HYPERGEOMETRIC AND BASIC HYPERGEOMETRIC SERIES AND INTEGRALS ASSOCIATED WITH ROOT SYSTEMS

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Abstract. We give an overview of some of the main results from the theories of hypergeometric and basic hypergeometric series and integrals associated with root systems. In particular, we list a number of summations, transformations and explicit evaluations for such multiple series and integrals. We concentrate on such results which do not directly extend to the elliptic level. This text is a provisional version of a chapter on hypergeometric and basic hypergeometric series and integrals associated with root systems for volume 2 of the new Askey–Bateman project which deals with "Multivariable special functions".

Contents

1. Introduction

Hypergeometric series associated with root systems first appeared implicitly in the 1972 work of Ališauskas, Jucys and Jucys [1] and Chacón, Ciftan and Biedenharn [10] in the context of the representation theory of the unitary groups, more precisely, as the multiplicity-free Wigner and Racah coefficients ($3j$ and $6j$-symbols) of the group $\mathrm{SU}(n+1)$. A few years later, Holman, Biedenharn and Louck [45] investigated these coefficients more explicitly as generalized hypergeometric series and obtained a first summation theorem for these. The series in question have explicit summands and contain the Weyl denominator

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of the root system A_n, and can thus be considered as hypergeometric series associated with this root system. (These series are not to be confused with the hypergeometric functions associated with root systems considered in Chapter 8 of this volume, which generalize the spherical functions on noncompact Riemannian symmetric spaces.) Subsequently, A_n hypergeometric series were shown to satisfy various extensions of well-known identities for classical hypergeometric series [31, 32, 44, 78]. For example, using the usual Pochhammer symbol notation for the shifted factorial (see (5.2.1)), Holman's [44] A_n extension of the terminating balanced Pfaff–Saalschütz ${}_3F_2$ summation is

$$\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n}\Bigg(\prod_{1\le i<j\le n}\frac{x_i+k_i-x_j-x_k}{x_i-x_j}\prod_{i,j=1}^n\frac{(-N_j+x_i-x_j)_{k_i}}{(1+x_i-x_j)_{k_i}}\\ \times\prod_{i=1}^n\frac{(a+x_i)_{k_i}\,(b+x_i)_{k_i}}{(c+x_i)_{k_i}\,(a+b-c+1-|N|+x_i)_{k_i}}\Bigg)\\ =\frac{(c-a)_{|N|}\,(c-b)_{|N|}}{\prod_{i=1}^n(c+x_i)_{N_i}\,(c-a-b+|N|-N_i-x_i)_{N_i}}, \tag{5.1.1}$$

where, throughout this chapter, $|N|:=N_1+\cdots+N_n$.

An extensive study of the *basic* analogue, or q-analogue, of A_n series was initiated by Milne in a series of papers [79, 80, 81]. The following application of the "fundamental theorem of A_n series" [80, Theorem 1.49],

$$\sum_{\substack{k_1,\dots,k_n\ge0\\|k|=N}}\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{i,j=1}^n\frac{(a_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}=\frac{(a_1\cdots a_n;q)_N}{(q;q)_N}, \tag{5.1.2}$$

where we are using the usual q-Pochhammer symbol notation for the q-shifted factorial (see (5.2.2)), demonstrates a phenomenon which is typical for the A_n theory: In (5.1.2), let $n\mapsto n+1$ and replace k_{n+1} by $N-(k_1+\cdots+k_n)$. Then, after further replacing the variables a_i by c_i, for $i=1,\dots,n$, a_{n+1} by $q^{-N}b/a$ and x_{n+1} by q^{-N}/a, respectively, the following terminating A_n ${}_6\phi_5$ summation is obtained:

$$\sum_{\substack{k_1,\dots,k_n\ge0\\|k|\le N}}\Bigg(\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{i=1}^n\frac{1-ax_iq^{k_i+|k|}}{1-ax_i}\prod_{i,j=1}^n\frac{(c_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\\ \times\prod_{i=1}^n\frac{(ax_i;q)_{|k|}\,(bx_i;q)_{k_i}}{(ax_iq/c_i;q)_{|k|}\,(ax_iq^{1+N};q)_{k_i}}\cdot\frac{(q^{-N};q)_{|k|}}{(aq/b;q)_{|k|}}\left(\frac{aq^{1+N}}{bc_1\cdots c_n}\right)^{|k|}\Bigg)\\ =\frac{(aq/bc_1\cdots c_n;q)_N}{(aq/b;q)_N}\prod_{i=1}^n\frac{(ax_iq;q)_N}{(ax_iq/c_i;q)_N}. \tag{5.1.3}$$

By application of the one-variable q-binomial theorem, it follows that another consequence of (5.1.2) is the following A_n extension of the nonterminating q-binomial theorem:

$$\sum_{k_1,\dots,k_n\geq 0}\prod_{1\leq i<j\leq n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{i,j=1}^{n}\frac{(a_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\cdot z^{|k|}=\frac{(a_1\cdots a_nz;q)_\infty}{(z;q)_\infty}, \tag{5.1.4}$$

valid for $|q|<1$ and $|z|<1$.

While A_n (basic) hypergeometric series have also been referred to as $\mathrm{SU}(n)$ or $\mathrm{U}(n)$ series, the terminology *(basic) hypergeometric series associated to the root system* A_n, or simply A_n *(basic) hypergeometric series*, is preferred by most authors nowadays.

A further important development was Gustafson's introduction of very-well-poised series for other root systems [33, 34]. Gustafson also introduced related multivariate integrals associated with root systems [14, 35, 36, 37, 38]. In this setting the multiple series or integrals are classified according to the type of specific factors (such as a Weyl denominator) appearing in the summand or integrand.

Most of the known results for multivariate (basic) hypergeometric series and integrals associated with root systems indeed concern classical root systems, while only sporadically summations or transformations for series or integrals associated with exceptional root systems have been obtained [13, 35, 47, 48, 50].

The root system classification appears to be very useful and one would hope that the various relations satisfied by the series or integrals can be interpreted in terms of root systems or even Lie theory. Although the type of series in question first arose (in the limit $q\to 1$) in the representation theory of compact Lie groups, many questions remain open about this connection. While a (quantum) group interpretation for the A_n type of series has been given by Rosengren in [109] — even for the elliptic extension of the series, surveyed in Chapter 6 of this volume — no analogous interpretations for the other root systems have yet been revealed.

In many instances various types (still referring to the root system classification) of series/integrals can be combined with each other after which one obtains series/integrals of some "mixed type" for which the correct classification is not really clear. The conclusion is that the root system classification of the series/integrals considered here is only rough and not always precise.

In terms of the rough classification in [6, Section 2], [8, Section 1], and [90, Section 5], a multivariate series $\sum_{k_1,\dots,k_{n+1}} S_{k_1,\dots,k_{n+1}}$ is considered to be an A_n *hypergeometric series*, if the summand $S_{k_1,\dots,k_{n+1}}$ contains the factor

$$\prod_{1\leq i<j\leq n+1}(x_i+k_i-x_j-k_j). \tag{5.1.5a}$$

It is considered to be an A_n *basic hypergeometric series*, if it contains the factor

$$\prod_{1\leq i<j\leq n+1}\big(x_iq^{k_i}-x_jq^{k_j}\big). \tag{5.1.5b}$$

If we take the sum over $k_1+\cdots+k_{n+1}=N$, we may replace k_{n+1} by $N-|k|$ (where, as before, $|k|=k_1+\cdots+k_n$) and the two respective products in (5.1.5a)/(5.1.5b) can be written as

$$\prod_{1\le i<j\le n}(x_i+k_i-x_j-k_j)\prod_{i=1}^{n}(a+x_i+k_i+|k|), \tag{5.1.6a}$$

where x_{n+1} was substituted by $-a-N$, and

$$\left(-aq^{|k|}\right)^{-n}\prod_{1\le i<j\le n}\left(x_iq^{k_i}-x_jq^{k_j}\right)\prod_{i=1}^{n}\left(1-ax_iq^{k_i+|k|}\right), \tag{5.1.6b}$$

where x_{n+1} was substituted by q^{-N}/a, respectively.

Likewise, a C_n *hypergeometric series* contains the factor

$$\prod_{1\le i<j\le n}(x_i+k_i-x_j-k_j)\prod_{1\le i\le j\le n}(x_i+k_i+x_j+k_j) \tag{5.1.7a}$$

and a C_n *basic hypergeometric series* the factor

$$\prod_{1\le i<j\le n}\left(x_iq^{k_i}-x_jq^{k_j}\right)\prod_{1\le i\le j\le n}\left(1-x_ix_jq^{k_i+k_j}\right). \tag{5.1.7b}$$

We omit giving similar definitions for other root systems. The above factors may be associated with the Weyl denominators $\prod_{\alpha>0}(1-e^{-\alpha})$ with the product taken over the positive roots in the root system. (Weyl denominators similarly appear in Chapter 8, Section 4.2, of this volume.)

A very similar classification applies to hypergeometric *integrals* associated with root systems, by considering certain factors of the integrand. For specific examples, see Section 3.

A special feature of the theory of hypergeometric or basic hypergeometric functions associated with root systems is that often there exist several different identities for one and the same root system that extend a particular one-variable identity. See the various A_n ${}_3\phi_2$ and A_n ${}_2\phi_1$ summations given by Milne in [88]. At this point we would also like to mention that various special A_n hypergeometric series possess rich structures of symmetry; these were made explicit by Kajihara [55].

This chapter deals to a large extent with the multivariate *basic* hypergeometric theory. The reason is that most results for ordinary hypergeometric series have basic hypergeometric analogues[1], so it does not make sense to treat the hypergeometric case separately.

A good amount of the theory of basic hypergeometric series associated with root systems has recently been generalized to the *elliptic* level. For an introduction to elliptic hypergeometric functions associated with root systems, see Chapter 6 of this volume. In the present chapter we emphasize some general facts about series associated with root systems which are important for understanding the nature of the series, but otherwise, to avoid overlap, mainly focus on parts of the theory which do *not* directly extend to the elliptic level. This in particular concerns identities obtained as confluent limits of more general identities and identities for nonterminating and/or multilateral series.

[1] Identities for basic hypergeometric series reduce to those for ordinary hypergeometric series by taking the limit $q\to 1$ in an appropriate manner [28, Section 1.2], see for instance the derivation of (5.2.31) from (5.2.30).

The following sections are devoted to multivariate identities, ranging from very simple identities to more complicated ones. In particular, in Sections 2 and 3 various summations, transformations, and integral evaluations are reviewed. Section 4 surveys the theory of basic hypergeometric series with Macdonald polynomial argument. The chapter concludes with Section 5, containing a brief discussion on applications of basic hypergeometric series associated with root systems.

Notation. In this chapter, we follow the convention in lines with slashed fractions that multiplication has priority over division (i.e., $c/ab = c/(ab)$, etc.). This convention is also silently used in Gasper and Rahman's book [28] which we often refer to.

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2. Some identities for (basic) hypergeometric series associated with root systems

A large number of identities for hypergeometric and basic hypergeometric series associated with root systems has appeared in the literature. Due to space limitations, we only provide a small representative selection of identities. Nevertheless, they are meant to give a flavor of the expressions which typically occur in the multivariate theory. For more details the reader is pointed to specific literature.

We use the following notations for shifted and q-shifted factorials (which are also referred to as Pochhammer and q-Pochhammer symbols, respectively):

$$(a)_k = \begin{cases} 1 & k = 0 \\ a(a+1)\dots(a+k-1) & k = 1,2,\dots, \\ \big((a-1)(a-2)\dots(a+k)\big)^{-1} & k = -1,-2,\dots, \end{cases} \tag{5.2.1}$$

$$(a;q)_k = \begin{cases} 1 & k = 0 \\ (1-a)(1-aq)\dots(1-aq^{k-1}) & k = 1,2,\dots, \\ \big((1-aq^{-1})(1-aq^{-2})\dots(1-aq^{k})\big)^{-1} & k = -1,-2,\dots, \end{cases} \tag{5.2.2a}$$

and

$$(a;q)_\infty = \prod_{i\ge 0}(1-aq^i). \tag{5.2.2b}$$

When dealing with products of shifted and q-shifted factorials, we frequently use the shorthand notations

$$(a_1,\dots,a_n)_j = (a_1)_j\cdots(a_n)_j \quad \text{and} \quad (a_1,\dots,a_n;q)_k = (a_1;q)_k\cdots(a_n;q)_k,$$

where j is an integer, and k is an integer or ∞.

This chapter reviews results for multivariate extensions associated with root systems of the following univariate series, whose definitions we give for self-containment.

Hypergeometric ${}_rF_s$ series and bilateral hypergeometric ${}_rH_s$ series are defined as

$$ {}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r)_k}{(1,b_1,\dots,b_s)_k}z^k, \tag{5.2.3a}$$

$$ {}_rH_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right]=\sum_{k=-\infty}^{\infty}\frac{(a_1,\dots,a_r)_k}{(b_1,\dots,b_s)_k}z^k. \tag{5.2.3b}$$

Similarly, basic hypergeometric ${}_r\phi_s$ series and bilateral basic hypergeometric ${}_r\psi_s$ series are defined as

$$ {}_r\phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right]=\sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(q,b_1,\dots,b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}z^k, \tag{5.2.4a}$$

$$ {}_r\psi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right]=\sum_{k=-\infty}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{s-r}z^k. \tag{5.2.4b}$$

See Slater's book [122] and Gasper and Rahman's book [28] for the conditions of theses series to terminate, to be balanced, and to be (very-)well-poised, and for various identities satisfied by these series.

2.1. **Some useful elementary facts.** We start with a few elementary ingredients which are useful for manipulating basic hypergeometric series associated with root systems.

(i) A fundamental ingredient (for inductive proofs and functional equations, etc.) is the following partial fraction decomposition [83, Section 7]:

$$\prod_{i=1}^n\frac{1-tx_iy_i}{1-tx_i}=y_1y_2\cdots y_n+\sum_{k=1}^n\frac{\prod_{i=1}^n(1-y_ix_i/x_k)}{(1-tx_k)\prod_{\substack{i=1\\ i\neq k}}^n(1-x_i/x_k)}. \tag{5.2.5}$$

In particular, this identity can be used to prove the fundamental theorem of A_n series in (5.1.2).

A slightly more general partial fraction decomposition was derived in [116, Lemma 3.2]. The identity in (5.2.5) can be obtained as a limiting case of an elliptic partial fraction decomposition of type A, cf. [133, p. 451, Example 3]. A related partial fraction decomposition of type D was established in [32, Lemma 4.14].

(ii) For simplifying products the following identity is useful [88, Lemma 4.3]:

$$\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_iq^{m_i}-x_jq^{m_j}}\prod_{i,j=1}^n\frac{(q^{m_i-k_j}x_i/x_j;q)_{k_i-m_i}}{(q^{1+m_i-m_j}x_i/x_j;q)_{k_i-m_i}}=(-1)^{|k|-|m|}q^{-\binom{|k|-|m|+1}{2}}. \tag{5.2.6}$$

(iii) When reversing the order of the summations

$$\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n}S_{k_1,\dots,k_n}=\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n}S_{N_1-k_1,\dots,N_n-k_n},$$

it is convenient to use the fact that the following variant of an "A_n q-binomial coefficient"

$$\prod_{i,j=1}^{n} \frac{(qx_i/x_j;q)_{N_i}}{(qx_i/x_j;q)_{k_i}\,(q^{1+k_i-k_j}x_i/x_j;q)_{N_i-k_i}} \tag{5.2.7}$$

(the usual q-binomial coefficient is given in (5.2.9)) remains unchanged after performing the simultaneous substitutions $k_i \mapsto N_i - k_i$ and $x_i \mapsto q^{-N_i}/x_i$, for $i = 1, \dots, n$ [114, Remark B.3].

