The monodromy of an elliptic difference equation

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Basic problem: We want to understand elliptic analogues of isomondromic deformations, which requires a good notion of monodromy.

Original idea (Etingof, Krichever, Rains): if $M(z)$ is a meromorphic fundamental matrix for the equation $v(qz) = A(z)v(z)$ and $A(pz) = A(z)$ (so the equation is p-elliptic), then $M(z)^{-1}M(pz)$ is q-elliptic. The resulting q-elliptic p-difference equation is not unique, but any two such equations are equivalent by a q-elliptic gauge xform.

Question: Can we make this more rigid?

Under certain genericity conditions, Krichever gets a unique equation by insisting that $M(z)$ be invertibly holomorphic (in a suitable annulus) and satisfy

$$M(qz) = A(z)M(z)\Delta$$

for a suitable diagonal matrix Δ. Equivalently, while a usual fundamental matrix identifies a vector space of solutions, this identifies a vector *bundle* of solutions.

Praagman uses a similar idea to construct a meromorphic fundamental matrix: construct a suitable vector bundle, then use the fact that any vector bundle is meromorphically trivial.

General approach*

Let $v(qz) = A(z)v(z)$ be any meromorphic q-difference equation (i.e., $A(z) \in \mathsf{GL}_n(\mathsf{Mer}(\mathbb{C}^*))$). We would like to define a sheaf of holomorphic solutions; since it's unreasonable to hope for any globally holomorphic solutions, we need to specify what this means locally.

More convenience, let's allow a general vector bundle; i.e., V is a vector bundle on $\mathbb{C}^*$, and $A : V \dashrightarrow q^*V$ an invertible meromorphic map. (A (meromorphic) "discrete connection" on V.)

*This is a generalized and (hopefully) simplified version of Section 13.2 of my "The noncommutative geometry of elliptic difference equations", arXiv:1607.08876. I'm leaning towards rewriting the section accordingly...

Let V be a vector bundle on the Riemann surface X. A *local condition* on V at a point $x \in X$ is a space σ_x ("lattice") of meromorphic germs of sections of V near x which has bounded poles and forms a module over the ring of analytic germs. (Equivalently, it's a coset of $\mathrm{GL}(V_x^{\mathrm{mer}})/\,\mathrm{GL}(V_x)$, where V_x is the module of germs of analytic sections, and V_x^{mer} is the space of germs of meromorphic sections). Call the local condition corresponding to the lattice V_x "regular", and otherwise "singular".

A *separated system of local conditions* consists of an assignment of a local condition to each point such that the set of points where the local condition is singular is discrete in X.*†

*I'll only ever deal with separated sytems of local conditions!

†Okounkov pointed out after my talk that when X is compact, the set of separated systems of local conditions has been studied as the "Beilinson-Drinfeld grassmannian" (of GL_n).

Observation: A system of local conditions on X is a global section of the sheaf $\mathcal{GL}(V \otimes \mathcal{K}_X)/\mathcal{GL}(V)$ (where $\mathcal{K}_X$ is the sheaf of meromorphic functions), and thus determines (by the connecting map on nonabelian cohomology) a class in $H^1(\mathcal{GL}(V))$.

In particular, a system of local conditions determines a vector bundle. More precisely, the sections of this bundle on an open set U consist of those meromorphic sections of V on U such that every germ is contained in the appropriate lattice. (This construction is shamelessly stolen from number theory, the "adèlic" construction of vector bundles on projective curves.*) Note that since this sheaf is a subsheaf of $V \otimes \mathcal{K}_X$, it comes with an injective meromorphic map to V.

*Also called "Weil uniformization". (comment added post-workshop)

Proposition. This establishes an equivalence between the category of vector bundles with separated systems of local conditions and the category of invertible meromorphic maps $W \dashrightarrow V$ of vector bundles.

Note that the morphisms $(V, \sigma) \to (W, \tau)$ consist of holomorphic maps $A : V \to W$ such that $A\sigma_x \subset \tau_x$; and $M : W \dashrightarrow V$ induces the system of local conditions MW_x on V. Since the category of meromorphic maps has well-behaved tensor products, this carries over to the category of bundles with local conditions, including symmetric powers, exterior powers, etc. The equivalence also respects holomorphic families (appropriately defined).

Important observation: There's an involution on the category of invertible meromorphic maps: $M \mapsto M^{-1}$, which induces an involution on the category of bundles with local conditions.

