BILATERAL IDENTITIES OF THE ROGERS–RAMANUJAN TYPE

MICHAEL J. SCHLOSSER*

Dedicated to the memory of Srinivasa Ramanujan

Abstract. We derive by analytic means a number of bilateral identities of the Rogers–Ramanujan type. Our results include bilateral extensions of the Rogers–Ramanujan and the Göllnitz–Gordon identities, and of related identities by Ramanujan, Jackson, and Slater. We give corresponding results for multisums including multilateral extensions of the Andrews–Gordon identities, of the Andrews–Bressoud generalization of the Göllnitz–Gordon identities, of Bressoud's even modulus identities, and other identities. Our closed form bilateral and multilateral summations appear to be the very first of their kind.

1. Introduction

For complex variables a and q with $|q|<1$ and $k\in\mathbb{Z}\cup\{\infty\}$, the q-shifted factorials are defined as follows (cf. [30]):

$$(a;q)_\infty:=\prod_{j=1}^{\infty}(1-aq^{j-1}),\qquad\text{and}\qquad (a;q)_k=\frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

Specifically,

$$(a;q)_k:=\begin{cases}1 & \text{for } k=0,\\ \prod_{j=1}^{k}(1-aq^{j-1}) & \text{for } k>0,\\ \prod_{j=1}^{-k}(1-aq^{-j})^{-1} & \text{for } k<0.\end{cases}$$

The variable q is referred to as the *base*. For brevity, we frequently use the compact notation

$$(a_1,\dots,a_m;q)_k=(a_1;q)_k\cdots(a_m;q)_k,$$

where m is a positive integer. Unless stated otherwise, all the summations in this paper converge absolutely (subject to the condition $|q|<1$, which we assume).

2010 *Mathematics Subject Classification.* Primary 11P84; Secondary 33D15.

Key words and phrases. Rogers–Ramanujan type identities, q-series, basic hypergeometric series, bilateral summations, multilateral summations.

*Research partly supported by FWF Austrian Science Fund grants F50-08 and P32305.

The first and second Rogers–Ramanujan identities,

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{1.1a}$$

$$\sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q;q)_k} = \frac{1}{(q^2,q^3;q^5)_\infty}, \tag{1.1b}$$

have a prominent history. They were first discovered and proved in 1894 by Rogers [49], and then independently rediscovered by the legendary Indian mathematician Ramanujan some time before 1913 (cf. Hardy [36]). They were also independently discovered and proved in 1917 by Schur [50]. About the pair of identities in (1.1) Hardy [38, p. xxxiv] remarked

> 'It would be difficult to find more beautiful formulae than the "Rogers–Ramanujan" identities, ...'

It is not clear how Ramanujan was led to discover (1.1). Bhatnagar [17] describes a method to conjecture these identities. A basic hypergeometric proof of (1.1) was found by Watson [56], who observed that these identities can be obtained from the (now called) Watson transformation by taking suitable limits and applying instances of Jacobi's triple product identity. Watson's proof is not the only early proof using hypergeometric machinery. In [49], Rogers obtained directly an identity which nowadays is called the "Rogers–Selberg identity" (and which happens to be a special case of the Watson transformation that was discovered much later) from which the two Rogers–Ramanujan identities follow by specialization and instances of Jacobi's triple product identity.

The Rogers–Ramanujan identities are deep identities which have found interpretations in combinatorics, number theory, orthogonal polynomials, probability theory, statistical mechanics, representations of Lie algebras, vertex operator algebras, knot theory, and conformal field theory [3, 8, 11, 15, 16, 20, 21, 18, 24, 27, 28, 29, 33, 44, 45, 46, 48, 50, 54]. (We do not claim that the list of areas given is complete. Moreover, the selected references are only representative samples of papers on interpretations of the Rogers–Ramanujan identities.) A recent highlight in the theory concerns the construction of these identities for higher-rank Lie algebras [35].

A pair of identities similar to (1.1) are the first and second Göllnitz–Gordon identities,

$$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k}{(q^2;q^2)_k} q^{k^2} = \frac{1}{(q,q^4,q^7;q^8)_\infty}, \tag{1.2a}$$

$$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k}{(q^2;q^2)_k} q^{k(k+2)} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty}. \tag{1.2b}$$

These appeared in a combinatorial study of integer partitions in unpublished work by Göllnitz in 1960 ([31], see also [32]) and were rediscovered in 1965 by Gordon [34]. However, they were recorded more than 40 years earlier by Ramanujan in his lost notebook, see [9, p. 36–37, Entries 1.7.11–12], and were also published in 1952 by Slater [52] as specific entries in her famous list of 130 identities of the Rogers–Ramanujan type. The systematic study of such identities had been initiated by Bailey [12, 13] a few years earlier. A more complete list of identities of the Rogers–Ramanujan type was recently given by McLaughlin, Sills and Zimmer [47]. Further such identities were given by Chu and Zhang [23]. McLaughlin, Sills and Zimmer's list is reproduced (with some typographical errors corrected) in Appendix A of Sills' recent book [51], which provides an excellent introduction to the Rogers–Ramanujan identities.