2.2. **Some terminating A_n q-binomial theorems.** The terminating q-binomial theorem can be written in the form (cf. [28, Ex. 1.2(vi)])

$$(z;q)_N = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} z^k, \tag{5.2.8}$$

where

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \frac{(q;q)_N}{(q;q)_k (q;q)_{N-k}} \tag{5.2.9}$$

is the q-binomial coefficient. (Here, N denotes a nonnegative integer.) In basic hypergeometric notation, this identity corresponds to a terminating ${}_1\phi_0$ summation. It can be immediately obtained from the nonterminating q-binomial theorem (or ${}_1\phi_0$ summation, cf. [28, Equation (II.3)]),

$${}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix}; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{5.2.10}$$

valid for $|q| < 1$ and $|z| < 1$. To obtain (5.2.8) from (5.2.10), replace a and z by q^{-n} and zq^n, respectively.

We have the following three multisum identities (for equivalent forms, where the Vandermonde determinant of type A (5.1.5b) explicitly appears in the summands, see [88, Theorems 5.44, 5.46, and 5.48]), each involving the A_n q-binomial coefficient in (5.2.7):

$$(z;q)_{|N|} = \sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \prod_{i,j=1}^{n} \frac{(qx_i/x_j;q)_{N_i}}{(qx_i/x_j;q)_{k_i}\,(q^{1+k_i-k_j}x_i/x_j;q)_{N_i-k_i}} \cdot (-1)^{|k|} q^{\binom{|k|}{2}} z^{|k|}, \tag{5.2.11a}$$

$$\prod_{i=1}^{n}(zx_i;q)_{N_i} = \sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \Bigg(\prod_{i,j=1}^{n} \frac{(qx_i/x_j;q)_{N_i}}{(qx_i/x_j;q)_{k_i}\,(q^{1+k_i-k_j}x_i/x_j;q)_{N_i-k_i}} \times (-1)^{|k|} q^{\sum_{i=1}^{n}\binom{k_i}{2}} z^{|k|} \prod_{i=1}^{n} x_i^{k_i} \Bigg), \tag{5.2.11b}$$

$$\prod_{i=1}^n (zq^{|N|-N_i}/x_i;q)_{N_i} = \sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \Bigg(\prod_{i,j=1}^n \frac{(qx_i/x_j;q)_{N_i}}{(qx_i/x_j;q)_{k_i}\,(q^{1+k_i-k_j}x_i/x_j;q)_{N_i-k_i}} \\ \times (-1)^{|k|} q^{\binom{|k|}{2}+\sum_{1\le i<j\le n}k_ik_j} z^{|k|} \prod_{i=1}^n x_i^{-k_i} \Bigg). \tag{5.2.11c}$$

The summations in (5.2.11b) and (5.2.11c) are related by reversal of sums as explained in Subsection 2.1(iii). In the last two identities the variable z is redundant. Inclusion turns both sides into polynomials in z.

Here are two other terminating A_n q-binomial theorems [88, Theorems 5.52 and 5.50]:

$$(z;q)_N = \sum_{\substack{k_1,\dots,k_n\ge 0\\ |k|\le N}} \prod_{1\le i<j\le n} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{i,j=1}^n (qx_i/x_j;q)_{k_i}^{-1} \cdot (q^{-N};q)_{|k|}\, q^{N|k|} z^{|k|}, \tag{5.2.11d}$$

$$(z;q)_N = \sum_{\substack{k_1,\dots,k_n\ge 0\\ |k|\le N}} \Bigg(\prod_{1\le i<j\le n} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{i,j=1}^n (qx_i/x_j;q)_{k_i}^{-1} \cdot (q^{-N};q)_{|k|} \\ \times (-1)^{(n-1)|k|} q^{N|k|-\binom{|k|}{2}+n\sum_{i=1}^n\binom{k_i}{2}} z^{|k|} \prod_{i=1}^n x_i^{nk_i-|k|} \Bigg). \tag{5.2.11e}$$

Note that Equations (5.2.11d) and (5.2.11e) are equivalent with respect to inverting the base $q \to q^{-1}$.

Yet another terminating A_n q-binomial theorem, implicit from in [8], is

$$\prod_{i=1}^n (z/x_i;q)_N = \sum_{\substack{k_1,\dots,k_n\ge 0\\ |k|\le N}} \Bigg(\prod_{1\le i<j\le n} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{i,j=1}^n (qx_i/x_j;q)_{k_i}^{-1} \cdot (q^{-N};q)_{|k|} \\ \times q^{N|k|+\sum_{1\le i<j\le n}k_ik_j} z^{|k|} \prod_{i=1}^n x_i^{-k_i} (z/x_i;q)_{|k|-k_i} \Bigg). \tag{5.2.11f}$$

Four of the terminating A_n q-binomial theorems of this subsection are special cases of nonterminating A_n q-binomial theorems. In particular, (5.2.11a) is a special case of (5.1.4). Similarly, (5.2.11b) is a special case of [70, Theorem 4.7]. Further, (5.2.11f) is a special case of [88, Theorem 5.42] (which is the $b_1 = \cdots = b_n = q$ special case of (5.2.26)). Finally, (5.2.11e) is a special case of [8, Theorem 5.19].

2.3. **Other terminating summations.** All the A_n q-binomial theorems (i.e., terminating ${}_1\phi_0$ summations) listed in the previous subsection (and others not listed here, as those implicit in [111, Section 7.7]) admit generalizations to summations involving more parameters. These include in particular various A_n ${}_2\phi_1$, ${}_3\phi_2$, ${}_6\phi_5$, or ${}_8\phi_7$ summations (see e.g. [88, 111, 119]).

The terminating balanced ${}_3\phi_2$ summation (or q-Pfaff–Saalschütz summation) is (cf. [28, Equation (II.12)])

$$ {}_3\phi_2\left[\begin{matrix} a, b, q^{-N} \\ c, abq^{1-N}/c \end{matrix}; q, q\right] = \frac{(c/a, c/b; q)_N}{(c, c/ab; q)_N}. \tag{5.2.12} $$

Here are two A_n ${}_3\phi_2$ summations (cf. [88]):

$$ \sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \prod_{1\le i<j\le n} \frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \prod_{i,j=1}^{n} \frac{(q^{-N_j} x_i/x_j; q)_{k_i}}{(qx_i/x_j; q)_{k_i}} \prod_{i=1}^{n} \frac{(ax_i; q)_{k_i}}{(cx_i; q)_{k_i}} \cdot \frac{(b; q)_{|k|}}{(abq^{1-|N|}/c; q)_{|k|}} q^{|k|} \\ = \frac{(c/a; q)_{|N|}}{(c/ab; q)_{|N|}} \prod_{i=1}^{n} \frac{(cx_i/b; q)_{N_i}}{(cx_i; q)_{N_i}}, \tag{5.2.13} $$

$$ \sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \prod_{1\le i<j\le n} \frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \prod_{i,j=1}^{n} \frac{(q^{-N_j} x_i/x_j; q)_{k_i}}{(qx_i/x_j; q)_{k_i}} \prod_{i=1}^{n} \frac{(ax_i, bx_i; q)_{k_i}}{(cx_i, abx_i q^{1-|N|}/c; q)_{k_i}} q^{|k|} \\ = (c/a, c/b; q)_{|N|} \prod_{i=1}^{n} (cx_i, cx_i q^{|N|-N_i}/ab; q)_{N_i}^{-1}. \tag{5.2.14} $$

Several other A_n ${}_3\phi_2$ summations are given in [88]. For instance, a simple polynomial argument applied to (5.2.13) yields

$$ \sum_{\substack{k_1,\dots,k_n\ge 0 \\ |k|\le N}} \prod_{1\le i<j\le n} \frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \prod_{i,j=1}^{n} \frac{(a_j x_i/x_j; q)_{k_i}}{(qx_i/x_j; q)_{k_i}} \prod_{i=1}^{n} \frac{(bx_i; q)_{k_i}}{(cx_i; q)_{k_i}} \cdot \frac{(q^{-N}; q)_{|k|}}{(a_1 \cdots a_n bq^{1-N}/c; q)_{|k|}} q^{|k|} \\ = \frac{(c/b; q)_N}{(c/a_1 \cdots a_n b; q)_N} \prod_{i=1}^{n} \frac{(cx_i/a_i; q)_N}{(cx_i; q)_N}. \tag{5.2.15} $$

Here is another terminating balanced ${}_3\phi_2$ summation ([92, Theorem 4.3] rewritten) which may be considered of "mixed-type":

$$ \sum_{\substack{k_1,\dots,k_n\ge 0 \\ |k|\le N}} \Bigg(\prod_{1\le i<j\le n} \frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \frac{1}{(x_i x_j; q)_{k_i+k_j}} \prod_{i,j=1}^{n} \frac{(a_j x_i/x_j, x_i x_j/a_j; q)_{k_i}}{(qx_i/x_j; q)_{k_i}} \\ \times \frac{(q^{-N}; q)_{|k|}}{\prod_{i=1}^{n} (bx_i q^{-N}, qx_i/b; q)_{k_i}} q^{|k|} \Bigg) = \prod_{i=1}^{n} \frac{(qa_i/bx_i, qx_i/a_i b; q)_N}{(q/bx_i, qx_i/b; q)_N}. \tag{5.2.16} $$

Again, a simple polynomial argument can be applied to transform this summation to another one, in this case to a sum over a rectangular region (see [92, Theorem 4.2]), which we do not state here explicitly.

Among the most general summations for basic hypergeometric series associated with root systems are various multivariate ${}_8\phi_7$ Jackson summations. In the univariate case, Jackson's terminating balanced very-well-poised ${}_8\phi_7$ summation is (cf. [28, Equation (II.22)])

$$
{}_8\phi_7\left[\begin{matrix} a,\, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, q^{-N} \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq^{N+1} \end{matrix}; q, q\right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_N}{(aq/b, aq/c, aq/d, aq/bcd; q)_N}, \tag{5.2.17}
$$

where $a^2q = bcdeq^{-N}$. Some of the multivariate Jackson summations have been extended to the level of elliptic hypergeometric series and are partly covered in Section 6.3 of this volume. One of the most important is the following A_n Jackson summation [84, Theorem 6.14]:

$$
\begin{aligned}
\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \Bigg(\prod_{1\le i<j\le n} \frac{x_iq^{k_i} - x_jq^{k_j}}{x_i - x_j} \prod_{i=1}^n \frac{1 - ax_iq^{k_i+|k|}}{1 - ax_i} \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j; q)_{k_i}}{(qx_i/x_j; q)_{k_i}} \\
\times \prod_{i=1}^n \frac{(ax_i;q)_{|k|}\,(dx_i, a^2x_iq^{1+N_i}/bcd; q)_{k_i}}{(ax_iq^{1+|N_i|};q)_{|k|}\,(ax_iq/b, ax_iq/c; q)_{k_i}} \cdot \frac{(b,c;q)_{|k|}}{(aq/d, bcdq^{-|N|}/a; q)_{|k|}} q^{|k|} \Bigg) \\
= \frac{(aq/bd, aq/cd; q)_{|N|}}{(aq/d, aq/bcd; q)_{|N|}} \prod_{i=1}^n \frac{(ax_iq, ax_iq/bc; q)_{N_i}}{(ax_iq/b, ax_iq/c; q)_{N_i}}.
\end{aligned} \tag{5.2.18}
$$

It was initially proved by partial fraction decompositions and functional equations; a more direct proof (which extends to the elliptic level) utilizes partial fraction decompositions and induction [104].

From (5.2.18), by multivariable matrix inversion, the following A_n Jackson summation was deduced in [119, Theorem 4.1]:

$$
\begin{aligned}
\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \Bigg(\prod_{1\le i<j\le n} \frac{x_iq^{k_i} - x_jq^{k_j}}{x_i - x_j} \prod_{i=1}^n \frac{(bcd/ax_i;q)_{|k|-k_i}\,(d/x_i;q)_{|k|}\,(a^2x_iq^{1+|N|}/bcd;q)_{k_i}}{(d/x_i;q)_{|k|-k_i}\,(bcdq^{-N_i}/ax_i;q)_{|k|}\,(ax_iq/d;q)_{k_i}} \\
\times \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \cdot \frac{(1-aq^{2|k|})}{(1-a)} \frac{(a,b,c;q)_{|k|}}{(aq^{1+|N|}, aq/b, aq/c; q)_{|k|}} q^{|k|} \Bigg) \\
= \frac{(aq, aq/bc; q)_{|N|}}{(aq/b, aq/c; q)_{|N|}} \prod_{i=1}^n \frac{(ax_iq/bd, ax_iq/bc; q)_{N_i}}{(ax_iq/d, ax_iq/bcd; q)_{N_i}}.
\end{aligned} \tag{5.2.19}
$$

(Its elliptic extension is deduced in [112].) Both summations, (5.2.18) and (5.2.19), which are summed over rectangular regions, can be turned to summations over a tetrahedral region (or simplex) $\{k_1,\dots,k_n \ge 0,\ |k| \le N\}$ by a polynomial argument, a standard procedure in the multivariate theory, used extensively in [88]. Other Jackson summations which have been extended to the elliptic level are the C_n Jackson summations in [14, Theorem 4.1] (independently derived in [92, Theorem 6.13]) and [115, Theorem 4.3], the D_n Jackson summations (also referred to as A_n Jackson summations by some authors) of [6, Theorem 7] and [113, Theorem 5.6], the A_n and D_n Jackson summations in [9, Section 11], and the BC_n Jackson summation in [19, Theorem 3] (also derived in [101, $p \to 0$ in Theorem 2.1]) which was originally conjectured in [126].

The following A_n Jackson summation is due to Gustafson and Rakha [41, Theorem 1.2] (but stated here as in [110] where it has been extended to the elliptic level):

$$\begin{aligned}\sum_{\substack{k_1,\dots,k_n\ge 0\\ |k|\le N}}\Bigg(&\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\,(x_ix_j;q)_{k_i+k_j}\prod_{1\le i,j\le n}(qx_i/x_j;q)_{k_i}^{-1}\prod_{i=1}^n\frac{1-ax_iq^{k_i+|k|}}{1-ax_i}\\ &\times\prod_{i=1}^n\frac{(ax_i;q)_{|k|}}{(aq/x_i;q)_{|k|-k_i}\,(ax_iq^{1+N_i};q)_{k_i}}\cdot\prod_{j=1}^4\frac{\prod_{i=1}^n(x_ib_j;q)_{k_i}}{(aq/b_j;q)_{|k|}}\cdot(q^{-N};q)_{|k|}\,q^{|k|}\Bigg)\\ &=\big(aq/b_1,aq/b_2,aq/b_3,aq/b_1b_2b_3X^2;q\big)_N^{-1}\prod_{i=1}^n\frac{(ax_iq;q)_N}{(aq/x_i;q)_N}\\ &\times\begin{cases}\big(aq/X,aq/b_1b_2X,aq/b_1b_3X,aq/b_2b_3X;q\big)_N, & \text{if } n \text{ is odd},\\ \big(aq/b_1X,aq/b_2X,aq/b_3X,aq/b_1b_2b_3X;q\big)_N, & \text{if } n \text{ is even},\end{cases}\end{aligned}\tag{5.2.20}$$

where $X=x_1\cdots x_n$, under the assumption that $a^2q^{N+1}=b_1b_2b_3b_4X^2$.