Given a q-difference equation $A : V \dashrightarrow q^*V$ on $\mathbb{C}^*$, a (naïve) meromorphic fundamental matrix corresponds to a meromorphic map $M : \mathcal{O}^n_{\mathbb{C}^*} \dashrightarrow V$ such that $q^*M = AM$. Any such matrix determines a system of local conditions σ on V, which moreover satisfies the consistency condition $q^*\sigma = A\sigma$.

There's an important source of such consistent systems of local conditions: For each $q^{\mathbb{Z}}$-orbit of $\mathbb{C}^*$, choose one point where we assign the regular local condition. If those points are bounded away from 0 and ∞, then the induced system of local conditions will be separated. Note that the chosen point only matters on those orbits where the equation is singular.

Given a system of local conditions on V which is consistent with A, we may construct the corresponding bundle W, and find that A induces an isomorphism $W \cong q^*W$. In particular, W is q-*equivariant*. Conversely, given invertible $M : W \dashrightarrow V$ with W a q-equivariant vector bundle, we may set $A := M^{-1}q^*M$ and thus obtain a q-difference equation on V.

Theorem. There is an equivalence between the category of triples (V, A, σ) of q-difference equations with consistent systems of local conditions and the category of pairs (W, σ) where W is a q-equivariant vector bundle and σ is a local condition on W.

Any q-equivariant sheaf descends through $\pi_q : \mathbb{C}^* \to \mathbb{C}^*/\langle q\rangle$, so we may replace this by pairs (W, σ) where W is a vector bundle on $\mathbb{C}^*/\langle q\rangle$ and σ is a local condition on $\pi_q^* W$.

Theorem. If $\mathsf{Sol}(A,\sigma)$ is the vector bundle on $\mathbb{C}^*/\langle q\rangle$ constructed in this way, then for any open subset $U \subset \mathbb{C}^*/\langle q\rangle$, $\Gamma(U;\mathsf{Sol}(A,\sigma))$ is naturally isomorphic to the space of solutions of the equation in $\Gamma(\pi_q^{-1}U;V)$.

We again have holomorphicity and consistency with tensor products, Schur functors, etc. (Note in particular that $\det W$ arises from a first-order equation on $\det V$.)

Proposition. There is a natural bijection between the space $\mathsf{Hom}(\mathsf{Sol}(A,\sigma),\mathsf{Sol}(B,\tau))$ of vector bundle morphisms and the space of maps $C : V_A \dashrightarrow V_B$ with $q^*CA = BC$ and $C\sigma \subset \tau$.

Proof. Write the morphism as a global section of $\mathsf{Sol}(A,\sigma)^* \otimes \mathsf{Sol}(B,\tau) \cong \mathsf{Sol}(B \otimes A^{-t}, \tau \otimes \sigma^*)$, and observe that the condition to be a global section of the latter is precisely the stated condition on C. $\square$

In particular, two equations with local conditions are gauge equivalent iff they have isomorphic sheaves of holomorphic solutions. (I.e., gauge equivalence == isomonodromy)

Example. Suppose $A(z) \in \mathsf{GL}_n(\mathbb{C}(z))$. There are two natural systems of local conditions: σ_0, in which every point in some punctured neighborhood of 0 is regular, and analogously σ_∞. This induces a meromorphic map

$$\mathsf{Sol}(A, \sigma_0) \dashrightarrow \mathsf{Sol}(A, \sigma_\infty).$$

If $A(0) = 1$, then $\mathsf{Sol}(A, \sigma_0)$ is trivial, and its global sections are holomorphic at 0, and similarly if $A(\infty) = 1$. If both hold, then the above meromorphic map becomes a matrix of q-elliptic functions, precisely Birkhoff's notion of monodromy.*

*Post-workshop comment: The example on the next slide shows that without these conditions, it may not be possible to recover the equation (up to rational gauge equivalence) from the map.

