AN EXTENSION OF A SUPERCONGRUENCE OF LONG AND RAMAKRISHNA

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Abstract. We prove two supercongruences for specific truncated hypergeometric series. These include an uniparametric extension of a supercongruence that was recently established by Long and Ramakrishna. Our proofs involve special instances of various hypergeometric identities including Whipple's transformation and the Karlsson–Minton summation.

1. Introduction

Let $(a)_n = a(a+1)\cdots(a+n-1)$ denote the Pochhammer symbol. For complex numbers $a_0, a_1, \ldots, a_r$ and $b_1, \ldots, b_r$, the (generalized) hypergeometric series ${}_{r+1}F_r$ is defined as

$$ {}_{r+1}F_r\left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_r \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k!(b_1)_k \cdots (b_r)_k} z^k. $$

Summation and transformation formulas for generalized hypergeometric series play an important part in the investigation of supercongruences. See, for instance, [7, 14, 15, 18–20, 22]. In particular, Long and Ramakrishna [18, Theorems 3 and 2] proved the following two supercongruences:

$$ \sum_{k=0}^{p-1} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} \equiv \begin{cases} -\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3}, & \text{if } p \equiv 1 \pmod 4, \\ -\frac{p^2}{16}\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3}, & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.1} $$

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and

$$\sum_{k=0}^{p-1}(6k+1)\frac{\left(\frac{1}{3}\right)_k^6}{k!^6} \equiv \begin{cases} -p\,\Gamma_p\left(\dfrac{1}{3}\right)^9 \pmod{p^6}, & \text{if } p\equiv 1 \pmod 6, \\ -\frac{10}{27}p^4\,\Gamma_p\left(\dfrac{1}{3}\right)^9 \pmod{p^6}, & \text{if } p\equiv 5 \pmod 6, \end{cases} \tag{1.2}$$

where $\Gamma_p(x)$ is the p-adic Gamma function. The restriction of the supercongruence (1.1) modulo p^2 was earlier established by Van Hamme [21, Equation (H.2)]. The supercongruence (1.2) is even stronger than a conjecure made by Van Hamme [21, Equation (D.2)] who asserted the corresponding supercongruence modulo p^4 for $p\equiv 1 \pmod 6$. Long and Ramakrishna also mentioned that (1.2) does not hold modulo p^7 in general.

The first purpose of this paper is to prove the following supercongruence. Note that the $r=\pm 1$ cases partially confirm the $d=5$ and $q\to 1$ case of [8, Conjectures 1 and 2].

Theorem 1. *Let $r\leqslant 1$ be an odd integer coprime with 5. Let p be a prime such that $p\equiv -\frac{r}{2} \pmod 5$ and $p\geqslant \frac{5-r}{2}$. Then*

$$\sum_{k=0}^{p-1}(10k+r)\frac{\left(\frac{r}{5}\right)_k^5}{k!^5} \equiv 0 \pmod{p^4}, \tag{1.3}$$

Recently, the second author [16] established the following supercongruence related to (1.2):

$$\sum_{k=0}^{p-1}(6k-1)\frac{\left(-\frac{1}{3}\right)_k^6}{k!^6} \equiv \begin{cases} 140p^4\,\Gamma_p\left(\dfrac{2}{3}\right)^9 \pmod{p^5}, & \text{if } p\equiv 1 \pmod 6, \\ 378p\,\Gamma_p\left(\dfrac{2}{3}\right)^9 \pmod{p^5}, & \text{if } p\equiv 5 \pmod 6, \end{cases} \tag{1.4}$$

where p is a prime.

The second purpose of this paper is to give the following common generalization of the second supercongruence in (1.2), restricted to modulo p^5, and the first supercongruence in (1.4).

