SUPERCONGRUENCES INVOLVING DOMB NUMBERS AND BINARY QUADRATIC FORMS

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ABSTRACT. In this paper, we prove two recently conjectured supercongruences (modulo p^3, where p is any prime greater than 3) of Zhi-Hong Sun on truncated sums involving the Domb numbers. Our proofs involve a number of ingredients such as congruences involving specialized Bernoulli polynomials, harmonic numbers, binomial coefficients, and hypergeometric summations and transformations.

1. INTRODUCTION

The *Domb numbers* $\{D_n\}$, defined by

$$D_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}$$

for non-negative integers n, first appeared in an extensive study by C. Domb [4] on interacting particles on crystal lattices. In particular, Domb showed that D_n counts the number of $2n$-step polygons on the diamond lattice.

The Domb numbes also appear in a variety of other settings, such as in the coefficients in several known series for $1/\pi$. For example, from [1, Equation (1.3)] we know that

$$\sum_{n=0}^{\infty} \frac{5n+1}{64^n} D_n = \frac{8}{\sqrt{3}\pi}.$$

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In [10, Theorem 3.1], M.D. Rogers showed the following generating function for the Domb numbers by applying a rather intricate method:

$$\sum_{n=0}^{\infty} D_n u^n = \frac{1}{1-4u} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{3k}{k} \left(\frac{u^2}{(1-4u)^3} \right)^k,$$

where $|u|$ is sufficiently small. Y.-P. Mu and Z.-W. Sun [9, Equation (1.11)] proved a congruence involving the Domb numbers by applying the telescoping method: For any prime $p > 3$, we have the supercongruence

$$\sum_{k=0}^{p-1} \frac{3k^2+k}{16^k} D_k \equiv -4p^4 q_p(2) \pmod{p^5},$$

where $q_p(a)$ denotes the Fermat quotient $(a^{p-1}-1)/p$.

In [5], J.-C. Liu proved a couple of conjectures of Z.-W. Sun and Z.-H. Sun. In particular he confirmed [5, Theorem 1.3] that for any positive integer n the two sums

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) D_k 8^{n-1-k} \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (2k+1) D_k (-8)^{n-1-k}$$

are also positive integers.

Z.-H. Sun [17, Conjecture 4.1] conjectured the following congruence for the Domb numbers: Let $p > 3$ be a prime. Then

$$D_{p-1} \equiv 64^{p-1} - \frac{p^3}{6} B_{p-3} \pmod{p^4},$$

where $\{B_n\}$ are the Bernoulli numbers given by

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \ \ (n \geq 2).$$

This conjecture was confirmed by the first author and J. Wang [7]. For more research on Domb numbers, we kindly refer the readers to [5, 8, 15, 18, 20] (and the references therein).

The main result of this paper is Theorem 1.1 which contains two supercongruences that were originally conjectured by Z.-H. Sun in [19, Conjecture 3.5, Conjecture 3.6]. What makes them interesting is that their formulations involve the binary quadratic form x^2+3y^2 for primes p that are congruent to 1 modulo 3. (It is well-known that any prime $p \equiv 1 \pmod 3$ can be expressed as $p = x^2+3y^2$ for some integers x and y, an assertion first made by Fermat and subsequently proved by Euler, see [3]. In his paper [19], Sun stated further conjectures of similar type, involving different moduli, and other binary quadratic forms.) First, Sun defined

$$R_3(p) = \left(1 + 2p + \frac{4}{3}(2^{p-1}-1) - \frac{3}{2}(3^{p-1}-1)\right)\binom{\frac{p-1}{2}}{\lfloor p/6 \rfloor}^2.$$

The two supercongruences which we will confirm are as follows.

Theorem 1.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} k^3 \frac{D_k}{4^k}$$
$$\equiv \begin{cases} -\frac{64}{45}x^2 + \frac{32}{45}p + \frac{43p^2}{90x^2} \pmod{p^3} & \texttt{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ \frac{28}{9}R_3(p) \pmod{p^2} & \texttt{if } p \equiv 2 \pmod{3} \textit{ and } p \neq 5, \end{cases}$$

$$\sum_{k=0}^{p-1} k^3 \frac{D_k}{16^k}$$
$$\equiv \begin{cases} \frac{4}{45}x^2 - \frac{2}{45}p + \frac{p^2}{45x^2} \pmod{p^3} & \texttt{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ -\frac{4}{9}R_3(p) \pmod{p^2} & \texttt{if } p \equiv 2 \pmod{3}. \end{cases}$$

Our preparations for the proof of this theorem consist of seven lemmas that we give in Section 2. These are used in Section 3, devoted to the actual proof of Theorem 1.1. As tools for establishing the results in Sections 2 and 3 we utilize some congruences from [6, 8] and several combinatorial identities that can be found and proved by the package `Sigma` [12] via the software `Mathematica`.