The analytic identities in (1.1) and (1.2) admit partition-theoretic interpretations (cf. [6]). Because of the specific form of the q-products on the right-hand sides, the identities in (1.1), resp. (1.2), are often classified as modulus 5 and modulus 8 identities, respectively.

Another identity intimately linked to Ramanujan's name is the following summation formula (cf. [30, Appendix (II.29)])

$$\sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(b;q)_k} z^k = \frac{(q, az, q/az, b/a; q)_\infty}{(b, z, b/az, q/a; q)_\infty}, \qquad |b/a| < |z| < 1. \tag{1.3}$$

This identity, commonly known as Ramanujan's ${}_1\psi_1$ summation, is a bilateral extension of the q-binomial theorem (cf. [30, Appendix (II.3)])

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z| < 1, \tag{1.4}$$

which is a fundamental identity in the theory of basic hypergeometric series. Hardy described (1.3), which Ramanujan had noted but did not publish, as "a remarkable formula with many parameters" [37, Eq. (12.12.2)]. Importantly, (1.3) contains Jacobi's triple product identity (3.1) as a limiting case, an identity which plays a key role in many of the proofs of identities of the Rogers–Ramanujan type (and which we also make heavy use of in this paper).

Knowing that the q-binomial theorem (1.4) extends to a bilateral summation, one can ask the same question about the Rogers–Ramanujan and Göllnitz–Gordon identities in (1.1) and (1.2). While some authors have studied properties of bilateral series which extend the series in (1.1) (see [2, 26, 41]), no closed form bilateral summations which include the evaluations in (1.1) (or (1.2)) as special cases have been obtained yet.

In this paper, we derive *bilateral* extensions of the Rogers–Ramanujan and Göllnitz–Gordon identities in (1.1) and (1.2) and provide a number of related results. Our main results for single series are given in Section 2, together with several noteworthy corollaries. The proofs of the main results of Section 2, namely Theorems 2.1, 2.6 and 2.8 are deferred to Section 3. The proofs are analytic and involve a method similar to that used by Watson in [56] to prove the classical Rogers–Ramanujan identities. In particular, we utilize suitable limiting cases of a bilateral basic hypergeometric transformation formula of Bailey in combination with special instances of Jacobi's triple product identity to establish the respective identities. In Section 4 multisum extensions of our results are given, which in particular include multilateral extensions of the Andrews–Gordon identities and of the Andrews–Bressoud generalization of the Göllnitz–Gordon identities, in addition to other multisum identities. We end our paper with some concluding remarks in Section 5.

2. Main results and corollaries in the single series case

Our first result is a bilateral extension of the two Rogers–Ramanujan identities in (1.1).

Theorem 2.1. *We have the following two bilateral summations:*

$$\sum_{k=-\infty}^{\infty}\frac{q^{k^2}}{(zq;q)_k}z^{2k}=\frac{(1/z;q)_\infty}{(1/z^2,z^2q;q)_\infty}(q^5;q^5)_\infty \times\left[(z^5q^3,z^{-5}q^2;q^5)_\infty+z^{-1}(z^5q^2,z^{-5}q^3;q^5)_\infty\right], \tag{2.1a}$$

$$\sum_{k=-\infty}^{\infty}\frac{q^{k(k+1)}}{(zq;q)_k}z^{2k}=\frac{(1/z;q)_\infty}{(1/z^2,z^2q;q)_\infty}(q^5;q^5)_\infty \times\left[(z^5q^4,z^{-5}q;q^5)_\infty+z^{-3}(z^5q,z^{-5}q^4;q^5)_\infty\right], \tag{2.1b}$$

for complex z such that $z\notin\{q^{-1},q^{-2},\dots\}$.

The $z\to1$ limit of (2.1a) gives (1.1a), while the $z\to1$ limit of (2.1b) gives (1.1b).

Remark 2.2. As was kindly brought to the author's attention by George Andrews after being shown an earlier version of this paper, a result related to the series on the left-hand side of (2.1b) was found by Andrews in 1970 [2, Thm. 3], namely: *Let*

$$g(z)=(-z;q)_\infty\sum_{k=-\infty}^{\infty}\frac{q^{k(k-1)}}{(-z;q)_k}z^{2k}, \tag{2.2a}$$

then

$$\frac{g(z)+g(-z)}{2}=\frac{(q^2,-z^2,-z^{-2}q^2;q^2)_\infty}{(q;q^2)_\infty(q^4,q^{16};q^{20})_\infty}. \tag{2.2b}$$

The last expression shows that the even part of $g(z)$ (or of $g(-z)$) can be expressed in closed form. The method used in [2] can also be used to express the odd part of $g(z)$ in closed form (which was not done in [2]), which together with (2.2a) can be used to obtain an evaluation for $g(z)$, similar to (2.1b).