Theorem 2. *Let $r\leqslant 1$ be an integer coprime with 3. Let p be a prime such that $p\equiv -r \pmod 3$ and $p\geqslant 3-r$. Then*

$$\sum_{k=0}^{p-1}(6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \equiv \frac{(-1)^{r+1}\,80rp^4}{81}\cdot\frac{\Gamma_p\big(1+\frac{r}{3}\big)^2}{\Gamma_p\big(1+\frac{2r}{3}\big)^3\,\Gamma_p\big(1-\frac{r}{3}\big)^4} \times\sum_{k=0}^{1-r}\frac{(r-1)_k\left(\frac{r}{3}\right)_k^3}{(1)_k\left(\frac{2r}{3}\right)_k^3} \pmod{p^5}. \tag{1.5}$$

Letting $r=1$ and $r=-1$ in (1.5) and using (1.9) and (1.11), we arrive at the $p\equiv 5 \pmod 6$ case of (1.2) modulo p^5 and the $p\equiv 1 \pmod 6$ case of (1.4), respectively.

Our proof of Theorem 1 will require Whipple's well-poised ${}_7F_6$ transformation formula (see [2, p. 28]):

$$
{}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e, & -n \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n \end{matrix} ;1\right]
$$

$$
=\frac{(a+1)_n(a-d-e+1)_n}{(a-d+1)_n(a-e+1)_n}\,{}_4F_3\left[\begin{matrix} 1+a-b-c, & d, & e, & -n \\ d+e-a-n, & 1+a-b, & 1+a-c & \end{matrix} ;1\right], \quad (1.6)
$$

where n is a non-negative integer, and Karlsson–Minton's summation formula (see, for example, [4, Equation (1.9.2)]):

$$
{}_{r+1}F_r\left[\begin{matrix} -n, & b_1+m_1, & \ldots, & b_r+m_r \\ & b_1, & \ldots, & b_r \end{matrix} ;1\right]=0, \quad (1.7)
$$

where $n, m_1, \ldots, m_r$ are non-negative integers and $n > m_1 + \cdots + m_r$. Our proof of Theorem 2 relies on a ${}_7F_6$ transformation formula slightly different from Whipple's ${}_7F_6$ transformation formula (1.6), obtained as a result from combining (1.6) with a ${}_4F_3$ transformation formula. The transformation was already utilized by the second author to prove (1.4).

Furthermore, in order to prove Theorem 2, we require some properties of the p-adic Gamma function, collected in the following two lemmas.

Lemma 1. [3, Section 11.6] *Suppose p is an odd prime and $x \in \mathbb{Z}_p$. Then*

$$
\Gamma_p(0)=1, \quad \Gamma_p(1)=-1, \quad (1.8)
$$

$$
\Gamma_p(x)\Gamma_p(1-x)=(-1)^{a_p(x)}, \quad (1.9)
$$

$$
\Gamma_p(x)\equiv\Gamma_p(y) \pmod p \quad \text{for } x\equiv y \pmod p, \quad (1.10)
$$

$$
\frac{\Gamma_p(x+1)}{\Gamma_p(x)}=\begin{cases} -x & \text{if } v_p(x)=0, \\ -1 & \text{if } v_p(x)>0, \end{cases} \quad (1.11)
$$

where $a_p(x) \in \{1, 2, \ldots, p\}$ with $x \equiv a_p(x) \pmod p$ and $v_p(\cdot)$ denotes the p-order.

Lemma 2. [18, Lemma 17, (4)] *Let p be an odd prime. If $a \in \mathbb{Z}_p, n \in \mathbb{N}$ such that none of $a, a+1, \ldots, a+n-1$ are in $p\mathbb{Z}_p$, then*

$$
(a)_n=(-1)^n\frac{\Gamma_p(a+n)}{\Gamma_p(a)}. \quad (1.12)
$$

In the following Sections 2 and 3, we give proofs of Theorems 1 and 2, respectively. The final Section 4 is devoted to a discussion and includes two conjectures.

2. Proof of Theorem 1

Motivated by the work of McCarthy and Osburn [19] and Mortenson [20], we take the following choice of parameters in (1.6). Let $a=\frac{r}{5}$, $b=\frac{r+5}{10}$, $c=\frac{r+3p}{5}$, $d=\frac{r+3ip}{5}$, $e=\frac{r-3ip}{5}$, and $n=\frac{3p-r}{5}$, where $i^2=-1$. Then we conclude that