2. Preliminary Lemmas

Recall that the Bernoulli polynomials $\{B_n(x)\}$ are given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (n = 0, 1, 2, \ldots),$$

where, as before, $\{B_n\}$ are the Bernoulli numbers. We will also use the classical Legendre symbol $\left(\frac{a}{q}\right)$ (for integer a and odd prime q). The following lemma involving the (generalized) harmonic numbers can be easily deduced from [13, Theorem 5.2 (c)], [14, Theorem 3.9 (ii), (iii), (iv)], [14, third equation on p. 302], and the simple identity

$$\sum_{1 \le k < \frac{2p}{3}} \frac{1}{k} = \sum_{1 \le k < \frac{2p}{3}} \frac{(-1)^{k-1}}{k} + \sum_{1 \le k < \frac{p}{3}} \frac{1}{k}.$$

Lemma 2.1. *Let $p>5$ be a prime. Then*

$$\begin{aligned}
H_{\frac{p-1}{2}} &\equiv -2q_p(2) \pmod p,\\
H_{\lfloor\frac{p}{6}\rfloor} &\equiv -2q_p(2)-\frac{3}{2}q_p(3) \pmod p,\\
H^{(2)}_{\lfloor\frac{p}{3}\rfloor} &\equiv \frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac{1}{3}\Big) \pmod p,\\
H_{\lfloor\frac{p}{3}\rfloor} &\equiv -\frac{3}{2}q_p(3)+\frac{3p}{4}q_p^2(3)-\frac{p}{6}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac{1}{3}\Big) \pmod {p^2},\\
H_{\lfloor\frac{2p}{3}\rfloor} &\equiv -\frac{3}{2}q_p(3)+\frac{3p}{4}q_p^2(3)+\frac{p}{3}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac{1}{3}\Big) \pmod {p^2}.
\end{aligned}$$

Lemma 2.2. *Let $p>2$ be a prime. If $0\le j\le (p-1)/2$, then we have*

$$\binom{3j}{j}\binom{p+j}{3j+1}\equiv \frac{p}{3j+1}(1-pH_{2j}+pH_j) \pmod {p^3}.$$

Proof. If $0\le j\le (p-1)/2$ and $j\neq (p-1)/3$, then we have

$$\begin{aligned}
\binom{3j}{j}\binom{p+j}{3j+1} &= \frac{(p+j)\cdots(p+1)p(p-1)\cdots(p-2j)}{j!(2j)!(3j+1)}\\
&\equiv \frac{pj!(1+pH_j)(-1)^{2j}(2j)!(1-pH_{2j})}{j!(2j)!(3j+1)}\\
&\equiv \frac{p}{3j+1}(1-pH_{2j}+pH_j) \pmod {p^3}.
\end{aligned}$$

If $j=(p-1)/3$, then by Lemma 2.1, we have

$$\begin{aligned}
&\binom{p-1}{\frac{p-1}{3}}\binom{p+\frac{p-1}{3}}{\frac{p-1}{3}}\\
&\equiv \left(1-pH_{\frac{p-1}{3}}+\frac{p^2}{2}(H^2_{\frac{p-1}{3}}-H^{(2)}_{\frac{p-1}{3}})\right)\left(1+pH_{\frac{p-1}{3}}+\frac{p^2}{2}(H^2_{\frac{p-1}{3}}-H^{(2)}_{\frac{p-1}{3}})\right)\\
&\equiv 1-p^2H^{(2)}_{\frac{p-1}{3}}\equiv 1-\frac{p^2}{2}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac{1}{3}\Big) \pmod {p^3}
\end{aligned}$$

and

$$1-pH_{\frac{2p-2}{3}}+pH_{\frac{p-1}{3}}\equiv 1-\frac{p^2}{2}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac{1}{3}\Big) \pmod {p^3}.$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $p>3$ be a prime. For any p-adic integer t, we have*

$$\binom{\frac{2p-2}{3}+pt}{\frac{p-1}{2}}\equiv\binom{\frac{2p-2}{3}}{\frac{p-1}{2}}\left(1+pt(H_{\frac{2p-2}{3}}-H_{\frac{p-1}{6}})\right) \pmod {p^2}.$$

Proof. Set $m = (2p-2)/3$. It is easy to check that

$$\begin{aligned}\binom{m+pt}{(p-1)/2} &= \frac{(m+pt)\cdots(m+pt-(p-1)/2+1)}{((p-1)/2)!}\\ &\equiv \frac{m\cdots(m-(p-1)/2+1)}{((p-1)/2)!}(1+pt(H_m - H_{m-(p-1)/2})\\ &= \binom{m}{(p-1)/2}(1+pt(H_m - H_{m-(p-1)/2}) \pmod{p^2}.\end{aligned}$$

which completes the proof of Lemma 2.3. □

Lemma 2.4. *Let $p > 3$ be a prime. If $p = x^2 + 3y^2 \equiv 1 \pmod 3$, then*

$$p\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(3k+4)16^k} \equiv \frac{4}{25}\left(4x^2 - 2p - \frac{p^2}{4x^2}\right) \pmod{p^3}.$$

Proof. By using `Sigma`, we establish the following identity:

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{n+k}{k}(-1)^k}{3k+4} = -\frac{1}{(3n-1)(3n+1)(3n+4)}\prod_{k=1}^{n}\frac{3k-1}{3k-2}.$$