Two similar multivariate ${}_8\phi_7$ summations (at the elliptic level) are established in [110], and two others are conjectured in [125, Conjectures 6.2 and 6.5]. The latter actually look more complicated, the sums running over pairs of partitions whose Ferrers diagrams differ by a horizontal strip (cf. [73]). Those summations may play a role in the construction of Askey–Wilson polynomials of type A.

An A_n Jackson summation of a quite different type, intimately related to Macdonald polynomials [73], has been derived in [118, Theorem 4.1]:

$$\begin{aligned}\sum_{\substack{k_1,\dots,k_n\ge 0\\ |k|\le N}}\Bigg(&\prod_{i,j=1}^n\frac{(qx_i/t_ix_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{1\le i<j\le n}\frac{(t_jx_i/x_j;q)_{k_i-k_j}}{(qx_i/t_ix_j;q)_{k_i-k_j}}\,\frac{1}{x_i-x_j}\\ &\times\det_{1\le i,j\le n}\left[(x_iq^{k_i})^{n-j}\left(1-t_i^{j-n-1}\frac{1-t_0x_iq^{k_i}}{1-t_0x_iq^{k_i}/t_i}\prod_{s=1}^n\frac{x_iq^{k_i}-x_s}{x_iq^{k_i}/t_i-x_s}\right)\right]\\ &\times\prod_{i=1}^n\frac{(dq^{-N}/t_0x_i;q)_{|k|-k_i}\,(t_0x_iq/t_i,bx_i,t_0^2x_iq^{1+N}/bdt_1\cdots t_n;q)_{k_i}}{(dt_iq^{-N}/t_0x_i;q)_{|k|-k_i}\,(t_0x_iq,t_0x_iq/dt_i,t_0x_iq^{1+N}/t_i;q)_{k_i}}\\ &\times\frac{(d,q^{-N};q)_{|k|}}{(bdq^{-N}/t_0,t_0q/bt_1\cdots t_n;q)_{|k|}}\,q^{\sum_{i=1}^n(2-i)k_i}\prod_{i=1}^n t_i^{(i-1)k_i+\sum_{j=i+1}^n k_j}\Bigg)\\ &=\frac{(t_0q/b,t_0q/bdt_1\cdots t_n;q)_N}{(t_0q/bd,t_0q/bt_1\cdots t_n;q)_N}\prod_{i=1}^n\frac{(t_0x_iq/t_i,t_0x_iq/d;q)_N}{(t_0x_iq,t_0x_iq/dt_i;q)_N}.\end{aligned}\tag{5.2.21}$$

A similar Jackson sum of type C_n has been conjectured in [118, Conjecture 4.5]. For the A_n identity in (5.2.21) and the similar C_n identity from [118] (conjectured) elliptic extensions have not yet been established. The difficulty stems from the special type of determinants (which do not allow termwise elliptic extension) appearing in the respective summands of the series.

The A_n Jackson summation in (5.2.21) is also remarkable in the sense that no corresponding multivariate Bailey transformation has yet been found or conjectured. (The same applies to [118, Conjecture 4.5].) The other Jackson summations which we have discussed in this subsection all can be generalized to transformations. Since in this chapter we are mainly concerned with identities that do not directly extend to the elliptic setting we are not reproducing any of the multivariate Bailey transformations here. (The only exception is the C_n nonterminating Bailey transformation in (5.2.48), as nonterminating series do not admit a direct elliptic extension, for the reason of convergence.) For a discussion on multivariate extensions of Bailey's ${}_{10}\phi_9$ transformation, see Chapter 6, Subsections 6.3.3 and 6.3.4 of this volume.

2.4. **Some multilateral summations.** Dougall's bilateral ${}_2H_2$ summation [21, Section 13] is

$$ {}_2H_2\left[\begin{matrix}a,b\\c,d\end{matrix};1\right]=\frac{\Gamma(1-a)\Gamma(1-b)\Gamma(c)\Gamma(d)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}, \tag{5.2.22}$$

where the series either terminates, or $\mathrm{Re}(c+d-a-b-1)>0$ for convergence. This identity does not admit a direct basic extension (as a closed form ${}_2\psi_2$ summation with general parameters does not exist). A related, similar looking identity is Ramanujan's ${}_1\psi_1$ summation theorem (cf. [28, Equation (II.29)]),

$$ {}_1\psi_1\left[\begin{matrix}a\\b\end{matrix};q,z\right]=\frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, \tag{5.2.23}$$

where $|b/a|<|z|<1$.

An A_n extension of Dougall's ${}_2H_2$ summation theorem was proved by Gustafson in [32, Theorem 1.11] by induction and residue calculus:

$$\begin{aligned}\sum_{k_1,\dots,k_n=-\infty}^{\infty}\prod_{1\le i<j\le n}\frac{x_i+k_i-x_j-k_j}{x_i-x_j}\prod_{i=1}^{n}\prod_{j=1}^{n+1}\frac{(a_j+x_i)_{k_i}}{(b_j+x_i)_{k_i}}\\ =\frac{\Gamma\Big(-n+\sum_{j=1}^{n+1}(b_j-a_j)\Big)\prod_{i=1}^{n}\prod_{j=1}^{n+1}\Gamma(1-a_j-x_i)\,\Gamma(b_j+x_i)}{\prod_{i,j=1}^{n+1}\Gamma(b_j-a_i)\prod_{1\le i<j\le n}\Gamma(1-x_i+x_j)\,\Gamma(1+x_i-x_j)},\end{aligned} \tag{5.2.24}$$

provided $\mathrm{Re}\Big(\sum_{j=1}^{n+1}(b_j-a_j)\Big)>n$.

Similarly, an A_n extension of Ramanujan's ${}_1\psi_1$ summation theorem was proved in [32, Theorem 1.17]:

$$\begin{aligned}\sum_{k_1,\dots,k_n=-\infty}^{\infty}\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{i,j=1}^{n}\frac{(a_jx_i/x_j;q)_{k_i}}{(b_jx_i/x_j;q)_{k_i}}\cdot z^{|k|}\\ =\frac{(a_1\cdots a_nz,q/a_1\cdots a_nz;q)_\infty}{(z,b_1\cdots b_nq^{1-n}/a_1\cdots a_nz;q)_\infty}\prod_{i,j=1}^{n}\frac{(b_jx_i/a_ix_j,qx_i/x_j;q)_\infty}{(qx_i/a_ix_j,b_jx_i/x_j;q)_\infty},\end{aligned} \tag{5.2.25}$$

where $|q|<1$ and $|b_1\cdots b_n q^{1-n}/a_1\cdots a_n|<|z|<1$.

The special case of (5.2.25), in which $b_1=\cdots=b_n=b$, was previously obtained in [82, Theorem 1.15]. Further applications of this first multilateral ${}_1\psi_1$ summation appear in [82, 83, 88].

Another A_n ${}_1\psi_1$ summation theorem was found in [95, Theorem 3.2]:

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{i,j=1}^{n}(b_jx_i/x_j;q)_{k_i}^{-1}\prod_{i=1}^{n}x_i^{nk_i-|k|}\\ \times(a;q)_{|k|}(-1)^{(n-1)|k|}q^{-\binom{|k|}{2}+n\sum_{i=1}^{n}\binom{k_i}{2}}z^{|k|}\Bigg)\\ =\frac{(az,q/az,b_1\cdots b_nq^{1-n}/a;q)_\infty}{(z,b_1\cdots b_nq^{1-n}/az,q/a;q)_\infty}\prod_{i,j=1}^{n}\frac{(qx_i/x_j;q)_\infty}{(b_jx_i/x_j;q)_\infty},\tag{5.2.26}$$

where $|q|<1$ and $|b_1\cdots b_nq^{1-n}/a|<|z|<1$. (The specified region of convergence can be determined by an analysis as carried out in [117, Appendix A].)

Another A_n extension of Ramanujan's ${}_1\psi_1$ summation is implicitly contained in [74]. Written out in explicit terms, it reads as [131, last identity]

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty}\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\frac{(x_i/tx_j;q)_{k_i-k_j}}{(qtx_i/x_j;q)_{k_i-k_j}}q^{-k_j}t^{k_i-k_j}\right)\cdot\frac{(a;q)_{|k|}}{(b;q)_{|k|}}z^{|k|}\\ =\frac{(az,q/az,b/a,qt;q)_\infty}{(z,b/az,q/a,b;q)_\infty}\prod_{i=1}^{n-1}\frac{(qt^{i+1};q)_\infty}{(t^i;q)_\infty}\prod_{i,j=1}^{n}\frac{(qx_i/x_j;q)_\infty}{(qtx_i/x_j;q)_\infty},\tag{5.2.27}$$

where $|q|<1$, $|t|<1$ and $|b/a|<|z|<1$.

Taking coefficients of z^N on both sides of (5.2.27) while appealing to (the univariate version of) Ramanujan's ${}_1\psi_1$ summation, we obtain the interesting identity

$$\sum_{\substack{-\infty\le k_1,\dots,k_n\le\infty\\|k|=N}}\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\frac{(x_i/tx_j;q)_{k_i-k_j}}{(qtx_i/x_j;q)_{k_i-k_j}}q^{-k_j}t^{k_i-k_j}\right)\\ =\frac{(qt;q)_\infty}{(q;q)_\infty}\prod_{i=1}^{n-1}\frac{(qt^{i+1};q)_\infty}{(t^i;q)_\infty}\prod_{i,j=1}^{n}\frac{(qx_i/x_j;q)_\infty}{(qtx_i/x_j;q)_\infty},\tag{5.2.28}$$

subject to $|q|<1$. Observe that the right-hand side is independent of N.

Some additional (simpler) A_n ${}_1\psi_1$ summations are given in [39], [115, Section 2] and [111, Section 6].

Bailey's very-well-poised ${}_6\psi_6$ summation is (cf. [28, Equation (II.33)])

$$
\begin{aligned}
&{}_6\psi_6\left[\begin{matrix} qa^{\frac12}, -qa^{\frac12}, b, c, d, e \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e \end{matrix}; q, q\right] \\
&\quad = \frac{(q, aq, q/a, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, a^2q/bcde; q)_\infty},
\end{aligned}
$$

where $|q| < 1$ and $|a^2q/bcde| < 1$. Several root system extensions of Bailey's ${}_6\psi_6$ summation formula exist. Due to the fundamental importance of the ${}_6\psi_6$ summation, we review several of these summations:

We start with an A_n extension of the ${}_6\psi_6$ summation [32, Theorem 1.15]:

$$
\begin{aligned}
\sum_{k_1,\dots,k_n=-\infty}^{\infty} &\Bigg(\prod_{1\le i<j\le n} \frac{x_iq^{k_i} - x_jq^{k_j}}{x_i - x_j} \prod_{i=1}^n \frac{1 - ax_iq^{k_i+|k|}}{1 - ax_i} \prod_{i,j=1}^n \frac{(b_jx_i/x_j; q)_{k_i}}{(ax_iq/e_jx_j; q)_{k_i}} \\
&\times \prod_{i=1}^n \frac{(e_ix_i; q)_{|k|}\,(cx_i; q)_{k_i}}{(ax_iq/b_i; q)_{|k|}\,(ax_iq/d; q)_{k_i}} \cdot \frac{(d; q)_{|k|}}{(aq/c; q)_{|k|}} \left(\frac{a^{n+1}q}{BcdE}\right)^{|k|} \Bigg) \\
&= \frac{(aq/Bc, a^nq/dE, aq/cd; q)_\infty}{(a^{n+1}q/BcdE, aq/c, q/d; q)_\infty} \prod_{i,j=1}^n \frac{(ax_iq/b_ie_jx_j, qx_i/x_j; q)_\infty}{(qx_i/b_ix_j, ax_iq/e_jx_j; q)_\infty} \\
&\qquad \times \prod_{i=1}^n \frac{(aq/ce_ix_i, ax_iq/b_id, ax_iq, q/ax_i; q)_\infty}{(ax_iq/b_i, q/e_ix_i, q/cx_i, ax_iq/d; q)_\infty}, \qquad (5.2.29)
\end{aligned}
$$

where $B = b_1 \dots b_n$ and $E = e_1 \dots e_n$, provided $|q| < 1$ and $|a^{n+1}q/BcdE| < 1$.

The multilateral identity above can also be written in a more compact form. We then have the A_n ${}_6\psi_6$ summation from [32, Theorem 1.15]:

$$
\begin{aligned}
\sum_{\substack{-\infty\le k_1,\dots,k_{n+1}\le\infty \\ k_1+\dots+k_{n+1}=0}} &\prod_{1\le i<j\le n+1} \frac{x_iq^{k_i} - x_jq^{k_j}}{x_i - x_j} \prod_{i,j=1}^{n+1} \frac{(a_jx_i/x_j; q)_{k_i}}{(b_jx_i/x_j; q)_{k_i}} \\
&= \frac{(b_1\cdots b_{n+1}q^{-n}, q/a_1\cdots a_{n+1}; q)_\infty}{(q, b_1\cdots b_{n+1}q^{-n}/a_1\cdots a_{n+1}; q)_\infty} \prod_{i,j=1}^{n+1} \frac{(qx_i/x_j, b_jx_i/a_ix_j; q)_\infty}{(b_jx_i/x_j, x_iq/a_ix_j; q)_\infty}, \qquad (5.2.30)
\end{aligned}
$$

provided $|q| < 1$ and $|b_1\cdots b_{n+1}q^{-n}/a_1\cdots a_{n+1}| < 1$. It is not difficult to see that (5.2.29) and (5.2.30) are equivalent by a change of variables.

If in (5.2.30) one replaces the parameters x_i, a_i and b_i, by q^{x_i}, q^{a_i} and q^{b_i}, respectively, and formally lets $q \to 1$, one obtains the following A_n ${}_5H_5$ summation from [32, Theorem 1.13] (given a direct proof there by functional equations without appealing to a $q \to 1$

limit):

$$\sum_{\substack{-\infty\le k_1,\dots,k_{n+1}\le\infty\\ k_1+\cdots+k_{n+1}=0}} \prod_{1\le i<j\le n+1} \frac{x_i+k_i-x_j-k_j}{x_i-x_j} \prod_{i,j=1}^{n+1} \frac{(a_j+x_i-x_j)_{k_i}}{(b_j+x_i-x_j)_{k_i}}$$

$$= \frac{\Gamma\Big(-n+\sum_{i=1}^{n+1}(b_i-a_i)\Big)}{\Gamma\Big(1-\sum_{i=1}^{n+1}a_i\Big)\,\Gamma\Big(-n+\sum_{i=1}^{n+1}b_i\Big)} \prod_{i,j=1}^{n+1} \frac{\Gamma(b_j+x_i-x_j)\,\Gamma(1-a_i+x_i-x_j)}{\Gamma(1+x_i-x_j)\,\Gamma(b_j-a_i+x_i-x_j)}, \qquad (5.2.31)$$

provided $\mathrm{Re}\Big(\sum_{i=1}^{n+1}(b_i-a_i)\Big)>n$.