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix}\frac{r}{5}, & 1+\frac{r}{10}, & \frac{r+3p}{5}, & \frac{r+3ip}{5}, & \frac{r-3ip}{5}, & \frac{r-3p}{5}\\ & \frac{r}{10}, & 1-\frac{3p}{5}, & 1-\frac{3ip}{5}, & 1+\frac{3ip}{5}, & 1+\frac{3p}{5}\end{matrix};1\right]\\
&=\frac{\left(1+\frac{r}{5}\right)_{\frac{3p-r}{5}}\left(1-\frac{r}{5}\right)_{\frac{3p-r}{5}}}{\left(1-\frac{3ip}{5}\right)_{\frac{3p-r}{5}}\left(1+\frac{3ip}{5}\right)_{\frac{3p-r}{5}}}\,{}_4F_3\left[\begin{matrix}\frac{5-r-6p}{10}, & \frac{r+3ip}{5}, & \frac{r-3ip}{5}, & \frac{r-3p}{5}\\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5}\end{matrix};1\right],
\end{aligned}
\tag{2.1}
$$

It is easy to see that, for $k\geqslant 0$ and any p-adic integer b,

$$
(a+bp)_k\,(a-bp)_k\,(a+bip)_k\,(a-bip)_k\equiv(a)_k^4 \pmod{p^4}. \tag{2.2}
$$

Hence, the left-hand side of (2.1) is congruent to

$$
\begin{aligned}
\sum_{k=0}^{\frac{3p-r}{5}}\frac{\left(1+\frac{r}{10}\right)_k\left(\frac{r}{5}\right)_k^5}{\left(\frac{r}{10}\right)_k(1)_k^5}&=\frac{1}{r}\sum_{k=0}^{\frac{3p-r}{5}}(10k+r)\frac{\left(\frac{r}{5}\right)_k^5}{k!^5}\\
&\equiv\frac{1}{r}\sum_{k=0}^{p-1}(10k+r)\frac{\left(\frac{r}{5}\right)_k^5}{k!^5}\pmod{p^4},
\end{aligned}
$$

where we have used the fact that $\frac{\left(\frac{r}{5}\right)_k}{k!}\equiv 0 \pmod{p}$ for $\frac{3p-r}{5}<k\leqslant p-1$ (the condition $p\geqslant\frac{5-r}{2}$ in the theorem is to guarantee $\frac{3p-r}{5}\leqslant p-1$). Since $\frac{3p-r}{5}\geqslant\frac{2p+r}{5}$, we have

$$
\begin{aligned}
\frac{\left(1+\frac{r}{5}\right)_{\frac{3p-r}{5}}\left(1-\frac{r}{5}\right)_{\frac{3p-r}{5}}}{\left(1-\frac{3ip}{5}\right)_{\frac{3p-r}{5}}\left(1+\frac{3ip}{5}\right)_{\frac{3p-r}{5}}}&=\frac{\left(1+\frac{r}{5}\right)_{\frac{3p-r}{5}}\left(1-\frac{r}{5}\right)_{\frac{3p-r}{5}}}{\left(1+\frac{9p^2}{25}\right)_{\frac{3p-r}{5}}}\\
&\equiv 0\pmod{p^2}.
\end{aligned}
$$

Finally, by the congruences

$$
(a+bip)_k\,(a-bip)_k\equiv(a+bp)_k\,(a-bp)_k\equiv(a)_k^2\pmod{p^2} \tag{2.3}
$$

for any p-adic integer b, we obtain

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}\frac{5-r-6p}{10}, & \frac{r+3ip}{5}, & \frac{r-3ip}{5}, & \frac{r-3p}{5}\\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5}\end{matrix};1\right] &\equiv {}_4F_3\left[\begin{matrix}\frac{5-r-6p}{10}, & \frac{r}{5}, & \frac{r}{5}, & \frac{r-3p}{5}\\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5}\end{matrix};1\right]\\
&\equiv {}_4F_3\left[\begin{matrix}\frac{5-r-6p}{10}, & \frac{r+p}{5}, & \frac{r-p}{5}, & \frac{r-3p}{5}\\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5}\end{matrix};1\right]\\
&= 0 \pmod{p^2},
\end{aligned}
$$

where we have utilized Karlsson–Minton's summation (1.7) with $n=\frac{3p-r}{5}$, $b_1=\frac{2r-3p}{5}$, $b_2=\frac{r+5}{10}$, $b_3=\frac{5-3p}{5}$, $m_1=\frac{1-r}{2}$, $m_2=\frac{2p+r-5}{10}$, and $m_3=\frac{2p+r-5}{5}$ in the last step.