Remark 2.3. While in Section 3 we show how one can derive the two identities in (2.1) and the the other results listed in this section by using a powerful transformation formula for bilateral basic hypergeometric series derived by Bailey (see (3.2)) without appealing to the classical Rogers–Ramanujan identities in (1.1), the anonymous Referee pointed out a way how one can easily prove (actually, verify) the identities in (2.1) by an analytic, functional equation approach and making use of (1.1). We sketch the details for the Referee's verification of (2.1a). The details for verifying (2.1b) are similar.

Verification proof of (2.1a). Both sides of (2.1a) satisfy the functional equation

$$f(z;q)=\frac{z^2q}{1-zq}\,f(zq;q).$$

This implies that the ratio of the left- and right-hand sides is an elliptic (multiplicatively q-periodic) function. The right-hand side of (2.1a) has poles at $z=-q^m$, $z=\pm q^{1/2+m}$ and $z=\pm q^{-1/2-m/2}$, for m a nonnegative integer.

Because of the functional equation it is enough to consider the poles at $z=-1$, $z=\pm q^{-1/2}$ and $z=\pm q^{-1}$. The four poles at $z=-1$, $z=q^{-1/2}$, $z=-q^{-1/2}$ and $z=-q^{-1}$ have zero residue and the only pole with non-zero residue is the pole at $z=q^{-1}$. Indeed, both sides of the identity have poles at $z=q^{-1-m}$ for nonnegative integers m. Focusing on $z=q^{-1}$ (again, by the q-periodicity this is enough) one checks that the residue on the sum side at $z=q^{-1}$ is

$$-q^{-2}\sum_{k>0}\frac{q^{(k-1)^2}}{(q;q)_{k-1}}=-q^{-2}\sum_{k\geq 0}\frac{q^{k^2}}{(q;q)_k}=\frac{-q^{-2}}{(q,q^4;q^5)_\infty},$$

by the Rogers–Ramanujan identity (1.1a). (Because $|q|<1$, computing the poles termwise is justified). This agrees with the residue of $z=q^{-1}$ on the right. We conclude that the ratio of the left- and right-hand sides is a constant. Taking $z\to 1$ shows that this constant is 1, which completes the verification of (2.1a). □

Similar verification proofs can be given for the other bilateral identities that we obtained which involve the variable z.

All results in Sections 3 and 4 are obtained by specializing a sole master identity, namely (3.2). It is feasible for an interested reader to check whether any other relevant specializations of (3.2) were missed (in our collection of bilateral Rogers–Ramanujan type identities) that would yield further noteworthy identities.

As consequence of Theorem 2.1, we obtain the following four bilateral summations:

Corollary 2.4 (Bilateral modulus 25 identities)**.**

$$\sum_{k=-\infty}^{\infty}\frac{q^{k(5k-3)}}{(q;q^5)_k}=\frac{(q^4;q^5)_\infty(q^{10},q^{15},q^{25};q^{25})_\infty}{(q^2,q^3;q^5)_\infty},\tag{2.3a}$$

$$\sum_{k=-\infty}^{\infty}\frac{q^{(k-1)(5k-1)}}{(q^2;q^5)_k}=\frac{(q^3;q^5)_\infty(q^5,q^{20},q^{25};q^{25})_\infty}{(q,q^4;q^5)_\infty},\tag{2.3b}$$

$$\sum_{k=-\infty}^{\infty}\frac{q^{k(5k-4)}}{(q^3;q^5)_k}=\frac{(q^2;q^5)_\infty(q^5,q^{20},q^{25};q^{25})_\infty}{(q,q^4;q^5)_\infty},\tag{2.3c}$$

$$\sum_{k=-\infty}^{\infty}\frac{q^{k(5k+3)}}{(q^4;q^5)_k}=\frac{(q;q^5)_\infty(q^{10},q^{15},q^{25};q^{25})_\infty}{(q^2,q^3;q^5)_\infty}.\tag{2.3d}$$

Combinatorial interpretations of the left- and right-hand sides of these identities can be given (see the discussion in Section 5); since the formulations are rather lengthy, we do not provide the details here.

To deduce the bilateral identities in Corollary 2.4, first replace q by q^5 in (2.1) and then observe that the respective $z=q^{-3}$ and $z=q^{-2}$ cases of (2.1a) give (2.3b) and (2.3c), whereas the respective $z=q^{-4}$ and $z=q^{-1}$ cases of (2.1b) give (2.3a) and (2.3d).

Remark 2.5. Tim Huber has kindly informed the author how the series appearing in Corollary 2.4 are related to weight 1/5 modular forms for $\Gamma(5)$. (See [39] for a theory of theta functions to the quintic base).