(In terms of classical identities for hypergeometric series, this evaluation is equivalent to the $(a,b,c) \mapsto (n+1, 4/3, 1)$ case of the Pfaff–Saalschütz summation [11, Appendix III, Equation (III.2)].) So modulo p^3, we have

$$\begin{aligned}&p\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(3k+4)16^k}\\ &\equiv p\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k}(-1)^k}{3k+4} + \binom{-\frac{1}{2}}{\frac{p-4}{3}}^2 + \binom{\frac{p-1}{2}}{\frac{p-4}{3}}\binom{\frac{p-1}{2}+\frac{p-4}{3}}{\frac{p-4}{3}}\\ &= \frac{4}{25-9p^2}\frac{\left(\frac{2}{3}\right)_{\frac{p-1}{2}}}{\left(\frac{1}{3}\right)_{\frac{p-1}{3}}\left(\frac{p}{3}+1\right)_{\frac{p-1}{6}}} + \binom{-\frac{1}{2}}{\frac{p-4}{3}}^2 + \binom{\frac{p-1}{2}}{\frac{p-4}{3}}\binom{\frac{p-1}{2}+\frac{p-4}{3}}{\frac{p-4}{3}},\end{aligned}$$

where we used the standard notation for the shifted factorial $(a)_n = \prod_{j=0}^{n-1}(a+j)$ (cf. [11, Section 1.1.1]). It is easy to check that

$$\binom{-\frac{1}{2}}{\frac{p-4}{3}}^2 = \frac{4(p-1)^2}{(2p-5)^2}\binom{-\frac{1}{2}}{\frac{p-1}{3}}^2, \tag{2.1}$$

$$\binom{\frac{p-1}{2}}{\frac{p-4}{3}}\binom{\frac{p-1}{2}+\frac{p-4}{3}}{\frac{p-4}{3}} = \frac{4(p-1)}{5(p+5)}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\binom{\frac{p-1}{2}+\frac{p-1}{3}}{\frac{p-1}{3}}.$$

These identities, together with [6, pp. 14], yield

$$
\begin{aligned}
&p\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(3k+4)16^k}\\
&\equiv\frac{4}{25-9p^2}\frac{(\frac{2}{3})_{\frac{p-1}{2}}}{(\frac{1}{3})_{\frac{p-1}{3}}(\frac{p}{3}+1)_{\frac{p-1}{6}}}+\frac{4(p-1)^2}{(2p-5)^2}\binom{-\frac{1}{2}}{\frac{p-1}{3}}^2\\
&\quad+\frac{4(p-1)}{5(p+5)}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\binom{\frac{p-1}{2}+\frac{p-1}{3}}{\frac{p-1}{3}}\\
&\equiv\frac{4}{25}\left(1+\frac{9p^2}{25}\right)(-1)^{\frac{p-1}{6}}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\binom{\frac{2p-2}{3}}{\frac{p-1}{2}}\\
&\quad\times\left(1-\frac{2p}{3}q_p(2)+\frac{5p^2}{9}q_p^2(2)+\frac{5p^2}{12}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac{1}{3}\Big)\right)\\
&\quad+\frac{4}{25}\left(1-\frac{6p}{5}-\frac{3p^2}{25}\right)\binom{\frac{p-1}{2}}{\frac{p-1}{3}}^2\\
&\qquad\times\left(1-\frac{3p}{2}q_p(3)+\frac{15p^2}{8}q_p^2(3)+\frac{5p^2}{24}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac{1}{3}\Big)\right)\\
&\quad-\frac{4}{25}\left(1-\frac{6p}{5}-\frac{6p^2}{25}\right)\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\binom{\frac{5p-5}{6}}{\frac{p-1}{3}}\pmod{p^3}.
\end{aligned}
$$

Again, by [6, pp. 14–15], we have

$$
\begin{aligned}
&p\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(3k+4)16^k}\\
&\equiv\frac{4}{25}\left(1+\frac{9p^2}{25}\right)\left(4x^2-2p-\frac{p^2}{4x^2}\right)\\
&\quad\times\left(1+\frac{2p}{3}q_p(2)-\frac{p^2}{9}q_p^2(2)+\frac{5p^2}{24}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac{1}{3}\Big)\right)\\
&\quad\times\left(1-\frac{2p}{3}q_p(2)+\frac{5p^2}{9}q_p^2(2)+\frac{5p^2}{12}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac{1}{3}\Big)\right)\\
&\quad+\frac{4}{25}\left(1-\frac{6p}{5}-\frac{3p^2}{25}\right)\left(4x^2-2p-\frac{p^2}{4x^2}\right)\\
&\qquad\times\left(1-\frac{3p}{2}q_p(3)+\frac{15p^2}{8}q_p^2(3)+\frac{5p^2}{24}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac{1}{3}\Big)\right)\\
&\qquad\times\Bigg(1-\frac{4p}{3}q_p(2)+\frac{3p}{2}q_p(3)+\frac{14p^2}{9}q_p^2(2)-2p^2q_p(2)q_p(3)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \frac{3p^2}{8} q_p^2(3) + \frac{p^2}{8}\left(\frac{p}{3}\right) B_{p-2}\Big(\frac{1}{3}\Big)\Bigg) \\
&- \frac{4}{25}\left(1 - \frac{6p}{5} - \frac{6p^2}{25}\right)\left(4x^2 - 2p - \frac{p^2}{4x^2}\right) \\
&\times \left(1 - \frac{4p}{3} q_p(2) + \frac{14p^2}{9} q_p^2(2) + \frac{23p^2}{24}\left(\frac{p}{3}\right) B_{p-2}\Big(\frac{1}{3}\Big)\right) \pmod{p^3}.
\end{aligned}
$$