Another A_n very-well-poised ${}_6\psi_6$ summation was derived in [119].

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{i,j=1}^{n} \frac{(c_jx_i/x_j;q)_{k_i}}{(ax_iq/e_jx_j;q)_{k_i}}$$

$$\times \prod_{i=1}^{n} \frac{(aq/bCx_i;q)_{|k|-k_i}\,(dE/a^{n-1}e_ix_i;q)_{|k|}\,(bx_i;q)_{k_i}}{(dE/a^nx_i;q)_{|k|-k_i}\,(ac_iq/bCx_i;q)_{|k|}\,(ax_iq/d;q)_{k_i}}$$

$$\times \frac{1-aq^{2|k|}}{1-a}\,\frac{(E/a^{n-1};q)_{|k|}}{(aq/C;q)_{|k|}} \left(\frac{a^{n+1}q}{bCdE}\right)^{|k|}\Bigg)$$

$$= \frac{(aq,q/a,aq/bd;q)_\infty}{(aq/C,a^{n+1}q/bCdE,a^{n-1}q/E;q)_\infty} \prod_{i,j=1}^{n} \frac{(qx_i/x_j,ax_iq/c_ie_jx_j;q)_\infty}{(qx_i/c_ix_j,ax_iq/e_jx_j;q)_\infty}$$

$$\times \prod_{i=1}^{n} \frac{(a^nx_iq/dE,aq/be_ix_i,aq/bCx_i,ax_iq/c_id;q)_\infty}{(a^{n-1}e_ix_iq/dE,q/bx_i,ax_iq/d,ac_iq/bCx_i;q)_\infty}, \qquad (5.2.32)$$

where $C=c_1\cdots c_n$ and $E=e_1\cdots e_n$, provided $|q|<1$ and $|a^{n+1}q/BcdE|<1$.

A C_n very-well-poised ${}_6\psi_6$ summation was derived in [33, Theorem 5.1].

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{1\le i\le j\le n} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}$$

$$\times \prod_{i,j=1}^{n} \frac{(c_jx_i/x_j,e_jx_ix_j;q)_{k_i}}{(ax_ix_jq/c_j,ax_iq/e_jx_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(bx_i,dx_i;q)_{k_i}}{(ax_iq/b,ax_iq/d;q)_{k_i}} \cdot \left(\frac{a^{n+1}q}{bCdE}\right)^{|k|}\Bigg)$$

$$= \prod_{1\le i<j\le n} (ax_ix_jq/c_ic_j,aq/e_ie_jx_ix_j;q)_\infty \prod_{1\le i\le j\le n} (ax_ix_jq,q/ax_ix_j;q)_\infty$$

$$\times \frac{(aq/bd;q)_\infty}{(a^{n+1}q/bCdE;q)_\infty} \prod_{i,j=1}^{n} \frac{(ax_iq/c_ie_jx_j,qx_i/x_j;q)_\infty}{(ax_iq/e_jx_j,q/e_jx_ix_j,ax_ix_jq/c_i,qx_i/c_ix_j;q)_\infty}$$

$$\times \prod_{i=1}^{n} \frac{(ax_iq/bc_i,aq/be_ix_i,ax_iq/c_id,aq/de_ix_i;q)_\infty}{(ax_iq/b,q/bx_i,ax_iq/d,q/dx_i;q)_\infty}, \qquad (5.2.33)$$

where $C = c_1 \cdots c_n$ and $E = e_1 \cdots e_n$, provided $|q| < 1$ and $|a^{n+1}q/bCdE| < 1$.

Here is an $\mathrm{B}_n^\vee$ (in Macdonald's [71] terminology for affine root systems; or labeled $\mathrm{A}_{2n-1}^{(2)}$ by Kac [52]) very-well-poised ${}_6\psi_6$ summation, obtained in [33, Theorem 6.1].

$$
\begin{aligned}
&\sum_{\substack{-\infty\le k_1,\dots,k_n\le\infty\\ |k|\equiv\sigma \pmod 2}} \Bigg(\prod_{1\le i<j\le n} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{1\le i\le j\le n} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \\
&\qquad\qquad \times \prod_{i,j=1}^n \frac{(c_jx_i/x_j, e_jx_ix_j;q)_{k_i}}{(ax_ix_jq/c_j, ax_iq/e_jx_j;q)_{k_i}} \cdot \left(-\frac{a^n}{bCdE}\right)^{|k|} \Bigg) \\
&= \prod_{1\le i<j\le n} (ax_ix_jq/c_ic_j, aq/e_ie_jx_ix_j;q)_\infty \prod_{1\le i\le j\le n} (ax_ix_jq, q/ax_ix_j;q)_\infty \\
&\quad \times \frac{(-q;q)_\infty}{(-a^n/CE;q)_\infty} \prod_{i,j=1}^n \frac{(ax_iq/c_ie_jx_j, qx_i/x_j;q)_\infty}{(ax_iq/e_jx_j, q/e_jx_ix_j, ax_ix_jq/c_i, qx_i/c_ix_j;q)_\infty} \\
&\qquad\qquad\qquad \times \prod_{i=1}^n \frac{(aqx_i^2/c_i^2, aq/e_i^2u_i^2;q^2)_\infty}{(aqx_i^2, q/ax_i^2;q^2)_\infty},
\end{aligned}
\tag{5.2.34}
$$

where $C = c_1 \cdots c_n$ and $E = e_1 \cdots e_n$, and where $\sigma = 0, 1$, provided $|q| < 1$ and $|a^n/CE| < 1$.

As observed in [125], the identity in (5.2.34) is closely connected to the $b = \sqrt{aq}$, $d = -\sqrt{aq}$ case of the identity in (5.2.33), where the sum evaluates to twice the product on the right-hand side of (5.2.34) (the latter being independent of σ).

Another C_n very-well-poised ${}_6\psi_6$ summation was established in [18, Equation (2.22)].

$$
\begin{aligned}
&\sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j} \prod_{1\le i\le j\le n} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \\
&\qquad \times \prod_{1\le i<j\le n} \frac{(tax_ix_j;q)_{k_i+k_j}\,(tx_i/x_j;q)_{k_i-k_j}}{(ax_ix_jq/t;q)_{k_i+k_j}\,(qx_i/tx_j;q)_{k_i-k_j}} \cdot \left(\frac{t^2}{q}\right)^{\sum_{i=1}^n (i-1)k_i} \\
&\qquad \times \prod_{i=1}^n \frac{(bx_i, cx_i, dx_i, ex_i;q)_{k_i}}{(ax_iq/b, ax_iq/c, ax_iq/d, ax_iq/e;q)_{k_i}} \cdot \left(\frac{t^{2-2n}a^2q}{bcde}\right)^{|k|} \Bigg) \\
&= \prod_{i,j=1}^n \frac{(qx_i/x_j;q)_\infty}{(qx_i/tx_j;q)_\infty} \frac{\prod_{1\le i\le j\le n}(ax_ix_jq, q/ax_ix_j;q)_\infty}{\prod_{1\le i<j\le n}(ax_ix_jq/t, q/tax_ix_j;q)_\infty} \prod_{i=1}^n \frac{(qt^{-i};q)_\infty}{(qt^{2-i-n}a^2/bcde;q)_\infty} \\
&\quad \times \prod_{i=1}^n \frac{(at^{1-i}q/bc, at^{1-i}q/bd, at^{1-i}q/be, at^{1-i}q/cd, at^{1-i}q/ce, at^{1-i}q/de;q)_\infty}{(q/bx_i, q/cx_i, q/dx_i, q/ex_i, ax_iq/b, ax_iq/c, ax_iq/d, ax_iq/e;q)_\infty},
\end{aligned}
\tag{5.2.35}
$$

provided $|q| < 1$, $|a^2q^{2-n}/bcde| < 1$ and $|t^{2-2n}a^2q/bcde| < 1$.

The next $\mathrm{B}_n^\vee$ (or $\mathrm{A}_{2n-1}^{(2)}$) ${}_6\psi_6$ summation (compare with (5.2.34)) was derived in [125, Theorem 4.1].

$$\sum_{\substack{-\infty\le k_1,\dots,k_n\le\infty\\ |k|\equiv\sigma \pmod 2}}\left(\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{1\le i\le j\le n}\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right.$$
$$\times\prod_{1\le i<j\le n}\frac{(tax_ix_j;q)_{k_i+k_j}\,(tx_i/x_j;q)_{k_i-k_j}}{(ax_ix_jq/t;q)_{k_i+k_j}\,(qx_i/tx_j;q)_{k_i-k_j}}\cdot\left(\frac{t^2}{q}\right)^{\sum_{i=1}^n(i-1)k_i}$$
$$\left.\times\prod_{i=1}^n\frac{(bx_i,cx_i;q)_{k_i}}{(ax_iq/b,ax_iq/c;q)_{k_i}}\cdot\left(-\frac{t^{2-2n}a}{bc}\right)^{|k|}\right)$$
$$=\frac{1}{2}\prod_{i,j=1}^n\frac{(qx_i/x_j;q)_\infty}{(qx_i/tx_j;q)_\infty}\frac{\prod_{1\le i\le j\le n}(ax_ix_jq,q/ax_ix_j;q)_\infty}{\prod_{1\le i<j\le n}(ax_ix_jq/t,q/tax_ix_j;q)_\infty}\prod_{i=1}^n\frac{(qt^{-i};q)_\infty}{(-t^{2-i-n}a/bc;q)_\infty}$$
$$\times\prod_{i=1}^n\frac{(at^{1-i}q/bc,-t^{1-i};q)_\infty\,(at^{2-2i}q/b,at^{2-2i}q/c;q^2)_\infty}{(q/bx_i,q/cx_i,ax_iq/b,ax_iq/c;q)_\infty\,(q/ax_i^2,aqx_i^2;q^2)_\infty},\tag{5.2.36}$$

where $\sigma=0,1$, provided $|q|<1$, $|aq^{1-n}/bc|<1$ and $|t^{2-2n}a/bc|<1$.

Similar to the relation between (5.2.34) and (5.2.33), the identity in (5.2.36) is closely connected to the $d=\sqrt{aq}$, $e=-\sqrt{aq}$ case of the identity in (5.2.35), where the sum evaluates to twice the product on the right-hand side of (5.2.36) (the latter being independent of σ).

Two other (simpler) C_n ${}_6\psi_6$ summations are given in [115, Theorem 3.4].

Multivariate analogues of Bailey's ${}_6\psi_6$ summation for *exceptional* root systems were derived in [34] (summation for G_2), [47] (summation for F_4; see also [48]), and [50] (further summations for G_2).

2.5. **Watson transformations.** The Watson transformation (cf. [28, Equation (III.18)]),

$${}_8\phi_7\left[\begin{matrix}a,\,qa^{\frac12},-qa^{\frac12},b,c,d,e,q^{-N}\\ a^{\frac12},-a^{\frac12},aq/b,aq/c,aq/d,aq/e,aq^{N+1}\end{matrix};q,\frac{a^2q^{N+2}}{bcde}\right]$$
$$=\frac{(aq,aq/de;q)_N}{(aq/d,aq/e;q)_N}\,{}_4\phi_3\left[\begin{matrix}aq/bc,d,e,q^{-N}\\ aq/b,aq/c,deq^{-N}/a\end{matrix};q,q\right],\tag{5.2.37}$$

is very useful. For instance, it can be used for a quick proof of the Rogers–Ramanujan identities, see [28, Section 2.7].

A number of Watson transformations for basic hypergeometric series associated with root systems exist. We only reproduce a few of them here.

In [31, Theorem 2.24], Gustafson applied the representation theory of $\mathrm{U}(n)$ to derive the first multivariable generalization of Whipple's classical transformation of an ordinary ($q=1$) terminating well-poised ${}_7F_6(1)$ into a terminating balanced ${}_4F_3(1)$. Its q-analogue, the first multivariable Watson transformation, was obtained in [84, Theorems 6.1 and 6.4] and [85, Theorems 6.1 and 6.4] by a direct, elementary proof utilizing q-difference equations and induction. Further details and applications are given in [84, 85, 87]. A more symmetrical A_n Watson transformation was derived in [89, Theorem 2.1], by means of the summation theorems and analysis from [88], where [88] provides an A_n generalization of much of the analysis in chapters one and two of Gasper and Rahman's book [28].

The following A_n Watson transformation was derived in [93, Theorem A.3].

$$\begin{aligned}
\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \Bigg(&\prod_{1\le i<j\le n} \frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \prod_{i,j=1}^n \frac{(q^{-N_j} x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^n \frac{1-ax_i q^{k_i+|k|}}{1-ax_i} \frac{(ax_i;q)_{|k|}}{(ax_i q^{1+N_i};q)_{|k|}} \\
&\times \prod_{i=1}^n \frac{(bx_i,cx_i;q)_{k_i}}{(ax_iq/d,ax_iq/e;q)_{k_i}} \cdot \frac{(d,e;q)_{|k|}}{(aq/b,aq/c;q)_{|k|}} \left(\frac{a^2q^{|N|+2}}{bcde}\right)^{|k|} \Bigg) \\
&= \frac{(aq/ce;q)_{|N|}}{(aq/c;q)_{|N|}} \prod_{i=1}^n \frac{(ax_iq;q)_{N_i}}{(ax_iq/e;q)_{N_i}} \\
&\times \sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \Bigg(\prod_{1\le i<j\le n} \frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \prod_{i,j=1}^n \frac{(q^{-N_j} x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \\
&\times \prod_{i=1}^n \frac{(cx_i;q)_{k_i}}{(ax_iq/d;q)_{k_i}} \cdot \frac{(aq/bd,e;q)_{|k|}}{(aq/b,ceq^{-|N|}/a;q)_{|k|}} q^{|k|} \Bigg).
\end{aligned} \tag{5.2.38}$$

For a very similar but different A_n Watson transformation, see the $f_i = q^{-N_i}$, $i = 1,\dots,n$, case of (5.2.50).

The following $\mathrm{C}_n \leftrightarrow \mathrm{A}_{n-1}$ Watson transformation was first derived in [92, Theorem 6.6].

$$\begin{aligned}
\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \Bigg(&\prod_{1\le i<j\le n} \frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \prod_{1\le i\le j\le n} \frac{1-x_ix_jq^{k_i+k_j}}{1-x_ix_j} \prod_{i,j=1}^n \frac{(q^{-N_j} x_i/x_j, x_ix_j;q)_{k_i}}{(qx_i/x_j, q^{1+N_i}x_ix_j;q)_{k_i}} \\
&\times \prod_{i=1}^n \frac{(bx_i,cx_i,dx_i,ex_i;q)_{k_i}}{(qx_i/b,qx_i/c,qx_i/d,qx_i/e;q)_{k_i}} \cdot \left(\frac{q^{|N|+2}}{bcde}\right)^{|k|} \Bigg) \\
= (q/bc;q)_{|N|} &\prod_{i=1}^n \frac{1}{(qx_i/b,qx_i/c;q)_{N_i}} \prod_{i,j=1}^n (qx_ix_j;q)_{N_i} \prod_{1\le i<j\le n} \frac{1}{(qx_ix_j;q)_{N_i+N_j}} \\
&\times \sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \Bigg(\prod_{1\le i<j\le n} \frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \prod_{i,j=1}^n \frac{(q^{-N_j} x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \\
&\times \prod_{i=1}^n \frac{(bx_i,cx_i;q)_{k_i}}{(qx_i/d,qx_i/e;q)_{k_i}} \cdot \frac{(q/de;q)_{|k|}}{(bcq^{-|N|};q)_{|k|}} q^{|k|} \Bigg).
\end{aligned} \tag{5.2.39}$$

This identity was utilized in [4] and in [5] to obtain identities for characters of affine Lie algebras.