3. Proof of Theorem 2

We can verify (1.5) for $r=1$ and $p=2$ by hand. In what follows, we assume that p is an odd prime. Recall the following transformation formula [16, Equation (4.2)]:

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix}t, & 1+\frac{t}{2}, & -n, & t-a, & t-b, & t-c, & 1-t-m+n+a+b+c\\ & \frac{t}{2}, & 1+t+n, & 1+a, & 1+b, & 1+c, & 2t+m-n-a-b-c\end{matrix};1\right]\\
&= \frac{(1+t)_n(a+b+2-m-t)_n(a+c+2-m-t)_n(b+c+2-m-t)_n}{(1+a)_n(1+b)_n(1+c)_n(a+b+c+1-m-2t)_n}\\
&\quad\times \frac{(a+b+1-m-t)(a+c+1-m-t)(b+c+1-m-t)}{(a+b+n+1-m-t)(a+c+n+1-m-t)(b+c+n+1-m-t)}\\
&\quad\times {}_4F_3\left[\begin{matrix}-m, & -n, & a+b+c+1-m-2t, & a+b+c+1+n-m-t\\ a+b+1-m-t, & a+c+1-m-t, & b+c+1-m-t\end{matrix};1\right]. \qquad (3.1)
\end{aligned}
$$

Let ζ be a fifth primitive root of unity. Setting $m=1-r$, $t=\frac{r}{3}$, $n=\frac{2p-r}{3}$, $a=\frac{2p\zeta}{3}$, $b=\frac{2p\zeta^2}{3}$ and $c=\frac{2p\zeta^3}{3}$ in (3.1) and using $1+\zeta+\zeta^2+\zeta^3+\zeta^4=0$, the left-hand side of (3.1) becomes

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix}1+\frac{r}{6}, & \frac{r}{3}, & \frac{r-2p}{3}, & \frac{r-2p\zeta}{3}, & \frac{r-2p\zeta^2}{3}, & \frac{r-2p\zeta^3}{3}, & \frac{r-2p\zeta^4}{3}\\ & \frac{r}{6}, & 1+\frac{2p}{3}, & 1+\frac{2p\zeta}{3}, & 1+\frac{2p\zeta^2}{3}, & 1+\frac{2p\zeta^3}{3}, & 1+\frac{2p\zeta^4}{3}\end{matrix};1\right]\\
&\equiv \frac{1}{r}\sum_{k=0}^{\frac{2p-r}{3}}(6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \pmod{p^5},
\end{aligned}
$$

where we have used the facts that none of the denominators in ${}_7F_6$ contain a multiple of p (the condition $p \geqslant 3-r$ in the theorem is to guarantee $\frac{2p-r}{3} \leqslant p-1$) and

$$(u+vp)_k\,(u+vp\zeta)_k\left(u+vp\zeta^2\right)_k\left(u+vp\zeta^3\right)_k\left(u+vp\zeta^4\right)_k \equiv (u)_k^5 \pmod{p^5}.$$

Furthermore, for $\frac{2p-r}{3} < k \leqslant p-1$ we have $\left(\frac{r}{3}\right)_k \equiv 0 \pmod{p}$. Thus,

$$\begin{aligned}
&{}_7F_6\left[\begin{matrix}1+\frac{r}{6}, & \frac{r}{3}, & \frac{r-2p}{3}, & \frac{r-2p\zeta}{3}, & \frac{r-2p\zeta^2}{3}, & \frac{r-2p\zeta^3}{3}, & \frac{r-2p\zeta^4}{3}\\ & \frac{r}{6}, & 1+\frac{2p}{3}, & 1+\frac{2p\zeta}{3}, & 1+\frac{2p\zeta^2}{3}, & 1+\frac{2p\zeta^3}{3}, & 1+\frac{2p\zeta^4}{3}\end{matrix};1\right]\\
&\quad\equiv \frac{1}{r}\sum_{k=0}^{p-1}(6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \pmod{p^5}.
\end{aligned}\tag{3.2}$$