It is easy to check that the right-side of the above congruence is congruent to $\frac{4}{25}\left(4x^2 - 2p - \frac{p^2}{4x^2}\right)$ modulo p^3. Therefore we immediately get the desired result stated in Lemma 2.4. □

Lemma 2.5. *Let $p > 3$ be a prime with $p = x^2 + 3y^2 \equiv 1 \pmod 3$ and let $k = (p-4)/3$. Then*

$$
\begin{aligned}
&\big(k(k+1)(k+3) + (2-k^2)(3k+1)p - (k+2)(3k+1)(3k+2)p^2\big) \\
&\times \frac{\binom{-\frac12}{k}^2}{(k+1)(3k+2)}\left(\frac{\binom{3k}{k}\binom{p+k}{3k+1}}{3k+4} - \frac{1 - pH_{2k} + pH_k}{3k+1}\right) \\
&\equiv -\frac{184p^2x^2}{125} \pmod{p^3}
\end{aligned}
$$

and

$$
\begin{aligned}
&\big(k(1+2k) + 2p(k+1)(3k+1) - 2p^2(3k+1)(3k+2)\big) \\
&\times \frac{\binom{-\frac12}{k}^2}{3k+2}\left(\frac{\binom{3k}{k}\binom{p+2k}{3k+1}}{3k+4} + \frac{1 + pH_{2k} - pH_k}{3k+1}\right) \\
&\equiv -\frac{184p^2x^2}{125} \pmod{p^3}.
\end{aligned}
$$

Proof. We only prove the first congruence; the proof of the second congruence is similar. It is easy to see that

$$
\begin{aligned}
&\frac{\binom{3k}{k}\binom{p+k}{3k+1}}{3k+4} - \frac{1 - pH_{2k} + pH_k}{3k+1} \\
&= \binom{3k+3}{k+1}\binom{p+k+1}{3k+4}\frac{2(2p-5)(p-1)^2}{(4p-1)(p-3)(p+2)(p+5)} \\
&\quad - \frac{1 - pH_{2k+2} + pH_{k+1} + \frac{p}{2k+2} + \frac{p}{2k+1} - \frac{p}{k+1}}{3k+1}.
\end{aligned}
$$

By Lemma 2.2 we have

$$
\binom{3k+3}{k+1}\binom{p+k+1}{3k+4} \equiv 1 - pH_{2k+2} + pH_{k+1} \pmod{p^3}.
$$

Thus,

$$\frac{\binom{3k}{k}\binom{p+k}{3k+1}}{3k+4}-\frac{1-pH_{2k}+pH_k}{3k+1}\equiv-\frac{207p^2}{100}\pmod{p^3}.$$

Together with (2.1) and $\binom{(p-1)/2}{(p-1)/3}\equiv 2x \pmod p$ (cf. [21]), this yields

$$\begin{aligned}&\big(k(k+1)(k+3)+(2-k^2)(3k+1)p-(k+2)(3k+1)(3k+2)p^2\big)\\ &\times\frac{\binom{-\frac12}{k}^2}{(k+1)(3k+2)}\left(\frac{\binom{3k}{k}\binom{p+k}{3k+1}}{3k+4}-\frac{1-pH_{2k}+pH_k}{3k+1}\right)\\ &\equiv-\frac{184p^2x^2}{125}\pmod{p^3},\end{aligned}$$

which completes the proof of Lemma 2.5. □

Lemma 2.6. *Let $p>2$ be a prime. If $0\le j\le (p-1)/2$, then we have*

$$\binom{3j}{j}\binom{p+2j}{3j+1}\equiv\frac{p(-1)^j}{3j+1}(1+pH_{2j}-pH_j)\pmod{p^3}.$$

and

$$1-pH_{\frac{2p-2}{3}}+pH_{\frac{p-1}{3}}\equiv 1-\frac{p^2}{2}\left(\frac{p}{3}\right)B_{p-2}\Big(\frac13\Big)\pmod{p^3}.$$

If $(p+1)/2\le j\le p-1$, then

$$\binom{3j}{j}\binom{p+2j}{3j+1}\equiv\frac{2p(-1)^j}{3j+1}\pmod{p^2}.$$

Proof. If $0\le j\le (p-1)/2$ and $j\neq (p-1)/3$, then we have

$$\begin{aligned}\binom{3j}{j}\binom{p+2j}{3j+1}&=\frac{(p+2j)\cdots(p+1)p(p-1)\cdots(p-j)}{(3j+1)j!(2j)!}\\ &\equiv\frac{p(2j)!(1+pH_{2j})(-1)^j(j)!(1-pH_j)}{(3j+1)j!(2j)!}\\ &=\frac{p(-1)^j}{3j+1}(1+pH_{2j}-pH_j)\pmod{p^3}.\end{aligned}$$