Several other Watson transformations are given in [6, 8, 9, 12]. One of them is the following (cf. [8, Theorem 4.10]):

$$\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n}\Bigg(\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\,\frac{(ax_ix_jq/c;q)_{k_i+k_j}}{(ex_ix_j;q)_{k_i+k_j}}\prod_{i=1}^n\frac{1-ax_iq^{k_i+|k|}}{1-ax_i}\,\frac{(aq/ex_i;q)_{|k|-k_i}}{(c/x_i;q)_{|k|-k_i}}$$
$$\times\prod_{i=1}^n\frac{(ax_i,c/x_i;q)_{|k|}\,(bx_i;q)_{k_i}}{(ax_iq^{1+N_i},aq^{1-N_i}/ex_i;q)_{|k|}\,(ax_iq/d;q)_{k_i}}$$
$$\times\prod_{i,j=1}^n\frac{(q^{-N_j}x_i/x_j,ex_ix_jq^{N_j};q)_{k_i}}{(qx_i/x_j,ax_ix_jq/c;q)_{k_i}}\cdot\frac{(d;q)_{|k|}}{(aq/b;q)_{|k|}}\left(\frac{q^2a^2}{bcde}\right)^{|k|}\Bigg)$$
$$=d^{-|N|}\prod_{i=1}^n\frac{(ax_iq,dex_i/a;q)_{N_i}}{(ex_i/a,ax_iq/d;q)_{N_i}}$$
$$\times\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n}\Bigg(\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\,\frac{(ax_ix_jq/c;q)_{k_i+k_j}}{(ex_ix_j;q)_{k_i+k_j}}\prod_{i=1}^n\frac{(ax_iq/bc;q)_{k_i}}{(dex_i/a;q)_{k_i}}$$
$$\times\prod_{i,j=1}^n\frac{(q^{-N_j}x_i/x_j,ex_ix_jq^{N_j};q)_{k_i}}{(qx_i/x_j,ax_ix_jq/c;q)_{k_i}}\cdot\frac{(d;q)_{|k|}}{(aq/b;q)_{|k|}}q^{|k|}\Bigg). \quad (5.2.40)$$

This multivariate Watson transformation cannot be simplified to any multivariate Jackson summation as a special case.

2.6. **Dimension changing transformations.** Heine's q-analogue of the classical Euler transformation of ${}_2F_1$ series is (cf. [28, Equation (III.3)])

$${}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right]=\frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/a,c/b\\c\end{matrix};q,\frac{abz}{c}\right], \quad (5.2.41)$$

valid for $|q|<1$, $|z|<1$ and $|abz/c|<1$. The following result, which was first derived by Kajihara [54], connects A_n and A_m basic hypergeometric series and reduces, for $n=m=1$, to the q-Euler transformation.

$$\sum_{k_1,\dots,k_n\ge0}\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{1\le i,j\le n}\frac{(a_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{\substack{1\le i\le n\\1\le l\le m}}\frac{(b_lx_iy_l;q)_{k_i}}{(cx_iy_l;q)_{k_i}}\cdot z^{|k|}$$
$$=\frac{(ABz/c^m;q)_\infty}{(z;q)_\infty}$$
$$\times\sum_{\kappa_1,\dots,\kappa_m\ge0}\prod_{1\le j<l\le m}\frac{y_jq^{\kappa_j}-y_lq^{\kappa_l}}{y_j-y_l}\prod_{1\le j,l\le m}\frac{(cy_j/b_ly_l;q)_{\kappa_j}}{(qy_j/y_l;q)_{\kappa_j}}\prod_{\substack{1\le i\le n\\1\le l\le m}}\frac{(cx_iy_l/a_i;q)_{\kappa_l}}{(cx_iy_l;q)_{\kappa_l}}\cdot\left(\frac{ABz}{c^m}\right)^{|\kappa|}, \quad (5.2.42)$$

where $A=a_1\cdots a_n$, and $B=b_1\cdots b_m$, provided $|q|<1$, $|z|<1$ and $|ABz/c^m|<1$.

Now let

$$\Phi_N^{n,m}\left(\begin{array}{c|c}\{a_i\}_n & \{b_l y_l\}_m\\ \{x_i\}_n & \{cy_l\}_m\end{array}\right)$$
$$:=\sum_{\substack{k_1,\dots,k_n\ge 0\\ |k|=N}}\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{1\le i,j\le n}\frac{(a_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{\substack{1\le i\le n\\ 1\le l\le m}}\frac{(b_lx_iy_l;q)_{k_i}}{(cx_iy_l;q)_{k_i}}. \tag{5.2.43}$$

The transformation in (5.2.42) was used to derive the following identity [56, Theorem 3.1] (which we state here in corrected form):

$$\sum_{K=0}^{N}\Phi_K^{n_2,m_2}\left(\begin{array}{c|c}\{f/e_t\}_{n_2} & \{fw_r/d_r\}_{m_2}\\ \{v_t\}_{n_2} & \{fw_r\}_{m_2}\end{array}\right)\Phi_{N-K}^{n_1,m_1}\left(\begin{array}{c|c}\{a_i\}_{n_1} & \{b_ly_l\}_{m_1}\\ \{x_i\}_{n_1} & \{cy_l\}_{m_1}\end{array}\right)\left(\frac{d_1\cdots d_{m_2}e_1\cdots e_{n_2}}{f^{n_2}}\right)^K$$
$$=\sum_{L=0}^{N}\Phi_L^{m_1,n_1}\left(\begin{array}{c|c}\{c/b_l\}_{m_1} & \{cx_i/a_i\}_{n_1}\\ \{y_l\}_{m_1} & \{cx_i\}_{n_1}\end{array}\right)\Phi_{N-L}^{m_2,n_2}\left(\begin{array}{c|c}\{d_r\}_{m_2} & \{e_tv_t\}_{n_2}\\ \{w_r\}_{m_2} & \{fv_t\}_{n_2}\end{array}\right)\left(\frac{a_1\cdots a_{n_1}b_1\cdots b_{m_1}}{c^{m_1}}\right)^L, \tag{5.2.44}$$

where $a_1\cdots a_{n_1}b_1\cdots b_{m_1}/c^{m_1}=d_1\cdots d_{m_2}e_1\cdots e_{n_2}/f^{n_2}$. This identity can be viewed as a multivariate extension of the Sears transformation [28, Equation (III.16)]) (to which it reduces for $n=m=1$ after some elementary manipulations). A transformation similar to (5.2.42) but connecting C_n and C_m basic hypergeometric series has been given in [61].

Several other transformations connecting sums of different dimension exist. For instance, in [103] the following reduction formula for a multilateral Karlsson–Minton type basic hypergeometric series associated with the root system A_n was derived. (A basic hypergeometric series is said to be of *Karlsson–Minton type* if the quotient of corresponding upper and lower parameters is a nonnegative integer power of q.)

$$\sum_{\substack{k_1,\dots,k_n=-\infty\\ k_1+\cdots+k_n=0}}^{\infty}\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{\substack{1\le i\le n\\ 1\le j\le p}}\frac{(x_iy_jq^{m_j};q)_{k_i}}{(x_iy_j;q)_{k_i}}\prod_{i,j=1}^{n}\frac{(x_ia_j;q)_{k_i}}{(x_ib_j;q)_{k_i}}$$
$$=\frac{(q^{1-|m|}/AX,q^{1-n}BX;q)_\infty}{(q,q^{1-|m|-n}B/A;q)_\infty}\prod_{i,j=1}^{n}\frac{(b_i/a_j,qx_i/x_j;q)_\infty}{(q/x_ia_j,x_ib_j;q)_\infty}\prod_{\substack{1\le i\le n\\ 1\le j\le p}}\frac{(q^{-m_j}b_i/y_j;q)_{m_j}}{(q^{1-m_j}/x_iy_j;q)_{m_j}}$$
$$\times\sum_{\kappa_1,\dots,\kappa_p=0}^{m_1,\dots,m_p}q^{|\kappa|}\frac{(q^n/BX;q)_{|\kappa|}}{(q^{1-|m|}/AX;q)_{|\kappa|}}\prod_{1\le i<j\le n}\frac{y_iq^{\kappa_i}-y_jq^{\kappa_j}}{y_i-y_j}\prod_{\substack{1\le i\le n\\ 1\le j\le p}}\frac{(y_j/a_i;q)_{\kappa_j}}{(qy_j/b_i;q)_{\kappa_j}}\prod_{i,j=1}^{p}\frac{(q^{-m_i}y_j/y_i;q)_{\kappa_j}}{(qy_j/y_i;q)_{\kappa_j}}, \tag{5.2.45}$$

where $A=a_1\cdots a_n$, $B=b_1\cdots b_n$, $X=x_1\cdots x_n$, provided $|q|<1$ and $|q^{1-|m|-n}B/A|<1$.

Similarly, in [102] the following reduction formula for a multilateral Karlsson–Minton type basic hypergeometric series associated with the root system C_n was derived.

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{1\le i\le j\le n}\frac{1-x_ix_jq^{k_i+k_j}}{1-x_ix_j}$$
$$\times\prod_{\substack{1\le i\le n\\1\le j\le p}}\frac{(x_iy_jq^{m_j},qx_i/y_j;q)_{k_i}}{(x_iy_j,q^{1-m_j}x_i/y_j;q)_{k_i}}\prod_{\substack{1\le i\le n\\1\le j\le 2n+2}}\frac{(x_ia_j;q)_{k_i}}{(qx_i/a_j;q)_{k_i}}\cdot\left(\frac{q^{1-|m|}}{A}\right)^{|k|}\Bigg)$$
$$=\frac{\prod_{1\le i\le j\le n}(qx_ix_j,q/x_ix_j;q)_\infty\prod_{i,j=1}^{n}(qx_i/x_j;q)_\infty}{\prod_{1\le i\le n,\,1\le j\le 2n+2}(qx_i/a_j,q/x_ia_j;q)_\infty}\frac{\prod_{1\le i<j\le 2n+2}(q/a_ia_j;q)_\infty}{(q/A;q)_\infty}$$
$$\times\frac{\prod_{1\le i\le 2n+2,\,1\le j\le p}(y_ja_i;q)_{m_j}}{\prod_{1\le i\le n,\,1\le j\le p}(y_jx_i,y_j/x_i;q)_{m_j}}\frac{\prod_{1\le i<j\le p}(y_iy_j;q)_{m_i+m_j}}{\prod_{i,j=1}^{p}(y_iy_j;q)_{m_i}}\frac{1}{(A;q)_{|m|}}$$
$$\times\sum_{\kappa_1,\dots,\kappa_p=0}^{m_1,\dots,m_p}\Bigg(\prod_{1\le i<j\le n}\frac{y_iq^{\kappa_i}-y_jq^{\kappa_j}}{y_i-y_j}\prod_{1\le i\le j\le n}\frac{1-y_iy_jq^{\kappa_i+\kappa_j-1}}{1-y_iy_jq^{-1}}$$
$$\times\prod_{\substack{1\le i\le 2n+2\\1\le j\le p}}\frac{(y_j/a_i;q)_{\kappa_j}}{(y_ja_i;q)_{\kappa_j}}\prod_{i,j=1}^{p}\frac{(q^{-1}y_iy_j,q^{-m_j}y_i/y_j;q)_{\kappa_i}}{(qy_i/y_j,q^{m_j}y_iy_j;q)_{\kappa_i}}\cdot\left(Aq^{|m|}\right)^{|\kappa|}\Bigg),\qquad(5.2.46)$$

where $A=a_1\cdots a_{2n+2}$, provided $|q|<1$ and $|q^{1-|m|}/A|<1$. A substantially more general transformation (involving four-fold multiple sums) was given by Masuda [77, Theorem 3].

Both (5.2.45) and (5.2.46) have many interesting consequences. In particular, these transformations form bridges between the one-variable and the multivariable theory and can be used to prove various summations and transformations for A_n and C_n basic hypergeometric series. For details, we refer the reader to Rosengren's papers [102, 103].

Other dimension changing transformations have been given (or conjectured) in [7, 29, 64, 99, 105].

2.7. **Multiterm transformations.**

Bailey's nonterminating balanced very-well-poised ${}_{10}\phi_9$ transformation is (cf. [28, Equation (III.39)])

$${}_{10}\phi_9\left[\begin{matrix}a,\,qa^{\frac12},-qa^{\frac12},b,c,d,e,f,g,h\\a^{\frac12},-a^{\frac12},aq/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h\end{matrix};q,q\right]$$
$$+\frac{(aq,b/a,c,d,e,f,g,h,bq/c,bq/d,bq/e,bq/f,bq/g,bq/h;q)_\infty}{(b^2q/a,a/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h,bc/a,bd/a,be/a,bf/a,bg/a,bh/a;q)_\infty}$$
$$\times\,{}_{10}\phi_9\left[\begin{matrix}b^2/a,\,qba^{-\frac12},-qba^{-\frac12},b,bc/a,bd/a,be/a,bf/a,bg/a,bh/a\\ba^{-\frac12},-ba^{-\frac12},bq/a,bq/c,bq/d,bq/e,bq/f,bq/g,bq/h\end{matrix};q,q\right]$$
$$=\frac{(aq,b/a,\lambda q/f,\lambda q/g,\lambda q/h,bf/\lambda,bg/\lambda,bh/\lambda;q)_\infty}{(\lambda q,b/\lambda,aq/f,aq/g,aq/h,bf/a,bg/a,bh/a;q)_\infty}$$

$$\times {}_{10}\phi_9\left[\begin{matrix}\lambda,\, q\lambda^{\frac12}, -q\lambda^{\frac12}, b, \lambda c/a, \lambda d/a, \lambda e/a, f, g, h\\ \lambda^{\frac12}, -\lambda^{\frac12}, \lambda q/b, aq/c, aq/d, aq/e, \lambda q/f, \lambda q/g, \lambda q/h\end{matrix}; q, q\right]$$
$$+\frac{(aq, b/a, f, g, h, bq/f, bq/g, bq/h, \lambda c/a, \lambda d/a, \lambda e/a, abq/\lambda c, abq/\lambda d, abq/\lambda e; q)_\infty}{(b^2q/\lambda, \lambda/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h, bc/a, bd/a, be/a, bf/a, bg/a, bh/a; q)_\infty}$$
$$\times {}_{10}\phi_9\left[\begin{matrix}b^2/\lambda,\, qb\lambda^{-\frac12}, -qb\lambda^{-\frac12}, b, bc/a, bd/a, be/a, bf/\lambda, bg/\lambda, bh/\lambda\\ b\lambda^{-\frac12}, -b\lambda^{-\frac12}, bq/\lambda, abq/c\lambda, abq/d\lambda, abq/e\lambda, bq/f, bq/g, bq/h\end{matrix}; q, q\right], \tag{5.2.47}$$

where $\lambda = a^2q/cde$, $a^3q^2 = bcdefgh$ and $|q| < 1$. This identity is at the top of the classical hierarchy of identities for basic hypergeometric series.