In particular, writing $q=e^{2\pi \mathrm{i}\tau}$ (where $\mathrm{i}^2=-1$), the following functions A,B are weight 1/5 modular forms for $\Gamma(5)$ with a fifth root of unity as a multiplier:

$$\begin{aligned}A(\tau)&=\frac{q^{1/5}(q;q)_\infty^{2/5}}{(q^2;q^3;q^5)_\infty}\\&=\frac{q^{1/5}(q;q)_\infty^{2/5}}{(q;q^5)_\infty(q^{10},q^{15};q^{25};q^{25})_\infty}\sum_{k=-\infty}^{\infty}\frac{q^{k(5k+3)}}{(q^4;q^5)_k}\\&=\frac{q^{1/5}(q;q)_\infty^{2/5}}{(q^4;q^5)_\infty(q^{10},q^{15};q^{25};q^{25})_\infty}\sum_{k=-\infty}^{\infty}\frac{q^{k(5k-3)}}{(q;q^5)_k},\end{aligned}\tag{2.4a}$$

$$\begin{aligned}B(\tau)&=\frac{(q;q)_\infty^{2/5}}{(q;q^4;q^5)_\infty}\\&=\frac{(q;q)_\infty^{2/5}}{(q^3;q^5)_\infty(q^5,q^{20};q^{25};q^{25})_\infty}\sum_{k=-\infty}^{\infty}\frac{q^{(k-1)(5k-1)}}{(q^2;q^5)_k}\\&=\frac{(q;q)_\infty^{2/5}}{(q^2;q^5)_\infty(q^2,q^{20};q^{25};q^{25})_\infty}\sum_{k=-\infty}^{\infty}\frac{q^{k(5k-4)}}{(q^3;q^5)_k}.\end{aligned}\tag{2.4b}$$

The graded ring of modular forms of integer weight for $\Gamma_1(5)$ is generated by A^5 and B^5. Moreover, a basis for the space of weight 1 modular forms on $\Gamma_1(5)$ is given by

$$A^5,\ A^4B,\ A^3B^2,\ A^2B^3,\ AB^4,\ B^5.$$

It is possible to make similar connections to modularity for some of the other series appearing in this section, in particular, for the series in Corollary 2.7.

Our next result is a bilateral extension of the two Göllnitz–Gordon identities in (1.2).

Theorem 2.6. *We have the following two bilateral summations:*

$$\sum_{k=-\infty}^{\infty} \frac{(-zq;q^2)_k}{(zq^2;q^2)_k} q^{k^2} z^k = \frac{(-zq,1/z;q^2)_\infty}{(z^2q^2,1/z^2;q^2)_\infty}(q^8;q^8)_\infty \times \left[(z^4q^5,z^{-4}q^3;q^8)_\infty + z^{-1}(z^4q^3,z^{-4}q^5;q^8)_\infty\right], \tag{2.5a}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-zq;q^2)_k}{(zq^2;q^2)_k} q^{k(k+2)} z^k = \frac{(-zq,1/z;q^2)_\infty}{(z^2q^2,1/z^2;q^2)_\infty}(q^8;q^8)_\infty \times \left[(z^4q^7,z^{-4}q;q^8)_\infty + z^{-3}(z^4q,z^{-4}q^7;q^8)_\infty\right], \tag{2.5b}$$

for complex z such that $z \notin \{q^{-2},q^{-4},q^{-6},\dots\} \cup \{-q^{-1},-q^{-3},-q^{-5},\dots\}$.

The $z \to 1$ limit of (2.5a) gives (1.2a), while the $z \to 1$ limit of (2.5b) gives (1.2b). As consequence of Theorem 2.6, we obtain the following four bilateral summations:

Corollary 2.7 (Bilateral modulus 32 identities)**.**

$$\sum_{k=-\infty}^{\infty} \frac{(-q^5;q^8)_k}{(q;q^8)_{k+1}} q^{(k+2)(4k+1)} = \frac{(q^7;q^8)_\infty(q^8;q^{16})_\infty(q^{32};q^{32})_\infty}{(q^5,q^6;q^8)_\infty(q^2;q^{16})_\infty}, \tag{2.6a}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-q^7;q^8)_k}{(q^3;q^8)_{k+1}} q^{k(4k+3)} = \frac{(q^5;q^8)_\infty(q^8;q^{16})_\infty(q^{32};q^{32})_\infty}{(q^2,q^7;q^8)_\infty(q^6;q^{16})_\infty}, \tag{2.6b}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-q;q^8)_k}{(q^5;q^8)_k} q^{k(4k-3)} = \frac{(q^3;q^8)_\infty(q^8;q^{16})_\infty(q^{32};q^{32})_\infty}{(q,q^6;q^8)_\infty(q^{10};q^{16})_\infty}, \tag{2.6c}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-q^3;q^8)_k}{(q^7;q^8)_k} q^{k(4k+7)} = \frac{(q;q^8)_\infty(q^8;q^{16})_\infty(q^{32};q^{32})_\infty}{(q^2,q^3;q^8)_\infty(q^{14};q^{16})_\infty}. \tag{2.6d}$$

To deduce the bilateral identities in Corollary 2.7, first replace q by q^4 in (2.5) and then observe that the respective $z = q^3$ and $z = q^{-3}$ cases of (2.5a) give (2.6b) and (2.6c), whereas the respective $z = q$ and $z = q^{-1}$ cases of (2.5b) give (2.6a) and (2.6d).