*During the talk, I pointed out that this gauge equivalence is not as strong as one might think, as it's a gauge equivalence over general meromorphic matrices, which need not imply *rational* gauge equivalence. Consider the equation† $v(qz) = \begin{pmatrix} 1 & -z \\ -z & 1+z^2 \end{pmatrix} v(z)$, with fundamental matrix (relative to $\sigma_0 = \sigma_\infty = 1$)

$$M(z) = \prod_{k\geq 0} A(q^k z)^{-1} = \begin{pmatrix} \sum_{0\leq k} \frac{q^{k(k-1)} z^{2k}}{(q;q)_{2k}} & \sum_{0\leq k} \frac{q^{k^2} z^{2k+1}}{(q;q)_{2k+1}} \\ \sum_{0\leq k} \frac{q^{k(k+1)} z^{2k+1}}{(q;q)_{2k+1}} & \sum_{0\leq k} \frac{q^{k^2} z^{2k}}{(q;q)_{2k}} \end{pmatrix}.$$

This is the unique (up to $\mathrm{GL}_2(\mathbb{C})$) meromorphic gauge equivalence between this equation and $v(qz) = v(z)$, and is manifestly not rational.

*(Added post-workshop)

†This is a version of Ismail's q-Airy equation.

For first-order equations $v(qz) = a(z)v(z)$, we can determine the line bundle as follows.* The system of local conditions determines a divisor of a function on $\mathbb{C}^*$, and Weierstrass tells us that there exists a function $f(z)$ with that divisor. Gauging by this function gives a new equation

$$\widehat{v}(qz) = \widehat{a}(z)\widehat{v}(z), \qquad \widehat{a}(z) = f(qz)^{-1}a(z)f(z)$$

with trivial local conditions. This implies $\widehat{a}(z)$ is nonvanishing and holomorphic, so $\widehat{a}(z) = Cz^k \exp(\sum_{l\neq 0} c_l z^l)$. Further gauging by $\exp(\sum_{l\neq 0} c_l z^l/(q^l - 1))$ reduces our equation to $v(qz) = Cz^k v(z)$, which is the standard expression for a line bundle on $\mathbb{C}^*/\langle q\rangle$.

Another approach is to take *any* meromorphic solution of the equation, and observe that its divisor differs from the desired divisor by a q-periodic divisor, corresponding to a divisor on $\mathbb{C}^*/\langle q\rangle$ and thus a line bundle.

*Corrected from the slide as presented

Holomorphic fundamental matrices

The vector bundles on an elliptic curve have been classified, and in each case there's a fairly natural choice of multiplier. In other words, every vector bundle can be expressed as $\mathsf{Sol}(\mu, \mathcal{O}^n_{\mathbb{C}^*})$ with $\mu \in \mathsf{GL}_n(\mathbb{C}[z, 1/z])$.*

Given a choice of multiplier (and an equation on the trivial bundle), an isomorphism $\mathsf{Sol}(A, \sigma) \cong \mathsf{Sol}(\mu, 1)$ corresponds to a "holomorphic" fundamental matrix M such that (a) every column of M satisfies the local conditions σ, and (b) $M(qz) = A(z)M(z)\mu(z)^{-1}$.

(Of course, the true (and canonical) holomorphic fundamental matrix is simply the map $M : \pi_q^* \mathsf{Sol}(A, \sigma) \dashrightarrow V \ldots$)

*This representation is far from unique, but we can mostly rigidify by insisting that it be an extension of suitable "hypergeometric" equations.

Elliptic equations

Suppose now that we're also given $|p| < 1$, and $A : V \dashrightarrow q^*V$ starts out on the Riemann surface $\mathbb{C}^*/\langle p \rangle$. Equivalently, we may take V to be a p-equivariant vector bundle and $A : V \dashrightarrow q^*V$ an equivariant meromorphic map. Call such an object a "p-elliptic q-difference equation". If we forget the equivariant structure, we obtain a q-difference equation on $\mathbb{C}^*$ and thus a notion of consistent system of local conditions. Denote the resulting category by $\mathcal{E}ll\mathcal{D}i\!f\!f_{p,q}$.

Theorem. There is a natural equivalence between the category $\mathcal{E}ll\mathcal{D}i\!f\!f_{p,q}$ and the category of triples (V, W, M) where $M : \pi_q^*W \dashrightarrow \pi_p^*V$ satisfies $p^*(q^*MM^{-1}) = q^*MM^{-1}$.

Inverting M gives the following. (There's also a contravariant version in which we transpose M instead.)

Corollary. There is an equivalence of categories (the "elliptic Riemann-Hilbert correspondence") $\mathsf{Sol}_{p,q} : \mathcal{E}ll\mathcal{D}\mathit{iff}_{p,q} \cong \mathcal{E}ll\mathcal{D}\mathit{iff}_{q,p}$, with a natural isomorphism $\mathsf{Sol}_{q,p}\,\mathsf{Sol}_{p,q} \cong \mathrm{id}$.