The following identity from [111, Corollar 4.1] (which was derived by determinant evaluations, following a method first used by Gustafson and Krattenthaler [39, 40] to derive A_n extensions of Heine's ${}_2\phi_1$ transformations, and subsequently used in a systematic manner in [115] and [111]) concerns a C_n extension of Bailey's four-term transformation in (5.2.47), where both sides of the identity involve 2^n nonterminating C_n basic hypergeometric series. Let $a^3q^{3-n} = bc_id_ie_ix_ifgh$ and $\lambda = a^2q/c_id_ie_ix_i$ for $i = 1, \dots, n$. Then there holds

$$\sum_{S\subseteq\{1,2,\dots,n\}}\Bigg[\left(\frac{b}{a}\right)^{\binom{n-|S|}{2}}\prod_{i\notin S}\frac{(ax_i^2q, c_ix_i, d_ix_i, e_ix_i; q)_\infty}{(ax_i/b, ax_iq/c_i, ax_iq/d_i, ax_iq/e_i; q)_\infty}$$
$$\times\prod_{i\notin S}\frac{(fx_i, gx_i, hx_i, b/ax_i, bq/c_i, bq/d_i, bq/e_i, bq/f, bq/g, bq/h; q)_\infty}{(ax_iq/f, ax_iq/g, ax_iq/h, b^2q/a, bc_i/a, bd_i/a, be_i/a, bf/a, bg/a, bh/a; q)_\infty}$$
$$\times\sum_{k_1,\dots,k_n=0}^{\infty}\Bigg(\prod_{\substack{1\le i<j\le n\\ i,j\in S}}\frac{(x_iq^{k_i}-x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)}\prod_{i\in S}\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}$$
$$\times\prod_{\substack{1\le i<j\le n\\ i,j\notin S}}\frac{(q^{k_i}-q^{k_j})(1-b^2q^{k_i+k_j}/a)}{(x_i-x_j)(1-ax_ix_j)}\prod_{i\notin S}\frac{1-b^2q^{2k_x}/a}{1-b^2/a}$$
$$\times\prod_{i\in S, j\notin S}\frac{(x_iq^{k_i}-bq^{y_j}/a)(1-bx_jq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)}$$
$$\times\prod_{i\in S}\frac{(ax_i^2, bx_i, c_ix_i, d_ix_i, e_ix_i, fx_i, gx_i, hx_i; q)_{k_i}}{(q, ax_iq/b, ax_iq/c_i, ax_iq/d_i, ax_iq/e_i, ax_iq/f, ax_iq/g, ax_iq/h; q)_{k_i}}$$
$$\times\prod_{i\notin S}\frac{(b^2/a, bx_i, bc_i/a, bd_i/a, be_i/a, bf/a, bg/a, bh/a; q)_{k_i}}{(q, bq/ax_i, bq/c_i, bq/d_i, bq/e_i, bq/f, bq/g, bq/h; q)_{k_i}}\cdot q^{|k|}\Bigg)\Bigg]$$
$$=\prod_{i=1}^{n}\frac{(ax_i^2q, b/ax_i, \lambda x_iq/f, \lambda x_iq/g, \lambda x_iq/h, bfq^{i-1}/\lambda, bgq^{i-1}/\lambda, bhq^{i-1}/\lambda; q)_\infty}{(\lambda x_i^2q, b/\lambda x_i, ax_iq/f, ax_iq/g, ax_iq/h, bfq^{i-1}/a, bgq^{i-1}/a, bhq^{i-1}/a; q)_\infty}$$
$$\times\prod_{1\le i<j\le n}\frac{1-\lambda x_ix_j}{1-ax_ix_j}\sum_{S\subseteq\{1,2,\dots,n\}}\Bigg[\left(\frac{b}{\lambda}\right)^{\binom{n-|S|}{2}}$$

$$\times \prod_{i\notin S} \frac{(\lambda x_i^2 q, \lambda c_i x_i/a, \lambda d_i x_i/a, \lambda e_i x_i/a, f x_i, g x_i, h x_i; q)_\infty}{(\lambda x_i/b, a x_i q/c_i, a x_i q/d_i, a x_i q/e_i, \lambda x_i q/f, \lambda x_i q/g, \lambda x_i q/h; q)_\infty}$$
$$\times \prod_{i\notin S} \frac{(b/\lambda x_i, abq/c_i\lambda, abq/d_i\lambda, abq/e_i\lambda, bq/f, bq/g, bq/h; q)_\infty}{(b^2 q/\lambda, bc_i/a, bd_i/a, be_i/a, bf/\lambda, bg/\lambda, bh/\lambda; q)_\infty}$$
$$\times \sum_{k_1,\ldots,k_n=0}^{\infty} \Bigg(\prod_{\substack{1\le i<j\le n\\ i,j\in S}} \frac{(x_i q^{k_i} - x_j q^{k_j})(1-\lambda x_i x_j q^{k_i+k_j})}{(x_i - x_j)(1-\lambda x_i x_j)} \prod_{i\in S} \frac{1-\lambda x_i^2 q^{2k_i}}{1-\lambda x_i^2}$$
$$\times \prod_{\substack{1\le i<j\le n\\ i,j\notin S}} \frac{(q^{k_i} - q^{k_j})(1-b^2 q^{k_i+k_j}/\lambda)}{(x_i - x_j)(1-\lambda x_i x_j)} \prod_{i\notin S} \frac{1-b^2 q^{2k_i}/\lambda}{1-b^2/\lambda}$$
$$\times \prod_{i\in S, j\notin S} \frac{(x_i q^{k_i} - b q^{k_j}/\lambda)(1-b x_i q^{k_i+k_j})}{(x_i - x_j)(1-\lambda x_i x_j)}$$
$$\times \prod_{i\in S} \frac{(\lambda x_i^2, b x_i, \lambda c_i x_i/a, \lambda d_i x_i/a, \lambda e_i x_i/a, f x_i, g x_i, h x_i; q)_{k_i}}{(q, \lambda x_i q/b, a x_i q/c_i, a x_i q/d_i, a x_i q/e_i, \lambda x_i q/f, \lambda x_i q/g, \lambda x_i q/h; q)_{k_i}}$$
$$\times \prod_{i\notin S} \frac{(b^2/\lambda, b x_i, bc_i/a, bd_i/a, be_i/a, bf/\lambda, bg/\lambda, bh/\lambda; q)_{k_i}}{(q, bq/\lambda x_i, abq/c_i\lambda, abq/d_i\lambda, abq/e_i\lambda, bq/f, bq/g, bq/h; q)_{k_i}} \cdot q^{|k|} \Bigg) \Bigg], \quad (5.2.48)$$

where $|q| < 1$.

The next identity from [94] concerns an A_n extension of the nonterminating Watson transformation (cf. [28, Equation (III.36)]),

$$\begin{aligned}
&{}_8\phi_7 \left[\begin{matrix} a,\, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, f \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix} ; q, \frac{a^2q^2}{bcdef} \right] \\
&= \frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty}\, {}_4\phi_3 \left[\begin{matrix} aq/bc, d, e, f \\ aq/b, aq/c, def/a \end{matrix} ; q, q \right] \\
&\quad + \frac{(aq, aq/bc, d, e, f, a^2q^2/bdef, a^2q^2/cdef; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, a^2q^2/bcdef, def/aq; q)_\infty} \\
&\qquad \times {}_4\phi_3 \left[\begin{matrix} aq/de, aq/df, aq/ef, a^2q^2/bcdefa \\ a^2q^2/bdef, a^2q^2/cdef, aq^2/def \end{matrix} ; q, q \right], \qquad (5.2.49)
\end{aligned}$$

where $|q| < 1$ and $|a^2q^2/bcdef| < 1$. In the multivariate case this is a transformation of a nonterminating very-well-poised A_n basic hypergeometric series into $n+1$ multiples of nonterminating balanced A_n basic hypergeometric series. It is interesting to point out that although the A_n ${}_8\phi_7$ series on the left-hand side of (5.2.50) is of the same type as that on the left-hand side of (5.2.38), this nonterminating A_n Watson transformation does *not* reduce to the terminating A_n Watson transformation in (5.2.38) as the A_n ${}_4\phi_3$ series on the respective right-hand sides are of different type. Specifically,

$$\sum_{k_1,\dots,k_n\ge 0}\Bigg(\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{i,j=1}^n\frac{(f_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{i=1}^n\frac{1-ax_iq^{k_i+|k|}}{1-ax_i}\,\frac{(ax_i;q)_{|k|}}{(ax_iq/f_i;q)_{|k|}}$$
$$\times\prod_{i=1}^n\frac{(bx_i,cx_i;q)_{k_i}}{(ax_iq/d,ax_iq/e;q)_{k_i}}\cdot\frac{(d,e;q)_{|k|}}{(aq/b,aq/c;q)_{|k|}}\left(\frac{a^2q^2}{bcdef_1\cdots f_n}\right)^{|k|}\Bigg)$$
$$=\frac{(aq/bf_1\cdots f_n,aq/cf_1\cdots f_n;q)_\infty}{(aq/b,aq/c;q)_\infty}\prod_{i=1}^n\frac{(ax_iq,af_iq/bcf_1\cdots f_nx_i;q)_\infty}{(aq/bcf_1\cdots f_nx_i,ax_iq/f_i;q)_\infty}$$
$$\times\sum_{k_1,\dots,k_n\ge 0}\Bigg(\prod_{1\le i<j\le n}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{i,j=1}^n\frac{(f_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$
$$\times\prod_{i=1}^n\frac{(ax_iq/de,bx_i,cx_i;q)_{k_i}}{(ax_iq/d,ax_iq/e,bcf_1\cdots f_nx_i/a;q)_{k_i}}\cdot q^{|k|}\Bigg)$$
$$+\frac{(q,a^2q^2/bcdf_1\cdots f_n,a^2q^2/bcef_1\cdots f_n;q)_\infty}{(a^2q^2/bcdef_1\cdots f_n,aq/b,aq/c;q)_\infty}\prod_{i=1}^n\frac{(ax_iq;q)_\infty}{(ax_iq/f_i;q)_\infty}$$
$$\times\sum_{s=1}^n\Bigg[q^{(n-1)k_s}\frac{(ax_sq/de,bx_s,cx_s;q)_\infty}{(bcf_1\cdots f_nx_s/aq,ax_sq/d,ax_sq/e;q)_\infty}\prod_{i=1}^n\frac{(f_ix_s/x_i;q)_\infty}{(qx_s/x_i;q)_\infty}$$
$$\times\sum_{k_1,\dots,k_n\ge 0}\Bigg(\prod_{\substack{1\le i\le n\\ i\ne s}}\frac{x_i}{x_i-x_s}\prod_{\substack{1\le i<j\le n\\ i,j\ne s}}\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\prod_{\substack{1\le i,j\le n\\ i\ne s}}\frac{(f_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$
$$\times\prod_{\substack{1\le i\le n\\ i\ne s}}\frac{1-bcf_1\cdots f_nx_iq^{k_i-y_s-1}/a}{1-bcf_1\dots f_nx_i/aq}\,\frac{(ax_iq/de,bx_i,cx_i;q)_{k_i}}{(bcf_1\cdots f_nx_i/a,ax_iq/d,ax_iq/e;q)_{k_i}}$$
$$\times\frac{(a^2q^2/bcdef_1\cdots f_n,aq/bf_1\cdots f_n,aq/cf_1\cdots f_n;q)_{k_s}}{(q,a^2q^2/bcdf_1\cdots f_n,a^2q^2/bcef_1\cdots f_n;q)_{k_s}}$$
$$\times\prod_{i=1}^n\frac{(af_iq/bcf_1\cdots f_nx_i;q)_{k_s}}{(aq^2/bcf_1\cdots f_nx_i;q)_{k_s}}\cdot q^{|k|}\Bigg)\Bigg],\qquad(5.2.50)$$

where $|q|<1$ and $|a^2q^2/bcdef_1\cdots f_n|<1$.

The $f_i=q^{-N_i}$, $i=1,\dots,n$, case of (5.2.50) gives a terminating A_n Watson transformation which is different from the one in (5.2.38). Milne and Newcomb [94] obtained yet another nonterminating A_n Watson transformation.

Ito [49] derived a BC_n extension of Slater's [121] general transformation formula for very-well-poised balanced ${}_{2r}\psi_{2r}$ series. Some interesting and potentially useful transformations for A_n basic hypergeometric series involving nested sums were recently given by Fang [23].

For further references to summations and transformations for basic hypergeometric series associated with root systems, see Milne's survey paper [90], and Milne and Newcomb's paper [94], and the references therein.

3. Hypergeometric and basic hypergeometric integrals associated with root systems

There exist a number of hypergeometric integral evaluations associated with root systems. Several of them can be viewed as extensions of Selberg's multivariate extension of 1944 of the classical beta integral evaluation [120],

$$\int_0^1 \cdots \int_0^1 \prod_{1\le i<j\le n} |z_i - z_j|^{2\gamma} \prod_{i=1}^n z_i^{\alpha-1}(1-z_i)^{\beta-1}\mathrm{d}z_i$$

$$= \prod_{i=1}^n \frac{\Gamma(\alpha+(i-1)\gamma)\,\Gamma(\beta+(i-1)\gamma)\,\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(n+i-2)\gamma)\,\Gamma(1+\gamma)}, \tag{5.3.1}$$

provided $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\gamma) + \max(\frac{1}{n}, \mathrm{Re}\frac{\alpha}{n-1}, \mathrm{Re}\frac{\beta}{n-1}) > 0$. The Selberg integral is used in many areas, see Chapter 11 of this volume and [25].

In 1982, Macdonald [72] conjectured related constant term identities associated with root systems together with q-analogues. Assume R to be a reduced root system of rank n with basis of simple roots $\{\alpha_1, \dots, \alpha_n\}$. Further, let e^α be the formal exponentials, for $\alpha \in R$, which form the group ring of the lattice generated by R, and let $d_1, \dots, d_n$ be the degrees of the fundamental invariants of the Weyl group $W(R)$. Then Macdonald conjectured [72, Conjecture 3.1] that for any nonnegative integer k the constant term, i.e. the term not containing any e^α, in

$$\prod_{\alpha\in R^+}\prod_{i=1}^k (1-q^{i-1}e^{-\alpha})(1-q^i e^\alpha) \tag{5.3.2}$$

(where R^+ is a system of positive roots in R) is

$$\prod_{i=1}^n \frac{(q;q)_{kd_i}}{(q;q)_k\,(q;q)_{k(d_i-1)}}. \tag{5.3.3}$$

(For the root system A_{n-1}, this exactly corresponds to the $t = q^k$ case of the squared norm evaluation of Macdonald polynomials indexed by $\lambda = (0, \dots, 0)$, the empty partition, in (5.4.1).) This conjecture can be reformulated in terms of reduced affine root systems and further strengthened. Generalizations with extra parameter were proposed by Morris [96]. A thorough account of the historic development of q-Selberg integrals and corresponding constant term identities is provided in [25, Section 2.3].