Notice that Equations (2.6a) and (2.6b) can be obtained from each other by replacing q by $-q$. The same relation also holds for Equations (2.6c) and (2.6d).

We would like to stress that the bilateral summations in Corollaries 2.4 and 2.7, which we believe to be new (and also *beautiful*, in line with Hardy's quote about (1.1) stated in the introduction), are *not* special cases of the following bilateral extension of the Lebesgue identity

$$\sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(bq;q)_k} q^{\binom{k+1}{2}} b^k = \frac{(q^2, abq, q/ab, bq^2/a; q^2)_\infty}{(bq, q/a; q)_\infty} \tag{2.7}$$

(which can be obtained from [30, Appendix (II.30), $c \to \infty$ followed by $(a,b) \mapsto (ab, a)$]).

A noteworthy special case of (2.7) due to Göllnitz [32], which should be compared to the Göllnitz–Gordon identities in (1.2), is obtained by letting $(a, b, q) \mapsto (-q, 1, q^2)$:

$$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k}{(q^2;q^2)_k} q^{k(k+1)} = \frac{1}{(q^2, q^3, q^7; q^8)_\infty}. \tag{2.8}$$

Another noteworthy special case of (2.7) is obtained by letting $(a, b) \mapsto (-q, 1)$:

$$\sum_{k=0}^{\infty} \frac{(-q;q)_k}{(q;q)_k} q^{\binom{k+1}{2}} = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}. \tag{2.9}$$

which is identity (8) in Slater's list.

Other bilateral identities of the Rogers–Ramanujan type are collected in the following theorem:

Theorem 2.8. *We have the following four bilateral summations:*

$$\sum_{k=-\infty}^{\infty} \frac{(-z;q)_k}{(z^2q;q^2)_k} q^{\binom{k}{2}} z^k = \frac{(-z;q)_\infty (q;q^2)_\infty}{(z^2;q)_\infty (q^2/z^2;q^2)_\infty} (q^3, z^3, z^{-3}q^3; q^3)_\infty, \tag{2.10a}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-z;q^2)_k}{(zq;q)_{2k}} q^{k(k+1)} z^k = \frac{(q/z;q)_\infty (-zq^2;q^2)_\infty}{(z^2q^2, q^2/z^2, q; q^2)_\infty} (q^6, -z^3q^3, -z^{-3}q^3; q^6)_\infty, \tag{2.10b}$$

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \frac{(-z;q^2)_k}{(z;q)_{2k}} q^{k(k-1)} z^k &= \frac{(q/z;q)_\infty (-z;q^2)_\infty}{(z^2, q^2/z^2, q; q^2)_\infty} (q^6;q^6)_\infty \\ &\quad \times \left[(-z^3q, -z^{-3}q^5; q^6)_\infty + z^2(-z^3q^5, -z^{-3}q; q^6)_\infty \right],\end{aligned} \tag{2.10c}$$

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \frac{q^{2k^2}}{(z;q)_{2k+1}} z^{2k} &= \frac{(q/z;q)_\infty}{(z^2, q^2/z^2, q; q^2)_\infty} (q^8;q^8)_\infty \\ &\quad \times \left[(-z^4q^3, -z^{-4}q^5; q^8)_\infty + z(-z^4q^5, -z^{-4}q^3; q^8)_\infty \right],\end{aligned} \tag{2.10d}$$

for complex z such that the series on the left-hand sides have no poles.

The case $q \mapsto q^2$, followed by $z \to q$, of (2.10a) reduces to identity (25) in Slater's list, which can be stated as

$$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k}{(q^4;q^4)_k} q^{k^2} = \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q;q)_\infty (q^4;q^4)_\infty (q^6;q^6)_\infty} = \frac{(q^3;q^3)_\infty}{(q^4;q^4)_\infty (q, q^5; q^6)_\infty}. \tag{2.11}$$

The $z \to 1$ case of (2.10b) reduces to identity (48) in Slater's list, which is

$$\sum_{k=0}^{\infty} \frac{(-1;q^2)_k}{(q;q)_{2k}} q^{k(k+1)} = \frac{(q^4;q^4)_\infty (q^6;q^6)_\infty^5}{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^2}. \tag{2.12}$$