If $j=(p-1)/3$, then by Lemma 2.1 and $H^{(2)}_{p-1-k}\equiv -H^{(2)}_k \pmod p$, we have

$$\begin{aligned}&\binom{p-1}{\frac{p-1}{3}}\binom{p+\frac{2p-2}{3}}{\frac{2p-2}{3}}\\ &\equiv\left(1-pH_{\frac{p-1}{3}}+\frac{p^2}{2}(H^2_{\frac{p-1}{3}}-H^{(2)}_{\frac{p-1}{3}})\right)\left(1+pH_{\frac{2p-2}{3}}+\frac{p^2}{2}(H^2_{\frac{2p-2}{3}}-H^{(2)}_{\frac{2p-2}{3}})\right)\end{aligned}$$

$$\equiv 1+p(H_{\frac{2p-2}{3}}-H_{\frac{p-1}{3}})=\frac{p(-1)^j}{3j+1}(1+pH_{2j}-pH_j) \pmod{p^3}.$$

If $(p+1)/2 \le j \le p-1$, then

$$\begin{aligned}
&\binom{3j}{j}\binom{p+2j}{3j+1}\\
&=\frac{(p+2j)\cdots(2p+1)(2p)(2p-1)\cdots(p+1)p(p-1)\cdots(p-j)}{(3j+1)j!(2j)!}\\
&\equiv\frac{2p^2(2j)\cdots(p+1)(p-1)!(-1)^j(j)!}{(3j+1)j!(2j)!}\equiv\frac{2p(-1)^j}{3j+1} \pmod{p^2},
\end{aligned}$$

which completes the proof of Lemma 2.6. □

Lemma 2.7. *Let $p>3$ be a prime with $p=x^2+3y^2\equiv 1 \pmod 3$. Then*

$$p\sum_{j=0}^{\frac{p-1}{2}}\frac{\binom{2j}{j}^2(H_{2j}-H_j)}{(3j+4)16^j}\equiv-\frac{18}{125}(4x^2-2p) \pmod{p^2}.$$

Proof. By using `Sigma`, we establish the following identity:

$$\begin{aligned}
\sum_{j=0}^{n}\frac{\binom{n}{j}\binom{n+j}{j}(-1)^j(H_{2j}-H_j)}{3j+4}&=-\frac{9(2n+1)}{10(3n-1)(3n+4)}\\
&+\frac{(\frac{2}{3})_n}{(3n-1)(3n+1)(3n+4)(\frac{1}{3})_n}\left(\frac{9}{10}+\sum_{k=1}^{n}\frac{(\frac{1}{3})_k}{k(\frac{2}{3})_k}\right).
\end{aligned}$$

Substituting $n=(p-1)/2$ into the above identity, then modulo p^2 we have

$$p\sum_{j=0}^{\frac{p-1}{2}}\frac{\binom{2j}{j}^2(H_{2j}-H_j)}{(3j+4)16^j}\equiv\frac{p(\frac{2}{3})_{\frac{p-1}{2}}}{\frac{3p-5}{2}\frac{3p-1}{2}\frac{3p+5}{2}(\frac{1}{3})_{\frac{p-1}{2}}}\left(\frac{9}{10}+\sum_{k=1}^{\frac{p-1}{2}}\frac{(\frac{1}{3})_k}{k(\frac{2}{3})_k}\right).$$

In view of [8, pp. 9] and [6, pp. 14–15], we have

$$\frac{(\frac{2}{3})_{\frac{p-1}{2}}}{(\frac{1}{3})_{\frac{p-1}{3}}(\frac{p}{3}+1)_{\frac{p-1}{6}}}\equiv 4x^2-2p \pmod{p^2},$$

$$\frac{(\frac{2}{3})_{\frac{p-1}{2}}}{(\frac{1}{3})_{\frac{p-1}{2}}}\sum_{k=1}^{\frac{p-1}{2}}\frac{(\frac{1}{3})_k}{k(\frac{2}{3})_k}\equiv 0 \pmod{p}.$$

Hence,

$$
\begin{aligned}
p\sum_{j=0}^{\frac{p-1}{2}}\frac{\binom{2j}{j}^2(H_{2j}-H_j)}{(3j+4)16^j} &\equiv \frac{9}{10}\frac{p(\frac{2}{3})_{\frac{p-1}{2}}}{\frac{3p-5}{2}\frac{3p-1}{2}\frac{3p+5}{2}(\frac{1}{3})_{\frac{p-1}{2}}}\\
&\equiv -\frac{9}{10}\frac{4}{25}(4x^2-2p) = -\frac{18}{125}(4x^2-2p) \pmod{p^2}.
\end{aligned}
$$

The proof of Lemma 2.7 is complete. □

3. Proof of Theorem 1.1

Our proof of Theorem 1.1 heavily relies on the following two transformation formulas due to H.-H. Chan and W. Zudilin [2] and Z.-H. Sun [15] respectively,

$$
\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n+2k}{3k}\binom{2k}{k}^2\binom{3k}{k}16^{n-k}, \tag{3.1}
$$

$$
\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+k}{3k}\binom{2k}{k}^2\binom{3k}{k}4^{n-2k}. \tag{3.2}
$$

Proof of Theorem 1.1. We first consider the first congruence in Theorem 1.1 in the case $p = x^2+3y^2 \equiv 1 \pmod 3$. By (3.2), we have

$$
\begin{aligned}
\sum_{k=0}^{p-1}k^3\frac{D_k}{4^k} &= \sum_{k=0}^{p-1}\frac{k^3}{4^k}\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k+j}{3j}\binom{2j}{j}^2\binom{3j}{j}4^{k-2j}\\
&= \sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}^2\binom{3j}{j}}{16^j}\sum_{k=2j}^{p-1}k^3\binom{k+j}{3j}.
\end{aligned}
$$