Note that we could also take $p^*(q^*MM^{-1}) = Cz^kq^*MM^{-1}$, corresponding to a twisted equation $A : V \dashrightarrow q^*V \otimes \mathcal{L}$ for a suitable line bundle.* This gives $\mathcal{E}ll\mathcal{D}\mathit{iff}_{p,q,Cz^k} \cong \mathcal{E}ll\mathcal{D}\mathit{iff}_{q,p,C^{-1}z^{-k}}$.

*This is twisting by a p,q-equivariant gerbe...

* I was asked during the talk what constraints I needed to impose on p and q, so naturally replied that I was assuming that p and q must be independent elements of $\mathbb{C}^*$. On further reflection, this constraint isn't actually used in the above construction! The only real constraint is that p and q must not lie on the unit circle (since then the quotient is not a Riemann surface). For the algebraic subcategories considered below, we further need that p and q lie on the same side of the unit circle, so that the singularities of the corresponding elliptic Gamma function are not dense.

*(Added post-workshop)

Symmetric equations

Suppose $A(\eta/qz)A(z) = 1$ and we're looking for solutions $v(\eta/z) = v(z)$. Everything above carries over at the level of equivariant bundles (make everything $z \mapsto \eta/z$-equivariant). Since the group no longer acts freely, not every equivariant bundle is a pullback, but this reduces to a fairly simple constraint at the ramified points (i.e., with nontrivial stabilizer).

In particular, we obtain an equivalence between the category of symmetric p-elliptic q-difference equations with suitable local conditions and the same category with p and q swapped.*

*We can also twist by a $z \mapsto \eta/z$-invariant gerbe: Cz^k with $k = 0$.

Applying this to the Lax pair for elliptic Painlevé* gives biholomorphic maps between various spaces of initial conditions for elliptic Painlevé, "integrating" elliptic Painlevé.

More generally, as long as the local conditions are determined in a sufficiently rigid way, we get maps from *algebraic* categories of elliptic difference equations into the *holomorphic* category $\mathcal{E}ll\mathcal{D}i\!f\!f_{p,q,Cz^k}$ (or its symmetric analogue). The algebraic categories have a further source of holomorphic equivalences: replace the curve $\mathbb{C}^*/\langle p\rangle$ by a modular transform. This interacts nicely with the elliptic RH correspondence, giving a family of holomorphic equivalences corresponding to the action of $\mathrm{SL}_3(\mathbb{Z})$ on $(\log q, \log p, \log 1)$. (Caveat: Every element of $\mathrm{SL}_3(\mathbb{Z})$ has a holomorphic equivalence over it, but I don't know if this is an actual action of $\mathrm{SL}_3(\mathbb{Z})\dots$)

*Technicality: with certain nearly-canonical choices of local conditions that I only understand for parameters in general position...

Possible application: Hypergeometric equations are rigid (determined by their singularities), so we can arrange for their monodromy to be hypergeometric, and thus explicitly computable. E.g., the monodromy of the equation satisfied by the order m elliptic beta integral is essentially the same, just with p and q swapped; the fundamental matrix is just a matrix of elliptic beta integrals with shifted parameters.*

Rigidity implies that every relation in $\mathrm{SL}_3(\mathbb{Z})$ preserves the original equation, and thus should induce a relation between the corresponding fundamental matrices: a nonabelian generalization of the results of Felder and Varchenko on elliptic Gamma.

*The existence of such an integral representation follows from the general theory: the only tricky part is controlling certain generalized Fourier transforms, but that's easy since we understand the singularities of the input...

Further open questions

(1) For singularities in general position, there's a natural way to associate local conditions to a choice for each singularity of a preimage in $\mathbb{C}^*$, giving a discrete set of ways to embed the algebraic category in $\mathcal{E}ll\mathcal{D}i\!f\!f$ (and a family of isomonodromy deformations on the algebraic image under RH). What's the right way to extend this to more general singularities?

(2) What's the right version of this correspondence in the hyperbolic limit $p, q \to 1$? There's no difficulty using this approach to produce analytic bundles on $\mathbb{C}/2\pi i\mathbb{Z}$ associated to ordinary difference equations, but we need to somehow force the bundles and maps to be algebraic on $\mathbb{C}^*$... (Sectors without singularities + growth conditions?)