The $z \to q$ cases of (2.10b) and (2.10c) reduce to identities (50) and (29) in Slater's list, namely

$$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k}{(q;q)_{2k+1}} q^{k(k+2)} = \frac{(q^2;q^2)_\infty (q^{12};q^{12})_\infty^2}{(q;q)_\infty (q^4;q^4)_\infty (q^6;q^6)_\infty}, \tag{2.13a}$$

and

$$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k}{(q;q)_{2k}} q^{k^2} = \frac{(q^6;q^6)_\infty^2}{(q;q)_\infty (q^{12};q^{12})_\infty}, \tag{2.13b}$$

respectively. The $z \to q^2$ case of (2.10c) reduces to identity (28) in Slater's list, i.e.,

$$\sum_{k=0}^{\infty} \frac{(-q^2;q^2)_k}{(q;q)_{2k+1}} q^{k(k+1)} = \frac{(q^3;q^3)_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty (q^6;q^6)_\infty}. \tag{2.14}$$

Multiplication of both sides of (2.10d) by $(1-z)$ and letting $z \to 1$ reduces to a sum by F.H. Jackson [43], also given by Slater as identity (39), which is

$$\sum_{k\geq 0} \frac{q^{2k^2}}{(q;q)_{2k}} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty (q^2,q^{14};q^{16})_\infty}. \tag{2.15}$$

The $z \to q$ case of (2.10d) reduces to identity (38) in Slater's list, namely

$$\sum_{k\geq 0} \frac{q^{2k(k+1)}}{(q;q)_{2k+1}} = \frac{1}{(q,q^4,q^7;q^8)_\infty (q^6,q^{10};q^{16})_\infty}. \tag{2.16}$$

The $z \to -1$ cases of (2.10b) and (2.10c) reduce, after replacing the summation index k by $-k$, to the identities

$$\sum_{k\geq 0} \frac{(-1;q)_{2k}}{(q^2;q^2)_k} q^k = \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q;q)_\infty^2 (q^6;q^6)_\infty} = \frac{1}{(q,q^2;q^3)_\infty (q,q^5;q^6)_\infty}, \tag{2.17a}$$

$$\sum_{k\geq 0} \frac{(-q;q)_{2k}}{(q^2;q^2)_k} q^k = \frac{(q^6;q^6)_\infty^2}{(q;q)_\infty (q^3;q^3)_\infty} = \frac{1}{(q,q^2;q^3)_\infty (q^3,q^3;q^6)_\infty}. \tag{2.17b}$$

Equation (2.17a) is given by Slater as identity (24), while (2.17b) is due to Ismail and Stanton [40, Thm. 7]. It is not difficult to transform the ${}_2\phi_1$ series (with vanishing lower parameter) on the left-hand sides of Equations (2.17) by suitable instances of the q-Pfaff transformation [30, Appendix (III.4)] to ${}_1\phi_1$ series, by which (2.17a) is seen to be equivalent to the $q \mapsto -q$ case of (2.12) and also to (2.11), while (2.17b) is then seen to be equivalent to an identity by Ramanujan (cf. [9, p. 87, Entry 4.2.11]) and also to the $q \mapsto -q$ case of (2.14).

As consequence of Equation (2.10c), we obtain the following two bilateral summations:

Corollary 2.9 (Bilateral modulus 6 identities)**.**

$$\sum_{k=-\infty}^{\infty} \frac{(q^5;q^6)_k}{(-q^2;q^3)_{2k+1}} (-1)^k q^{k(3k+2)} = \frac{(q^5,q^6;q^6)_\infty}{(q;q^3)_\infty (q^3,q^4;q^6)_\infty}, \tag{2.18a}$$

$$\sum_{k=-\infty}^{\infty} \frac{(q;q^6)_k}{(-q;q^3)_{2k}} (-1)^k q^{k(3k-2)} = \frac{(q,q^6;q^6)_\infty}{(q^2;q^3)_\infty (q^2,q^3;q^6)_\infty}. \tag{2.18b}$$

To deduce the bilateral identities in Corollary 2.9, first replace q by q^3 in (2.10c) and then observe that the respective $z = -q^{-1}$ and $z = -q$ cases give (2.18a) and (2.18b).

The identities in Corollary 2.9 become even nicer if the summation index k is replaced by $-k$:

Corollary 2.9′ (Bilateral modulus 6 identities)**.**

$$\sum_{k=-\infty}^{\infty} \frac{(-q;q^3)_{2k-1}}{(q;q^6)_k} q^{3k-2} = \frac{(q^5,q^6;q^6)_\infty}{(q;q^3)_\infty (q^3,q^4;q^6)_\infty}, \tag{2.19a}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-q^2;q^3)_{2k}}{(q^5;q^6)_k} q^{3k} = \frac{(q,q^6;q^6)_\infty}{(q^2;q^3)_\infty (q^2,q^3;q^6)_\infty}. \tag{2.19b}$$

Further, as consequence of Equation (2.10d), we obtain the following four bilateral summations:

Corollary 2.10 (Bilateral modulus 32 identities)**.**

$$\sum_{k=-\infty}^{\infty} \frac{q^{2k(4k-3)}}{(q;q^8)_k (-q^5;q^8)_k} = \frac{(q^4,q^7;q^8)_\infty (q^{32};q^{32})_\infty}{(q^2,q^3;q^8)_\infty (q^{14};q^{16})_\infty}, \tag{2.20a}$$

$$\sum_{k=-\infty}^{\infty} \frac{q^{2k(4k+3)}}{(q^3;q^8)_{k+1} (-q^7;q^8)_k} = \frac{(q^4,q^5;q^8)_\infty (q^{32};q^{32})_\infty}{(q,q^6;q^8)_\infty (q^{10};q^{16})_\infty}, \tag{2.20b}$$

$$\sum_{k=-\infty}^{\infty} \frac{q^{2k(4k-3)}}{(q^5;q^8)_k (-q;q^8)_k} = \frac{(q^3,q^4;q^8)_\infty (q^{32};q^{32})_\infty}{(q^2,q^7;q^8)_\infty (q^6;q^{16})_\infty}, \tag{2.20c}$$

$$\sum_{k=-\infty}^{\infty} \frac{q^{2k(4k+3)}}{(q^7;q^8)_k (-q^3;q^8)_{k+1}} = \frac{(q,q^4;q^8)_\infty (q^{32};q^{32})_\infty}{(q^5,q^6;q^8)_\infty (q^2;q^{16})_\infty}. \tag{2.20d}$$

To deduce the bilateral identities in Corollary 2.10, first replace q by $-q^4$ in (2.10d) and then observe that the respective $z = q^3$ and $z = q^{-3}$ cases give (2.20b) and (2.20c).

The identities in (2.20a) and (2.20d) follow by replacing q by $-q$ in (2.20c) and (2.20b), respectively.

3. Derivations of the main results in the single series case

A rich source of material on basic hypergeometric series is Gasper and Rahman's classic textbook [30]. In particular, we refer to that book for standard notions (such as that of a bilateral basic hypergeometric ${}_r\psi_s$ series), and to Appendix I of that book for the elementary manipulations of q-shifted factorials, which we employ without explicit mention.

An identity which we make crucial use of is Jacobi's triple product identity (cf. [30, (II.28)])

$$\sum_{k=-\infty}^{\infty} q^{\binom{k}{2}}(-z)^k = (q, z, q/z; q)_\infty. \tag{3.1}$$

Our starting point for deriving bilateral summations of the Rogers–Ramanujan type is the following transformation of a general bilateral ${}_2\psi_2$ series into a multiple of a very-well-poised ${}_6\psi_8$ series due to Bailey [14, (3.2)] (see also [30, Exercise 5.11, second identity]).

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \frac{(e, f; q)_k}{(aq/c, aq/d; q)_k} \left(\frac{aq}{ef}\right)^k &= \frac{(q/c, q/d, aq/e, aq/f; q)_\infty}{(aq, q/a, aq/cd, aq/ef; q)_\infty} \\ &\times \sum_{k=-\infty}^{\infty} \frac{(1 - aq^{2k})(c, d, e, f; q)_k}{(1 - a)(aq/c, aq/d, aq/e, aq/f; q)_k} q^{k^2} \left(\frac{a^3q}{cdef}\right)^k,\end{aligned} \tag{3.2}$$

valid for $|aq/cd| < 1$ and $|aq/ef| < 1$. Bailey obtained this transformation by bilateralizing Watson's transformation (cf. [30, (III.18)]) using the same method (replacing n by $2n$, shifting the summation index $k \mapsto k + n$, suitably shifting parameters and taking the limit $n \to \infty$), applied by Cauchy [22] in his second proof of Jacobi's triple product identity.

In (3.2) we now let $f \to \infty$ and perform the simultaneous substitutions $(a, c, d, e) \mapsto (az, az/b, az/c, a)$. This yields the following transformation of a general ${}_1\psi_2$ series into a multiple of a very-well-poised ${}_5\psi_8$ series.

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \frac{(a; q)_k}{(bq, cq; q)_k} q^{\binom{k+1}{2}}(-z)^k &= \frac{(bq/az, cq/az, zq; q)_\infty}{(azq, q/az, bcq/az; q)_\infty} \\ &\times \sum_{k=-\infty}^{\infty} \frac{(1 - azq^{2k})(az/b, az/c, a; q)_k}{(1 - az)(bq, cq, zq; q)_k} q^{3\binom{k}{2}} \left(-bczq^2\right)^k,\end{aligned} \tag{3.3}$$

valid for $|bcq/az| < 1$.

Theorems 2.1, 2.6 and 2.8 all appear by combining special instances of (3.3) with (3.1).