By using `Sigma`, we establish the following identity:

$$
\sum_{k=2j}^{n-1}k^3\binom{k+j}{3j} = \frac{\Sigma_1}{(j+1)(3j+2)(3j+4)}\binom{n+j}{3j+1},
$$

where

$$
\begin{aligned}
\Sigma_1 = {}& j(j+1)(j+3) + n(2-j^2)(3j+1) - n^2(j+2)(3j+1)(3j+2)\\
&+ n^3(j+1)(3j+1)(3j+2).
\end{aligned}
$$

Thus,

$$\sum_{k=0}^{p-1} k^3 \frac{D_k}{4^k} = \sum_{j=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2 \binom{3j}{j} \binom{p+j}{3j+1}}{16^k} \frac{\Sigma_1}{(j+1)(3j+2)(3j+4)}.$$

Let

$\Sigma_2 = k(k+1)(k+3) + p(2-k^2)(3k+1) - p^2(k+2)(3k+1)(3k+2).$

In view of Lemma 2.2, we have for $p \equiv 1 \pmod 3$ the supercongruence

$$\sum_{k=0}^{p-1} k^3 \frac{D_k}{4^k} \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2 \binom{3k}{k} \binom{p+k}{3k+1}}{16^k} \frac{\Sigma_2}{(k+1)(3k+2)(3k+4)}$$

$$\equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k} \frac{p(1-pH_{2k}+pH_k)\Sigma_2}{(k+1)(3k+1)(3k+2)(3k+4)} + S_1 \pmod{p^3},$$

where S_1 is defined by the following expression with $k = (p-4)/3$,

$$S_1 = \big(k(k+1)(k+3) + (2-k^2)(3k+1)p - (k+2)(3k+1)(3k+2)p^2\big) \times \frac{\binom{-\frac{1}{2}}{k}^2}{(k+1)(3k+2)} \left(\frac{\binom{3k}{k}\binom{p+k}{3k+1}}{3k+4} - \frac{1-pH_{2k}+pH_k}{3k+1} \right).$$

In view of [16, Equation (3.5)] and [8], we have

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(k+1)16^k} \equiv 0 \pmod{p^2}, \qquad \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2 (H_{2k}-H_k)}{(3k+1)16^k} \equiv 0 \pmod{p}, \tag{3.3}$$

$$\frac{2}{3} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2 (H_{2k}-H_k)}{(3k+2)16^k} \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{p\binom{2k}{k}^2}{(3k+2)16^k} \equiv -\frac{p^2}{x^2} \pmod{p^3}. \tag{3.4}$$

Hence by Lemma 2.5 and [6, Theorem 1.2], we have

$$\sum_{k=0}^{p-1} k^3 \frac{D_k}{4^k} + \frac{184p^2x^2}{125} \equiv \frac{p}{27} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k} \left(\frac{-8}{3k+1} + \frac{21}{3k+2} - \frac{10}{3k+4} \right)$$

$$+ \frac{p^2}{3} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k} \left(\frac{-3}{k+1} + \frac{7}{3k+2} + \frac{1}{3k+4} \right)$$

$$- p^3 \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k} \left(\frac{1}{k+1} - \frac{2}{3k+4} \right)$$

$$
\begin{aligned}
&- \frac{p^2}{27}\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2(H_{2k}-H_k)}{16^k}\left(\frac{-8}{3k+1}+\frac{21}{3k+2}-\frac{10}{3k+4}\right)\\
&- \frac{p^3}{3}\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2(H_{2k}-H_k)}{16^k}\left(\frac{-3}{k+1}+\frac{7}{3k+2}+\frac{1}{3k+4}\right)\\
\equiv{}& \left(-\frac{8}{27}-\frac{10}{27}\frac{4}{25}\right)\left(4x^2-2p-\frac{p^2}{4x^2}\right)-\frac{21}{27}\frac{p^2}{x^2}+\frac{p}{3}\frac{4}{25}(4x^2-2p)\\
&+2p^2\frac{16x^2}{25}-\frac{10p}{27}\frac{18}{125}(4x^2-2p)+\frac{21}{27}\frac{3}{2}\frac{p^2}{x^2}+\frac{p^2}{3}\frac{18}{125}4x^2\\
\equiv{}& -\frac{64x^2}{45}+\frac{32p}{45}+\frac{43p^2}{90x^2}+\frac{184p^2x^2}{125} \pmod{p^3}.
\end{aligned}
$$

Thus we immediately obtain the desired result

$$
\sum_{k=0}^{p-1} k^3\frac{D_k}{4^k} \equiv -\frac{64x^2}{45}+\frac{32p}{45}+\frac{43p^2}{90x^2} \pmod{p^3}. \tag{3.5}
$$