On the other hand, we determine the terminating hypergeometric series on the right-hand side of (3.1) modulo p:

$$\begin{aligned}
&\frac{(a+b+1-m-t)(a+c+1-m-t)(b+c+1-m-t)}{(a+b+n+1-m-t)(a+c+n+1-m-t)(b+c+n+1-m-t)}\\
&\times {}_4F_3\left[\begin{matrix}-m, & -n, & a+b+c+1-m-2t, & a+b+c+1+n-m-t\\ a+b+1-m-t, & a+c+1-m-t, & & b+c+1-m-t\end{matrix};1\right]\\
&\equiv 8\sum_{k=0}^{1-r}\frac{(r-1)_k\left(\frac{r}{3}\right)_k^3}{(1)_k\left(\frac{2r}{3}\right)_k^3} \pmod{p}.
\end{aligned}\tag{3.3}$$

Moreover,

$$\begin{aligned}
&\frac{(1+t)_n(a+b+2-m-t)_n(a+c+2-m-t)_n(b+c+2-m-t)_n}{(1+a)_n(1+b)_n(1+c)_n(a+b+c+1-m-2t)_n}\\
&=\frac{\left(1+\frac{r}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}}{(-1)^{\frac{2p-r}{3}}\left(1+\frac{2p\zeta}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^2}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^3}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^4}{3}\right)_{\frac{2p-r}{3}}}.
\end{aligned}\tag{3.4}$$

Note that

$$\left(1+\frac{r}{3}\right)_{\frac{2p-r}{3}} = \frac{2p}{3}\left(1+\frac{r}{3}\right)_{\frac{2p-r-3}{3}}, \tag{3.5}$$

and

$$\left(1+\frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}$$
$$=\frac{5p^3}{27}\left(1+\frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{p-2r-3}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{p-2r-3}{3}}$$
$$\times\left(1+\frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{p-2r-3}{3}}\left(\frac{3+p(2\zeta+2\zeta^2+1)}{3}\right)_{\frac{p+r}{3}}$$
$$\times\left(\frac{3+p(2\zeta+2\zeta^3+1)}{3}\right)_{\frac{p+r}{3}}\left(\frac{3+p(2\zeta^2+2\zeta^3+1)}{3}\right)_{\frac{p+r}{3}}. \tag{3.6}$$

Combining (3.5) and (3.6), we arrive at

$$\frac{\left(1+\frac{r}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}}{(-1)^{\frac{2p-r}{3}}\left(1+\frac{2p\zeta}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^2}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^3}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^4}{3}\right)_{\frac{2p-r}{3}}}$$
$$\equiv\frac{(-1)^{\frac{2p-r}{3}}10p^4}{81}\cdot\frac{\left(1+\frac{r}{3}\right)_{\frac{2p-r-3}{3}}\left(1+\frac{2r}{3}\right)^3_{\frac{p-2r-3}{3}}(1)^3_{\frac{p+r}{3}}}{(1)^4_{\frac{2p-r}{3}}}\pmod{p^5}. \tag{3.7}$$

It follows from (3.2)–(3.4) and (3.7) that

$$\sum_{k=0}^{p-1}(6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6}\equiv\frac{(-1)^{\frac{2p-r}{3}}80rp^4}{81}\cdot\frac{\left(1+\frac{r}{3}\right)_{\frac{2p-r-3}{3}}\left(1+\frac{2r}{3}\right)^3_{\frac{p-2r-3}{3}}(1)^3_{\frac{p+r}{3}}}{(1)^4_{\frac{2p-r}{3}}}$$
$$\times\sum_{k=0}^{1-r}\frac{(r-1)_k\left(\frac{r}{3}\right)_k^3}{(1)_k\left(\frac{2r}{3}\right)_k^3}\pmod{p^5}. \tag{3.8}$$

