b_i b_j)_{k_i+k_j}}{\prod_{j=1}^6 q^{\binom{k_j}{2}} b_j^{k_j}}.$$

Asymmetric expression

The case $\lambda = b_6$ is integration against Nasrallah–Rahman:

$$\mathbf{J}(f) = \frac{(q)_\infty \prod_{1\le i<j\le 5}(b_i b_j)_\infty}{2\prod_{j=1}^5 (q/b_j b_6)_\infty} \oint f\left(\frac{z+z^{-1}}{2}\right) \frac{(z^{\pm 2})_\infty (qz^{\pm}/b_6)_\infty}{\prod_{j=1}^5 (b_j z^{\pm})_\infty} \frac{dz}{2\mathrm{i}\pi z}.$$

As we have mentioned, Rahman found biorthogonal system of ${}_{10}W_9$-functions with

$$\mathbf{J}(q_k r_l) = C\delta_{kl}.$$

The contour of integration depends on k and l (just as for Spiridonov's functions).

Discrete expression

We found new two-parameter family of discrete integrals:

$$\mathbf{J}(f) = \frac{(1-\lambda^2)\left(\prod_{j=1}^6 \theta(\mu b_j) - q\mu^{-2}\prod_{j=1}^6 \theta(\mu/b_j)\right)\prod_{j=1}^6 (q\lambda^{\pm}/b_j)_\infty}{(q)_\infty \theta(\lambda^2)\theta(\mu^2)\theta(\lambda\mu)\theta(\mu/\lambda)\prod_{1\le i<j\le 6}(q/b_i b_j)_\infty}$$
$$\times \sum_{x=-\infty}^{\infty} \frac{1-\lambda^2 q^{2x}}{1-\lambda^2} q^x \prod_{j=1}^6 \frac{(\lambda b_j)_x}{(q\lambda/b_j)_x} f\left(\frac{\lambda q^x + \lambda^{-1} q^{-x}}{2}\right) + (\lambda \leftrightarrow \mu).$$

In contrast to continuous integrals, this holds for all f in domain of $\mathbf{J}$.

Back to discrete-continuous integrals

Consider E_6:

$$\sum_{x=-\infty}^{\infty} \oint \frac{(1-q^x z^2)(1-q^x z^{-2})}{q^x z^{6x}} \times \prod_{j=1}^{6} \frac{(q^{1+x/2}/b_j z)_\infty (q^{1-x/2} z/b_j)_\infty}{(q^{N_j+x/2} b_j z)_\infty (q^{N_j-x/2} b_j/z)_\infty} \frac{dz}{2\mathrm{i}\pi z} = \frac{2}{\prod_{j=1}^{6} q^{\binom{N_j}{2}} b_j^{N_j}} \prod_{1\le i<j\le 6} \frac{(q/b_i b_j)_\infty}{(b_i b_j q^{N_i+N_j})_\infty},$$

$b_1 \cdots b_6 = q$ and $N_1 + \cdots + N_6 = 0$.

Replace b_j by $b_j q^{k_j}$ and N_j by $N_j - l_j$, where $\sum_j k_j = \sum_j l_j = 0$. Then write $c_j = b_j q^{N_j}$.

E_6 with parameter shifts

We obtain

$$\sum_{x=-\infty}^{\infty} \oint \frac{(1-q^x z^2)(1-q^x z^{-2})}{q^x z^{6x}} \times \prod_{j=1}^{6} \frac{(q^{1+x/2}/b_j z)_\infty (q^{1-x/2} z/b_j)_\infty}{(q^{x/2} c_j z)_\infty (q^{-x/2} c_j/z)_\infty} \times \prod_{j=1}^{6} (b_j (q^{-x/2} z)^{\pm})_{k_j} (c_j (q^{x/2} z)^{\pm})_{l_j} \frac{dz}{2\mathrm{i}\pi z}$$
$$= \frac{2}{\prod_{j=1}^{6} q^{\binom{k_j}{2}+\binom{l_j}{2}} b_j^{k_j} c_j^{l_j}} \prod_{1\le i<j\le 6} \frac{(q/b_i b_j)_\infty (b_i b_j)_{k_i+k_j} (c_i c_j)_{l_i+l_j}}{(c_i c_j)_\infty}.$$

Decoupling phenomenon

Let $\mathbf{J}'$ denote $\mathbf{J}$ with parameter shifts $b_j \mapsto c_j = b_j q^{N_j}$.
If f is in domain of $\mathbf{J}$ and g in domain of $\mathbf{J}'$, we have "calculus student's formula"

$$\sum_{x=-\infty}^{\infty} \oint \frac{(1-q^x z^2)(1-q^x z^{-2})}{q^x z^{6x}} \times \prod_{j=1}^{6} \frac{(q^{1+x/2}/b_j z)_\infty (q^{1-x/2} z/b_j)_\infty}{(q^{x/2} c_j z)_\infty (q^{-x/2} c_j/z)_\infty} \times f\left(\frac{q^{-x/2}z + q^{x/2}z^{-1}}{2}\right) g\left(\frac{q^{x/2}z + q^{-x/2}z^{-1}}{2}\right) \frac{dz}{2\mathrm{i}\pi z} = 2 \prod_{1\le i<j\le 6} \frac{(q/b_i b_j)_\infty}{(c_i c_j)_\infty} \mathbf{J}(f)\mathbf{J}'(g).$$

Double-index biorthogonality

If q_k, r_k are Rahman's functions with parameters b_j, q_k', r_k' with parameters $c_j = b_j q^{N_j}$, then

$$\sum_{x=-\infty}^{\infty} \oint \frac{(1-q^x z^2)(1-q^x z^{-2})}{q^x z^{6x}} \times \prod_{j=1}^{6} \frac{(q^{1+x/2}/b_j z)_\infty (q^{1-x/2} z/b_j)_\infty}{(q^{x/2} c_j z)_\infty (q^{-x/2} c_j/z)_\infty} \times (q_{k_1} r_{l_1})\left(\frac{q^{-x/2}z + q^{x/2}z^{-1}}{2}\right) \times (q'_{k_2} r'_{l_2})\left(\frac{q^{x/2}z + q^{-x/2}z^{-1}}{2}\right) \frac{dz}{2\mathrm{i}\pi z} = C\, \delta_{k_1 l_1} \delta_{k_2 l_2}.$$

Other discrete-continuous beta integrals

E_4 is similarly related to Askey–Wilson polynomials, but there are convergence problems (only finitely many polynomials).

E_3 is related to a system of biorthogonal rational functions due to Al-Salam and Ismail, and to another system due to van de Bult and Rains.

Concluding remarks

Recent progress on elliptic hypergeometric functions has led to conceptually new phenomena (discrete-continuous beta integrals, decoupling) for the trigonometric case.

Both from physics and mathematics perspective, it is interesting to find *multivariable* discrete-continuous integrals.

Seems we need multivariable extensions of Jackson–Slater transformations, nonterminating Jackson (and Saalschütz) summation. This is an underdeveloped area.