Now we are ready to prove the case $p \equiv 2 \pmod 3$ with $p > 5$. Similar to before,

$$
\begin{aligned}
&\sum_{k=0}^{p-1} k^3\frac{D_k}{4^k} \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2\binom{3k}{k}\binom{p+k}{3k+1}}{16^k}\frac{k(k+1)(k+3)+p(2-k^2)(3k+1)}{(k+1)(3k+2)(3k+4)}\\
&\equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\frac{p(1-pH_{2k}+pH_k)(k(k+1)(k+3)+p(2-k^2)(3k+1))}{(k+1)(3k+1)(3k+2)(3k+4)}\\
&\equiv \frac{p}{27}\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(\frac{-8}{3k+1}+\frac{21}{3k+2}-\frac{10}{3k+4}\right)\\
&\quad+\frac{p^2}{3}\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(\frac{-3}{k+1}+\frac{7}{3k+2}+\frac{1}{3k+4}\right)\\
&\quad-\frac{p^2}{27}\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2(H_{2k}-H_k)}{16^k}\left(\frac{-8}{3k+1}+\frac{21}{3k+2}-\frac{10}{3k+4}\right) \pmod{p^2}.
\end{aligned}
$$

In view of [8], we have

$$
\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(3k+1)16^k} \equiv 0 \pmod p,
$$

and in view of Lemma 2.4, we have

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(3k+4)16^k} \equiv 0 \pmod{p}.$$

Thus,

$$\sum_{k=0}^{p-1} k^3 \frac{D_k}{4^k} \equiv \frac{7p}{9} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(3k+2)16^k} + \frac{7p^2}{3} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(3k+2)16^k} - \frac{7p^2}{9} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2 (H_{2k}-H_k)}{(3k+2)16^k} \pmod{p^2}.$$

In view of [8, Equation (4.2)], we have

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2 (H_{2k}-H_k)}{16^k(3k+2)} \equiv 3 \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k(3k+2)} \pmod{p}, \tag{3.6}$$

and

$$p \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(3k+2)16^k} \equiv 4R_3(p) \pmod{p^2}. \tag{3.7}$$

Hence

$$\sum_{k=0}^{p-1} k^3 \frac{D_k}{4^k} \equiv \frac{28}{9} R_3(p) \pmod{p^2}. \tag{3.8}$$

Now we consider the other congruences in Theorem 1.1. Similar to above, by (3.1), we have

$$\begin{aligned}\sum_{k=0}^{p-1} k^3 \frac{D_k}{16^k} &= \sum_{k=0}^{p-1} \frac{k^3}{16^k} \sum_{j=0}^{k} (-1)^j \binom{k+2j}{3j} \binom{2j}{j}^2 \binom{3j}{j} 16^{k-j} \\ &= \sum_{j=0}^{p-1} \frac{\binom{2j}{j}^2 \binom{3j}{j}}{(-16)^j} \sum_{k=j}^{p-1} k^3 \binom{k+2j}{3j}.\end{aligned}$$

By using `Sigma`, we establish the following identity:

$$\sum_{k=j}^{n-1} k^3 \binom{k+2j}{3j} = \frac{\Sigma_3}{(3j+2)(3j+4)} \binom{n+2j}{3j+1},$$

where

$$\Sigma_3 = j(1+2j) + 2n(j+1)(3j+1) - 2n^2(3j+1)(3j+2) + n^3(3j+1)(3j+2).$$

Let

$$\Sigma_4 = j(1+2j) + 2p(j+1)(3j+1) - 2p^2(3j+1)(3j+2).$$

Thus, if $p \equiv 1 \pmod 3$, then modulo p^3, we have

$$\begin{aligned}
&\sum_{k=0}^{p-1} k^3 \frac{D_k}{16^k} - \frac{1}{18p(p+1)} \binom{-1/2}{\frac{2p-2}{3}}^2 \binom{2p-2}{\frac{2p-2}{3}} \binom{p+\frac{4p-4}{3}}{2p-1} \\
&\equiv \sum_{j=0}^{\frac{p-1}{2}} \frac{\binom{2j}{j}^2 \binom{3j}{j} \binom{p+2j}{3j+1}}{(-16)^j} \frac{\Sigma_4}{(3j+2)(3j+4)} \\
&\equiv \sum_{j=0}^{\frac{p-1}{2}} \frac{\binom{2j}{j}^2 p(-1)^j (1+pH_{2j} - pH_j)}{(-16)^j} \frac{\Sigma_4}{(3j+1)(3j+2)(3j+4)} + S_2,
\end{aligned}$$

where S_2 is defined by the following expression with $k = (p-4)/3$,

$$\begin{aligned}
S_2 = &-\frac{\binom{-\frac{1}{2}}{k}^2 (k(1+2k) + 2p(k+1)(3k+1) - 2p^2(3k+1)(3k+2))}{3k+2} \\
&\times \left(\frac{\binom{3k}{k}\binom{p+2k}{3k+1}}{3k+4} + \frac{1+pH_{2k} - pH_k}{3k+1} \right).
\end{aligned}$$