Of particular interest are those multiple integrals which in the univariate case reduce to the Askey–Wilson integral [2]:

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}} \frac{(z^2, 1/z^2; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty}\frac{\mathrm{d}z}{z} = \frac{2\,(abcd;q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \tag{5.3.4}$$

where $|q| < 1$, $|a| < 1$, $|b| < 1$, $|c| < 1$, $|d| < 1$, and $\mathbb{T}$ is the positively oriented unit circle. The Askey–Wilson integral is responsible for the orthogonality of the Askey–Wilson polynomials, which sit at the top of the q-Askey scheme of q-orthogonal polynomials. Such multivariate integral evaluations were first obtained by Gustafson in the early 1990s. In the following, we list some of the Askey–Wilson integral evaluations associated with root systems. Many of these or related integrals arise as constant term identities for (extensions of) Macdonald polynomials. This provides a natural link of the material presented here with Chapter 9 of this volume.

All these multivariate Askey–Wilson integral evaluations can be further generalized to multivariate extensions of the Nassrallah–Rahman integral evaluation [28, Equation (6.4.1)] (which has one more parameter then the Askey–Wilson integral evaluation). The latter admit elliptic extensions. They are treated in Section 6.2 of this volume together with some further extensions to integral transformations.

In the following, let $\mathbb{T}^n$ be the positively oriented n-dimensional complex torus. In [37, Theorem 6.1], the following A_n Askey–Wilson integral evaluation was derived:

$$\begin{aligned}\frac{1}{(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n}&\frac{\prod_{1\le i<j\le n+1}(z_i/z_j,z_j/z_i;q)_\infty}{\prod_{i,j=1}^{n+1}(a_i/z_j,b_iz_j;q)_\infty}\prod_{i=1}^n\frac{\mathrm{d}z_i}{z_i}\\&=\frac{(n+1)!\left(\prod_{i=1}^{n+1}a_ib_i;q\right)_\infty}{(q;q)_\infty^n\left(\prod_{i=1}^{n+1}a_i,\prod_{i=1}^{n+1}b_i;q\right)_\infty\prod_{i,j=1}^{n+1}(a_ib_j;q)_\infty},\end{aligned}\tag{5.3.5}$$

where $\prod_{i=1}^{n+1}z_i=1$, provided $|q|<1$, $|a_i|<1$ and $|b_i|<1$, for $1\le i\le n+1$.

A considerably more complicated A_n Askey–Wilson integral evaluation, depending on the parity of n, was given in [41, Theorem 1.1]:

$$\begin{aligned}&\frac{1}{(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n}\frac{\prod_{1\le i<j\le n+1}(z_i/z_j,z_j/z_i;q)_\infty}{\prod_{i,j=1}^{n+1}(a_i/z_j;q)_\infty\prod_{1\le i<j\le n+1}(bz_iz_j;q)_\infty}\prod_{i=1}^{n+1}\frac{(S/z_i;q)_\infty}{\prod_{j=1}^3(bc_jz_i;q)_\infty}\frac{\mathrm{d}z_i}{z_i}\\
&=\begin{cases}\dfrac{(n+1)!\left(b^{(n+4)/2}\prod_{i=1}^{n+1}a_i\prod_{j=1}^3c_j;q\right)_\infty\prod_{i=1}^{n+1}(S/a_i;q)_\infty}{(q;q)_\infty^n\left(\prod_{i=1}^{n+1}a_i,b^{(n+4)/2}\prod_{j=1}^3c_j;q\right)_\infty\prod_{j=1}^3(b^{(m+2)/2}c_j;q)_\infty}\\
\quad\times\dfrac{\prod_{j=1}^3\left(b^{(n+2)/2}c_j\prod_{i=1}^{n+1}a_i;q\right)_\infty}{\prod_{i=1}^{n+1}\prod_{j=1}^3(ba_ic_j;q)_\infty\prod_{1\le i<j\le n+1}(ba_ia_j;q)_\infty}, & \text{for } n \text{ even},\\[2ex]
\dfrac{(n+1)!\left(b^{(n+1)/2}\prod_{i=1}^{n+1}a_i;q\right)_\infty\prod_{i=1}^{n+1}(S/a_i;q)_\infty}{(q;q)_\infty^n\left(b^{(n+1)/2},\prod_{i=1}^{n+1}a_i;q\right)_\infty\prod_{i=1}^{n+1}\prod_{j=1}^3(ba_ic_j;q)_\infty}\\
\quad\times\dfrac{\prod_{j=1}^3\left(b^{(n+3)/2}\prod_{i=1}^{n+1}a_i\prod_{\substack{1\le k\le 3\\k\ne j}}c_k;q\right)_\infty}{\prod_{1\le i<j\le 3}(b^{(n+3)/2}c_ic_j;q)_\infty\prod_{1\le i<j\le n+1}(ba_ia_j;q)_\infty}, & \text{for } n \text{ odd}.\end{cases}\end{aligned}\tag{5.3.6}$$

where $\prod_{i=1}^{n+1} z_i = 1$ and $S = b^{n+2}\prod_{i=1}^{n+1} a_i \prod_{j=1}^{3} c_j$, provided $|q| < 1$, $|b| < 1$, $|a_i| < 1$ and $|c_j| < 1$, for $1 \le i \le n+1$ and $1 \le j \le n$.

The following C_n Askey–Wilson integral evaluation was derived in [37, Theorem 7.1]:

$$\frac{1}{(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n} \frac{\prod_{1\le i<j\le n}(z_i/z_j, z_j/z_i, z_iz_j, 1/z_iz_j;q)_\infty}{\prod_{i=1}^{2n+2}\prod_{j=1}^{n}(a_iz_j, a_i/z_j;q)_\infty} \prod_{i=1}^{n}(z_i^2, 1/z_i^2;q)_\infty \frac{\mathrm{d}z_i}{z_i}$$
$$= \frac{2^n n!\left(\prod_{i=1}^{2n+2} a_i;q\right)_\infty}{(q;q)_\infty^n \prod_{1\le i<j\le 2n+2}(a_ia_j;q)_\infty}, \tag{5.3.7}$$

provided $|q| < 1$ and $|a_i| < 1$ for $1 \le i \le n$.

Another C_n Askey–Wilson integral evaluation was given in [35, Equation (2)]:

$$\frac{1}{(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n} \prod_{1\le i<j\le n} \frac{(z_i/z_j, z_j/z_i, z_iz_j, 1/z_iz_j;q)_\infty}{(bz_i/z_j, bz_j/z_i, bz_iz_j, b/z_iz_j;q)_\infty} \prod_{i=1}^{n} \frac{(z_i^2, 1/z_i^2;q)_\infty}{\prod_{j=1}^{4}(a_jz_i, a_j/z_j;q)_\infty} \frac{\mathrm{d}z_i}{z_i}$$
$$= \frac{2^n n!\,(b;q)_\infty^n}{(q;q)_\infty^n} \prod_{i=1}^{n} \frac{\left(b^{n+i-2}\prod_{j=1}^{4} a_j;q\right)_\infty}{(b^i;q)_\infty \prod_{1\le j<k\le 4}(a_ja_kb^{i-1};q)_\infty}, \tag{5.3.8}$$

provided $|q| < 1$, $|b_i| < 1$ and $|a_j| < 1$, for $1 \le i \le n$ and $1 \le j \le 4$. By suitably specializing the variables a_j for $1 \le j \le 4$, this multivariate integral evaluation can be used (see [35]) to prove Morris' [96] generalizations of the Macdonald conjectures [96] for the affine root systems C_n, $\mathrm{C}_n^\vee$, BC_n, B_n $\mathrm{B}_n^\vee$ and D_n (using Macdonald's classification in [71]).

The multivariate integral evaluation in (5.3.8) explicitly describes the normalization factor of the orthogonality measure for the Macdonald–Koornwinder polynomials (see [62] and Chapter 9 of this volume), the BC_n generalization of the Askey–Wilson polynomials.

In [37, Theorem 8.1] an Askey–Wilson integral evaluation for the root system G_2 was given:

$$\frac{1}{(2\pi\mathrm{i})^2}\int_{\mathbb{T}^2} \frac{\prod_{\substack{1\le i,j\le 3\\ i\ne j}}(z_i/z_j;q)_\infty \prod_{j=1}^{3}(z_j, 1/z_j;q)_\infty}{\prod_{i=1}^{4}\prod_{j=1}^{3}(a_iz_j, a_i/z_j;q)_\infty} \frac{\mathrm{d}z_1}{z_1}\frac{\mathrm{d}z_2}{z_2}$$
$$= \frac{12\left(\prod_{i=1}^{4} a_i^2;q\right)_\infty \prod_{i=1}^{4}(a_i;q)_\infty}{(q;q)_\infty^2\left(\prod_{i=1}^{4} a_i;q\right)_\infty \prod_{1\le i\le j\le 4}(a_ia_j;q)_\infty \prod_{1\le i<j<k\le 4}(a_ia_ja_k;q)_\infty}, \tag{5.3.9}$$

where $\prod_{j=1}^{3} z_j = 1$ and $|a_i| < 1$ for $1 \le i \le 4$.

All these basic hypergeometric integral evaluations can be specialized to ordinary hypergeometric integral evaluations by taking suitable limits. In particular, if in (5.3.5) one replaces the parameters a_i by q^{a_i}, and b_i by q^{b_i}, for $1 \le i \le n+1$, and then takes the limit as $q \to 1^-$, one obtains the following multidimensional Mellin–Barnes integral [35, Theorem 9.1]

$$\frac{1}{(2\pi\mathrm{i})^n}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\cdots\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\prod_{i,j=1}^{n+1}\Gamma(a_i-z_j)\Gamma(b_i+z_j)}{\prod_{\substack{1\le i,j\le n+1\\ i\neq j}}\Gamma(z_i-z_j)}\prod_{i=1}^{n}\frac{\mathrm{d}z_i}{z_i}$$
$$=\frac{(n+1)!\,\Gamma(a_1+\cdots+a_{n+1})\,\Gamma(b_1+\cdots+b_{n+1})\prod_{i,j=1}^{n+1}\Gamma(a_i+b_j)}{\Gamma(a_1+\cdots+a_{n+1}+b_1+\cdots+b_{n+1})},\quad (5.3.10)$$

where $\sum_{i=1}^{n+1} z_i = 0$, provided $\mathrm{Re}(a_i) > 0$ and $\mathrm{Re}(b_i) > 0$, for $1 \le i \le n+1$. (For a generalization of (5.3.10), obtained by taking a suitable $q \to 1^-$ limit from an A_n Nassrallah–Rahman integral that extends (5.3.5), see [36, Theorem 5.1]).

Here we reproduced only a few of the many existing integral evaluations. More can be found in the papers [37, 38, 99, 125] (to list just a few relevant references). An interesting integral transformation with F_4 symmetry has been given in [13]. For further discussion of integral identities (evaluations and transformations) associated with root systems, where such identities are considered at the elliptic level, see Section 6.2 of this volume.

4. Basic hypergeometric series with Macdonald polynomial argument

The series considered here were first introduced by Macdonald (in unpublished work [76] of 1987), and by Kaneko [59].

Important special cases were considered earlier. Basic hypergeometric series with Schur polynomial argument (the Schur polynomials correspond to the $q = t$ case of the Macdonald polynomials) were studied by Milne [86] who derived ${}_1\phi_0$, ${}_2\phi_1$ and ${}_1\psi_1$ summations and several transformations for such series. Hypergeometric series with Jack polynomial argument (the Jack polynomials indexed by α correspond to the $q = t^\alpha$, $t \to 1$ specialization of the Macdonald polynomials) were studied by Herz, Constantine and Muirhead [11, 43, 97] for $\alpha = 2$ (the zonal polynomial case) and for arbitrary α by Korányi, Yan and Kaneko [58, 63, 134, 135].

For a thorough treatment of Macdonald polynomials (by which we mean the GL_n type symmetric Macdonald polynomials in the terminology of Chapter 9 of this volume; for the general root system case see [75] and Chapter 9 of this volume), see [73, Chapter VI] and Sections 9.1.1 and 9.3.7 of this volume. (Macdonald's book [73] also deals thoroughly with important special cases of the Macdonald polynomials, including in particular the aforementioned Schur, zonal and Jack polynomials.) See Chapter 10 of this volume for a survey on combinatorial aspects of these multivariate polynomials.

Let Λ_n denote the ring of symmetric functions in the variables $z = (z_1, \dots, z_n)$ over $\mathbb{C}$. Further, we assume two nonzero generic parameters q, t satisfying $|q|, |t| < 1$. The Macdonald polynomials $P_\lambda(z; q, t)$ (often shortened to P_λ or $P_\lambda(z)$ as long as no ambiguity arises), indexed by partitions λ of length $l(\lambda) \le n$, form an orthogonal basis of Λ_n. They can be defined as the unique family of symmetric polynomials whose expansion in terms of the monomial symmetric functions $m_\lambda(z)$ is uni-upper-triangular with respect to the dominance order $<$ of partitions,

$$P_\lambda(z;q,t) = m_\lambda(z) + \sum_{\mu<\lambda} c_{\lambda\mu}(q,t) m_\mu(z)$$

(with $c_{\lambda\mu}(q,t)$ being a rational function in q and t), being orthogonal with respect to the scalar product [73, Chapter VI, Section 9]

$$\langle f,g\rangle = \frac{1}{n!(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n} f(z)\overline{g(z)}\,\Delta_{q,t}(z)\prod_{i=1}^n \frac{\mathrm{d}z_i}{z_i}, \qquad \text{for } f,g\in\Lambda_n,$$

where

$$\Delta_{q,t}(z) = \prod_{\substack{1\le i,j\le n\\ i\ne j}} \frac{(z_i/z_j;q)_\infty}{(tz_i/z_j;q)_\infty}.$$

As in Section 3, $\mathbb{T}^n$ is the positively oriented n-dimensional complex torus. The squared norm evaluation of P_λ is [73, Chapter VI, Section 9, Example 1.(d)]

$$\langle P_\lambda, P_\lambda\rangle = \prod_{1\le i<j\le n} \frac{(q^{\lambda_i-\lambda_j}t^{j-i}, q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1}, q^{\lambda_i-\lambda_j+1}t^{j-i-1};q)_\infty}. \tag{5.4.1}$$

In [73, Chapter VI] Macdonald develops most of the theory for the polynomials $P_\lambda(z;q,t)$ using a different (albeit, up to normalization, equivalent) scalar product (which we are not displaying here) that is more algebraic in nature and does not require the conditions $|q|<1$ and $|t|<1$. Rather than considering symmetric functions over $\mathbb{C}$, Macdonald assumes q and t to be indeterminate and considers symmetric functions over $\mathbb{Q}(q,t)$. The above scalar product has the advantage that the structure of the root system A_{n-1} is clearly visible. This aspect of the theory generalizes to other root systems, see [75] and Chapter 9 of this volume.