By Lemmas 1 and 2, we have

$$\frac{\left(1+\frac{r}{3}\right)_{\frac{2p-r-3}{3}}\left(1+\frac{2r}{3}\right)^3_{\frac{p-2r-3}{3}}(1)^3_{\frac{p+r}{3}}}{(1)^4_{\frac{2p-r}{3}}}$$
$$\overset{(1.12)}{=}\frac{(-1)^{\frac{2p-r}{3}+r}\Gamma_p\left(\frac{2p}{3}\right)\Gamma_p\left(\frac{p}{3}\right)^3\Gamma_p\left(1+\frac{p+r}{3}\right)^3\Gamma_p(1)}{\Gamma_p\left(1+\frac{r}{3}\right)\Gamma_p\left(1+\frac{2r}{3}\right)^3\Gamma_p\left(1+\frac{2p-r}{3}\right)^4}$$

$$\overset{(1.10)}{\equiv} \frac{(-1)^{\frac{2p-r}{3}+r}\,\Gamma_p(0)^4\,\Gamma_p\left(1+\frac{r}{3}\right)^2\Gamma_p(1)}{\Gamma_p\big(1+\frac{2r}{3}\big)^3\,\Gamma_p\big(1-\frac{r}{3}\big)^4} \pmod p$$

$$\overset{(1.8)}{\equiv} \frac{(-1)^{\frac{2p-r}{3}+r+1}\,\Gamma_p\big(1+\frac{r}{3}\big)^2}{\Gamma_p\big(1+\frac{2r}{3}\big)^3\,\Gamma_p\big(1-\frac{r}{3}\big)^4}. \tag{3.9}$$

The proof of (1.5) then follows from (3.8) and (3.9).

4. Discussion

We know that many supercongruences have nice q-analogues (see [5, 6, 8–13, 17, 23]). For example, we have the following conjectural q-analogue of (1.3): for the same p and r as in Theorem 1,

$$\sum_{k=0}^{p-1}[10k+r]\frac{(q^r;q^5)_k^5}{(q^5;q^5)_k^5}q^{\frac{5(3-r)k}{2}} \equiv 0 \pmod{[p]^4}, \tag{4.1}$$

where $[n]=1+q+\cdots+q^{n-1}$ is the q-integer and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the q-shifted factorial.

Although there are q-analogues of Whipple's well-poised ${}_7F_6$ transformation and of Karlsson–Minton's summation (see [4, Appendix (II.27) and (III.18)]), we are unable to give a proof of (4.1). This is because we only know a q-analogue of (2.3) (see [10, Lemma 1]) but do not know any q-analogues of (2.2). Besides, we do not know how to prove (4.1) by using the method of 'creative microscoping' devised in [11] either.

While in Theorem 2 we were able to provide a common generalization of the second supercongruence in (1.2) (restricted to modulo p^5) and the first supercongruence in (1.4), it appears to be rather difficult to extend Theorem 1 to a higher supercongruence involving the p-adic Gamma function in the spirit of Theorem 2, even in the special cases $r=1$ or $r=-1$.

We end our paper with two further conjectures for future research. Conjecture 1 concerns a stronger version of Theorem 2 and includes the second supercongruence in (1.2) as a special case. Conjecture 2 concerns a common generalization of the first supercongruence in (1.2) and the second supercongruence in (1.4).

Conjecture 1. *The supercongruence* (1.5) *holds modulo* p^6 *for any prime* $p>3$.

Conjecture 2. *Let* $r\leqslant 1$ *be an integer coprime with* 3. *Let* $p\geqslant 7$ *be a prime such that* $p\equiv r \pmod 3$ *and* $p\geqslant 3-2r$. *Then*

$$\sum_{k=0}^{p-1}(6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \equiv \frac{(-1)^r\,8rp}{3}\cdot\frac{\Gamma_p\big(1+\frac{r}{3}\big)^2}{\Gamma_p\big(1+\frac{2r}{3}\big)^3\,\Gamma_p\big(1-\frac{r}{3}\big)^4} \times\sum_{k=0}^{1-r}\frac{(r-1)_k\left(\frac{r}{3}\right)_k^3}{(1)_k\left(\frac{2r}{3}\right)_k^3} \pmod{p^6}. \tag{4.2}$$

We remark that, to prove Conjecture 1, we need to determine the ${}_4F_3$ series on the left-hand side of (3.3) modulo p^2. But this ${}_4F_3$ series contains 5th roots of unity and lacks the symmetry which we would require for being able to deal with it. Moreover, by using (3.1) and the same method as in the proof of Theorem 2, we can only show that (4.2) holds modulo p^2. A new technique is needed to prove Conjecture 2.

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