Hence, similar to above, by (3.3), (3.4) and Lemma 2.5, we have

$$\begin{aligned}
&\sum_{k=0}^{p-1} k^3 \frac{D_k}{16^k} - \frac{1}{18p(p+1)} \binom{-1/2}{\frac{2p-2}{3}}^2 \binom{2p-2}{\frac{2p-2}{3}} \binom{p+\frac{4p-4}{3}}{2p-1} \\
&\equiv \frac{p}{27} \sum_{j=0}^{\frac{p-1}{2}} \frac{\binom{2j}{j}^2}{16^j} \left(\frac{-1}{3j+1} + \frac{-3}{3j+2} + \frac{10}{3j+4} \right) \\
&\quad + \frac{p^2}{3} \sum_{j=0}^{\frac{p-1}{2}} \frac{\binom{2j}{j}^2}{16^j} \left(\frac{1}{3j+2} + \frac{1}{3j+4} \right) - 2p^3 \sum_{j=0}^{\frac{p-1}{2}} \frac{\binom{2j}{j}^2}{(3j+4)16^j} \\
&\quad + \frac{p^2}{27} \sum_{j=0}^{\frac{p-1}{2}} \frac{\binom{2j}{j}^2}{16^j} \left(\frac{-1}{3j+1} + \frac{-3}{3j+2} + \frac{10}{3j+4} \right) (H_{2j} - H_j) \\
&\quad + \frac{p^3}{3} \sum_{j=0}^{\frac{p-1}{2}} \frac{\binom{2j}{j}^2}{16^j} \left(\frac{1}{3j+2} + \frac{1}{3j+4} \right) (H_{2j} - H_j) + \frac{184p^2x^2}{125} \\
&\equiv \left(-\frac{1}{27} + \frac{10}{27}\frac{4}{25} \right) \left(4x^2 - 2p - \frac{p^2}{4x^2} \right) + \frac{1}{9}\frac{p^2}{x^2} + \frac{p}{3}\frac{4}{25}(4x^2 - 2p)
\end{aligned}$$

$$
\begin{aligned}
&-2p^2\frac{16x^2}{25}-\frac{10p}{27}\frac{18}{125}(4x^2-2p)+\frac{1}{9}\frac{3}{2}\frac{p^2}{x^2}-\frac{p^2}{3}\frac{18}{125}4x^2+\frac{184p^2x^2}{125}\\
\equiv\ &\frac{4x^2}{45}-\frac{2p}{45}+\frac{49p^2}{180x^2}\pmod{p^3}.
\end{aligned}
$$

It is easy to see that

$$\binom{2p-2}{\frac{2p-2}{3}}\binom{p+\frac{4p-4}{3}}{2p-1}\equiv -2p\pmod{p^2}.$$

In view of [8, pp. 18], we have

$$\binom{-1/2}{\frac{2p-2}{3}}^2\equiv\frac{9p^2}{4x^2}\pmod{p^3}.$$

These yield

$$
\begin{aligned}
\sum_{k=0}^{p-1}k^3\frac{D_k}{16^k}&\equiv\frac{4x^2}{45}-\frac{2p}{45}+\frac{49p^2}{180x^2}-\frac{p^2}{4x^2}\\
&=\frac{4x^2}{45}-\frac{2p}{45}+\frac{p^2}{45x^2}\pmod{p^3}. \qquad (3.9)
\end{aligned}
$$

If $p\equiv 2\pmod 3$ with $p>5$ (the case $p=5$ can be checked directly), then modulo p^2, we have

$$
\begin{aligned}
&\sum_{k=0}^{p-1}k^3\frac{D_k}{16^k}\\
\equiv&\sum_{j=0}^{\frac{p-1}{2}}\frac{\binom{2j}{j}^2}{16^j}\frac{pj(1+2j)+2p^2(j+1)(3j+1)+p^2j(2j+1)(H_{2j}-H_j)}{(3j+1)(3j+2)(3j+4)}.
\end{aligned}
$$

Hence, similar to above, we have

$$
\begin{aligned}
\sum_{k=0}^{p-1}k^3\frac{D_k}{16^k}\equiv&\frac{p}{27}\sum_{j=0}^{\frac{p-1}{2}}\frac{\binom{2j}{j}^2}{16^j}\left(\frac{-1}{3j+1}+\frac{-3}{3j+2}+\frac{10}{3j+4}\right)\\
&+\frac{p^2}{3}\sum_{j=0}^{\frac{p-1}{2}}\frac{\binom{2j}{j}^2}{16^j}\left(\frac{1}{3j+2}+\frac{1}{3j+4}\right)\\
&+\frac{p^2}{27}\sum_{j=0}^{\frac{p-1}{2}}\frac{\binom{2j}{j}^2}{16^j}\left(\frac{-1}{3j+1}+\frac{-3}{3j+2}+\frac{10}{3j+4}\right)(H_{2j}-H_j)\\
\equiv&-\frac{p}{9}\sum_{j=0}^{\frac{p-1}{2}}\frac{\binom{2j}{j}^2}{(3j+2)16^j}+\frac{p^2}{3}\sum_{j=0}^{\frac{p-1}{2}}\frac{\binom{2j}{j}^2}{(3j+2)16^j}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{p^2}{9}\sum_{j=0}^{\frac{p-1}{2}} \frac{\binom{2j}{j}^2}{(3j+2)16^j}(H_{2j}-H_j) \\
&\equiv -\frac{p}{9}\sum_{j=0}^{\frac{p-1}{2}} \frac{\binom{2j}{j}^2}{(3j+2)16^j} \equiv -\frac{4}{9}R_3(p) \pmod{p^2}.
\end{aligned}
$$

This, together with (3.5), (3.8) and (3.9), completes the proof of Theorem 1.1. □

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