The $P_\lambda(z;q,t)$ are homogeneous in $z=(z_1,\dots,z_n)$ of degree $|\lambda|$. They satisfy the stability property

$$P_\lambda(z_1,\dots,z_n;q,t) = P_\lambda(z_1,\dots,z_n,0;q,t).$$

Further, they satisfy [73, Chapter VI, Equation (4.17)]

$$P_\lambda(z;q,t) = (z_1\cdots z_n)^{\lambda_n} P_{\lambda-\lambda_n}(z;q,t), \tag{5.4.2}$$

where $\lambda-\lambda_n := (\lambda_1-\lambda_n,\dots,\lambda_{n-1}-\lambda_n,0)$ for any partition λ with $l(\lambda)\le n$.

For any partition λ, and $f\in\Lambda_n$ let $u_\lambda:\Lambda_n\to\mathbb{C}$ be the evaluation homomorphism defined by

$$u_\lambda\big(f(z)\big) = \begin{cases} f(z)\Big|_{z_i=q^{\lambda_i}t^{n-i},\,1\le i\le n} & \text{for } l(\lambda)\le n,\\ 0 & \text{otherwise.}\end{cases}$$

The following evaluation symmetry (cf. [73, Chapter VI, Equation (6.6)]), first proved by Koornwinder in unpublished work, is very useful (in particular, for interchanging summations in the process of deriving transformations):

$$u_0(P_\lambda)\,u_\lambda(P_\mu) = u_0(P_\mu)\,u_\mu(P_\lambda). \tag{5.4.3}$$

For any partition λ, let

$$(a;q,t)_\lambda = \prod_{i\ge 1}(at^{1-i};q)_{\lambda_i},$$

and

$$n(\lambda) = \sum_{i\ge 1}(i-1)\lambda_i = \sum_{i\ge 1}\binom{\lambda_i'}{2},$$

where λ' denotes the conjugate partition of λ. We also use the shorthand notation

$$(a_1,\dots,a_n;q,t)_\lambda = (a_1;q,t)_\lambda \cdots (a_n;q,t)_\lambda.$$

Further, for $l(\lambda)\le n$, we define

$$c_\lambda(q,t) = \prod_{i=1}^n (t^{n-i+1};q)_{\lambda_i} \prod_{1\le i<j\le n} \frac{(t^{j-i};q)_{\lambda_i-\lambda_j}}{(t^{j-i+1};q)_{\lambda_i-\lambda_j}}, \tag{5.4.4a}$$

$$c'_\lambda(q,t) = \prod_{i=1}^n (qt^{n-i};q)_{\lambda_i} \prod_{1\le i<j\le n} \frac{(qt^{j-i-1};q)_{\lambda_i-\lambda_j}}{(qt^{j-i};q)_{\lambda_i-\lambda_j}}. \tag{5.4.4b}$$

An important normalization of the Macdonald polynomials is given by

$$Q_\lambda(z;q,t) = b_\lambda(q,t)P_\lambda(z;q,t),$$

where

$$b_\lambda(q,t) = \frac{h_\lambda(q,t)}{c'_\lambda(q,t)}.$$

The Q_λ are exactly the polynomials dual to P_λ with respect to scalar product mentioned right after Equation (5.4.1). The two normalizations of Macdonald polynomials appear jointly in the Cauchy identity

$$\sum_\lambda P_\lambda(z;q,t)Q_\lambda(y;q,t) = \prod_{i,j=1}^n \frac{(tz_iy_j;q)_\infty}{(z_iy_j;q)_\infty}. \tag{5.4.5}$$

Let a be an indeterminate and define the homomorphism

$$\epsilon_{a;t} : \Lambda_n \to \mathbb{C}[a]$$

by its action on the power sum symmetric functions $p_r = p_r(z_1,\dots,z_n) := \sum_{i=1}^n z_i^r$ for $r\ge 1$ (which algebraically generate Λ_n), namely

$$\epsilon_{a;t}(p_r) = \frac{1-a^r}{1-t^r},$$

for each $r\ge 1$. For $a=t^n$, we have $\epsilon_{t^n;t}(f) = f(1,t,\dots,t^{n-1})$ for any $f\in\Lambda_n$.

The following evaluations are useful (cf. [59, Theorem 3.3]):

$$\epsilon_{a;t}\big(P_\lambda(z;q,t)\big) = t^{n(\lambda)}\frac{(a;q,t)_\lambda}{c_\lambda(q,t)}, \qquad \epsilon_{a;t}\big(Q_\lambda(z;q,t)\big) = t^{n(\lambda)}\frac{(a;q,t)_\lambda}{c'_\lambda(q,t)}. \tag{5.4.6}$$

Basic hypergeometric series with Macdonald polynomial argument are defined as

$$ {}_r\Phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,t,z\right]=\sum_{\lambda}\Big((-1)^{|\lambda|}q^{n(\lambda')}t^{-n(\lambda)}\Big)^{s+1-r}\,\frac{t^{n(\lambda)}}{c'_\lambda(q,t)}\frac{(a_1,\dots,a_r;q,t)_\lambda}{(b_1,\dots,b_s;q,t)_\lambda}P_\lambda(z;q,t), \tag{5.4.7}$$

provided that the series converges.

Application of the homomorphism $\epsilon_{a;t}$ with respect to y to both sides of the Cauchy identity in (5.4.5) immediately gives the following q-binomial theorem for Macdonald polynomials:

$$ {}_1\Phi_0\left[\begin{matrix}a\\ -\end{matrix};q,t,z\right]=\prod_{i=1}^{n}\frac{(az_i;q)_\infty}{(z_i;q)_\infty}, \tag{5.4.8}$$

which converges for $|z_i|<1$, $1\le i\le n$. It is interesting that the right-hand side is independent of t.

Baker and Forrester [3] made use of the q-binomial theorem for Macdonald polynomials and the evaluation symmetry (5.4.3) to derive the following multivariate generalization of the Heine transformation:

$$ {}_2\Phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,t,xt^\delta\right]=\prod_{i=1}^{n}\frac{(bt^{1-i},axt^{n-i};q)_\infty}{(ct^{1-i},xt^{n-i};q)_\infty}\cdot{}_2\Phi_1\left[\begin{matrix}c/b,xt^{n-1}\\ axt^{n-1}\end{matrix};q,t,bt^{1-n}t^\delta\right], \tag{5.4.9}$$

valid for $|x|<1$ and $|bt^{1-n}|<1$, where xt^δ stands for the argument $(x,xt,\dots,xt^{n-1})$. Notice that this transformation involves specialized Macdonald polynomials (which factorize since $P_\lambda(xt^\delta)=x^{|\lambda|}t^{n(\lambda)}(t^n;q)_\lambda/c_\lambda(q,t)$ due to homogeneity, and the specialization (5.4.6)) on both sides. A multivariate generalization of the first iterate of the Heine transformation involving unspecialized interpolation Macdonald polynomials was given by Lascoux, Rains and Warnaar [66, Corollary 10.2]. A further extension was obtained by Lascoux and Warnaar [67, Corollary 6.3] as a special case of of multivariate extension of the q-Kummer–Thomae–Whipple transformation [67, Corollary 6.2].

For $x=ct^{1-n}/ab$ the right-hand side of (5.4.9) reduces to a ${}_1\Phi_0$ series which can be summed using (5.4.8). This gives a multivariate extension of the q-Gauß summation:

$$ {}_2\Phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,t,\frac{ct^{1-n}}{ab}t^\delta\right]=\prod_{i=1}^{n}\frac{(ct^{1-i}/b,ct^{1-i}/a;q)_\infty}{(ct^{1-i}/ab,ct^{1-i};q)_\infty}, \tag{5.4.10}$$

valid for $|ct^{1-n}/ab|$. More general q-Gauß summations involving unspecialized (non-)symmetric Macdonald polynomials were given by Lascoux, Rains and Warnaar [66], and by Lascoux and Warnaar [67, Corollary 5.4].

For general unspecialized argument $z=(z_1,\dots,z_n)$ Baker and Forrester [3], building on work of Kaneko [59], proved the following multivariate extension of the Euler transformation (or equivalently, of the second iterate of the Heine transformation):

$$ {}_2\Phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,t,z\right]=\prod_{i=1}^{n}\frac{(abz_i/c;q)_\infty}{(z_i;q)_\infty}\cdot{}_2\Phi_1\left[\begin{matrix}c/a,c/b\\ c\end{matrix};q,t,abz/c\right], \tag{5.4.11}$$

valid for $|z_i|<1$ and $|abz_i/c|<1$, $1\le i\le n$. A nonsymmetric extension was given in [66, Corollary 10.3].

We list two other results from [3]. Let N be a nonnegative integer. The q-Pfaff–Saalschütz summation for basic hypergeometric series with specialized Macdonald polynomial argument is

$$ {}_3\Phi_2\left[\begin{matrix}a,b,q^{-N}\\ c,abq^{1-N}t^{n-1}/c\end{matrix};q,t,qt^\delta\right]=\prod_{i=1}^n\frac{(ct^{1-i}/a,ct^{1-i}/b;q)_N}{(ct^{1-i},ct^{1-i}/ab;q)_N}. \tag{5.4.12}$$

This can be generalized to a multivariate Sears' transformation with specialized Macdonald polynomial arguments:

$$\begin{aligned} &{}_4\Phi_3\left[\begin{matrix}a,b,c,q^{-N}\\ d,e,ft^{n-1}\end{matrix};q,t,qt^\delta\right]\\ &=a^{nN}\prod_{i=1}^n\frac{(et^{1-i}/a,ft^{n-i}/a;q)_N}{(et^{1-i},ft^{n-i};q)_N}\cdot{}_4\Phi_3\left[\begin{matrix}a,d/b,d/c,q^{-N}\\ d,aq^{1-N}t^{n-1}/e,aq^{1-N}/f\end{matrix};q,t,qt^\delta\right], \end{aligned}\tag{5.4.13}$$

where $def=abcq^{1-N}$. In [98, Section 4] Rains proved extensions of (5.4.12) and (5.4.13) for Macdonald polynomials indexed by partitions of skew shape. Extensions of (5.4.12) and (5.4.13) to nonsymmetric Macdonald polynomials were given in [66, Theorem 6.6 and Proposition 6.8].

Kaneko [59] developed q-difference equations for basic hypergeometric series with Macdonald polynomial argument and related them to q-Selberg integrals. Warnaar [127] proved various generalizations of q-Selberg integral evaluations and constant term identities, including a q-analogue of the Hua–Kadell formula for Jack polynomials (cf. [46, Theorem 5.2.1] and [53, Theorem 2]). Rains and Warnaar [100, Section 5.3] obtained further multivariate ${}_4\Phi_3$ transformations.

We complete this section with a multivariate extension of Ramanujan's ${}_1\psi_1$ summation formula due to Kaneko [60], which is a t-extension of an earlier result by Milne [86] (namely, for basic hypergeometric series with Schur function argument).

Let $\mathbb{Z}^n_\geq=\{(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n\mid\lambda_1\geq\cdots\geq\lambda_n\}$. By (5.4.2), the Macdonald polynomials $P_\lambda(z;q,t)$ can be defined for any $\lambda\in\mathbb{Z}^n_\geq$. Bilateral basic hypergeometric series with Macdonald polynomial argument are defined as

$$\begin{aligned} {}_r\Psi_{s+1}\left[\begin{matrix}a_1,\dots,a_r\\ b,b_1,\dots,b_s\end{matrix};q,t,z\right]=\sum_{\lambda\in\mathbb{Z}^n_\geq}\Bigg(&\left((-1)^{|\lambda|}q^{n(\lambda')}t^{-n(\lambda)}\right)^{s+1-r}\frac{(q;q)_\infty^n}{(b;q)_\infty^n}\prod_{i=1}^n\frac{(bq^{\lambda_i}t^{n-i};q)_\infty}{(q^{\lambda_i+1}t^{n-i};q)_\infty}\\ &\times\frac{t^{n(\lambda)}}{c'_\lambda(q,t)}\frac{(a_1,\dots,a_r;q,t)_\lambda}{(b_1,\dots,b_s;q,t)_\lambda}P_\lambda(z;q,t)\Bigg), \end{aligned}\tag{5.4.14}$$

provided that the series converges. With this notation, Kaneko's ${}_1\psi_1$ summation for Macdonald polynomials is

$${}_1\Psi_1\left[\begin{matrix}a\\ b\end{matrix};q,t,z\right]=\prod_{i=1}^n\frac{(qt^{n-i},bt^{1-i}/a,az_i,q/az_i;q)_\infty}{(bqt^{1-i},qt^{n-i}/a,z_i,bt^{1-n}/az_i;q)_\infty}, \tag{5.4.15}$$

subject to $|bt^{1-n}/a|<|z_i|<1$, for $1\leq i\leq n$.

In [130, Theorem 2.6], Warnaar gives a generalization of (5.4.15) involving a pair of Macdonald polynomials in two independent sets of variables. Other identities of this type are obtained in [128].

5. Remarks on applications

As mentioned in the introduction, hypergeometric series associated to root systems first arose in the context of $3j$ and $6j$ symbols for the unitary groups [1, 10, 45]. This initiated their study and that of their basic analogues from a pure mathematics point of view.

Basic hypergeometric series associated with root systems have found applications in various areas. We list a few occurrences, making no claim about completeness. First of all, such series, in particular multivariate ${}_6\psi_6$ summations associated with root systems, were used to give elementary proofs of the Macdonald identities [33, 80]. More generally, these series were used for deriving expansions of various special powers of the eta function [5, 68, 69, 89, 132] and for establishing infinite families of exact formulae for sums of squares and of triangular numbers [91, 106, 107]. Basic hypergeometric series associated with root systems were also employed in the enumeration of plane partitions [29, 65, 108]. Applications to Macdonald polynomials were given in [57, 118]. Basic hypergeometric integrals associated to root systems were used in the construction of BC_n orthogonal polynomials and BC_n biorthogonal rational functions that generalize the Macdonald polynomials, see [62, 98]. Watson transformations (and related transformations) associated to root systems were used in [4, 5, 12, 30, 136] to derive multiple Rogers–Ramanujan identities and characters for affine Lie algebras. For applications to quantum groups, see [109]. Basic hypergeometric series of Macdonald polynomial argument were used to construct Selberg-type integrals for A_{n-1} [129]. Also, hypergeometric series with Jack and zonal polynomial argument appeared in studies on random matrices [17, 24] and Selberg integrals [58, 63].

Very recently the subject has gained growing attention by physicists working in spin models and in quantum field theory. In particular, it was shown in [15, 16] that Gustafson's multivariate hypergeometric integrals appear naturally in the integrable spin models. Further, it was shown [42, 51] that the partition functions in $3d$ field theories can be expressed in terms of specific basic hypergeometric integrals. As made explicit in [22], these partition functions can also be obtained by reduction from the $4d$ superconformal indices which, according to [20], can be identified with elliptic hypergeometric integrals. (The latter are reviewed in Chapter 6 of this volume.) Accordingly, multivariate basic hypergeometric integrals and series associated with various symmetry groups (or gauge groups, in the terminology of quantum field theory) appear as explicit expressions for the respective partition functions [22, 123, 124]. Several of these are new and await further mathematical study. These partition functions can also be interpreted as solutions of the Yang–Baxter equation, see [26, 27].

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