Proof of Theorem 2.1. In (3.3), we first let $c \to 0$, perform the substitutions $(b, z) \mapsto (z, bz/a)$ and let $a \to \infty$. We obtain

$$\sum_{k=-\infty}^{\infty} \frac{q^{k^2}(bz)^k}{(zq;q)_k} = \frac{(q/b;q)_\infty}{(bzq, q/bz;q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(1-bzq^{2k})}{(1-bz)} \frac{(b;q)_k}{(zq;q)_k} q^{5\binom{k}{2}} \left(-b^2z^3q^2\right)^k. \tag{3.4}$$

The $b = z$ case of (3.4) reduces to

$$\sum_{k=-\infty}^{\infty} \frac{q^{k^2}z^{2k}}{(zq;q)_k} = \frac{(1/z;q)_\infty \, z^{-1}}{(z^2q, 1/z^2;q)_\infty} \sum_{k=-\infty}^{\infty} (1+zq^k) q^{5\binom{k}{2}} \left(-z^5q^2\right)^k,$$

which after two applications of (3.1) yields (2.1a). Similarly, the $b = zq$ case of (3.4) reduces to

$$\sum_{k=-\infty}^{\infty} \frac{q^{k(k+1)}z^{2k}}{(zq;q)_k} = \frac{(1/z;q)_\infty}{(z^2q, 1/z^2;q)_\infty} \sum_{k=-\infty}^{\infty} (1-z^2q^{1+2k}) q^{5\binom{k}{2}} \left(-z^5q^4\right)^k,$$

which after two applications of (3.1) yields (2.1b). □

In the remaining proofs we only sketch the most relevant steps.

Proof of Theorem 2.6. In (3.3), we first let $c \to 0$, replace q by q^2 and set $(a, b, z) \mapsto (-zq, z, -zq^{-1})$. The result, after two applications of (3.1), is (2.5a). Now (2.5b) can readily be obtained from (2.5a) by replacing z by $-1/zq$ and reversing the sum. □

Proof of Theorem 2.8. The identity (2.10a) follows from (3.3) by making the substitution $(a, b, c, z) \mapsto (-z, zq^{-1/2}, -zq^{-1/2}, -z)$, and applying (3.1). The identity (2.10b) follows from (3.3) by replacing q by q^2, setting $(a, b, c, z) \mapsto (-z, z, zq^{-1}, -z)$, and applying (3.1). The identity (2.10c) follows from (3.3) by replacing q by q^2, setting $(a, b, c, z) \mapsto (-z, zq^{-1}, zq^{-2}, -zq^{-2})$, and applying (3.1) twice. The identity (2.10d) follows from (3.3) by replacing q by q^2, setting $(b, c, z) \mapsto (z, zq^{-1}, z^2/a)$ followed by taking $a \to \infty$, applying (3.1) twice and dividing both sides by $(1-z)$. □

4. Multisum extensions

Here we derive multisum extensions of the results from Section 2. Throughout we assume $r \geq 2$. We write $k = (k_1, \dots, k_{r-1})$ and define $k_r := 0$. Further, we define

$$\Lambda^{(r-1)} := \{k \in \mathbb{Z}^{r-1} \mid \infty > k_1 \geq \cdots \geq k_{r-1} > -\infty\}$$

in order to compactly specify the range of our multilateral summations.

Our multisum extension of Theorem 2.1 is a multilateral extension of the Andrews–Gordon identities [4], which, for integers r and i with $r \geq 2$ and $1 \leq i \leq r$, can be written as

$$\sum_{k \in \Lambda^{(r-1)}} \frac{q^{\sum_{j=1}^{r-1} k_j^2 + \sum_{j=i}^{r-1} k_j}}{(q;q)_{k_{r-1}} \prod_{j=1}^{r-2} (q;q)_{k_j - k_{j+1}}} = \frac{(q^i, q^{2r+1-i}, q^{2r+1}; q^{2r+1})_\infty}{(q;q)_\infty}. \tag{4.1}$$

An elementary, inductive proof of (4.1) was given by Andrews in [7].

Our multisum extension of Theorem 2.6 is a multilateral extension of Andrews generalized Göllnitz–Gordon identity [5, 6] (the $i=r$ case of identity (4.2) below), which has been extended to a family of identities for integers r and i with $r\geq 2$ and $1\leq i\leq r$, by Bressoud [20]. It can be written as

$$\sum_{k\in\Lambda^{(r-1)}}\frac{(-q^{1-k_1};q^2)_{k_1}q^{2\sum_{j=1}^{r-1}k_j^2+2\sum_{j=i}^{r-1}k_j}}{(q^2;q^2)_{k_{r-1}}\prod_{j=1}^{r-2}(q^2;q^2)_{k_j-k_{j+1}}}=\frac{(q^2;q^4)_\infty(q^{2i-1},q^{4r+1-2i},q^{4r};q^{4r})_\infty}{(q;q)_\infty}. \tag{4.2}$$

The identities in (4.1) and (4.2) reduce to (1.1) and (1.2), respectively, for $r=2$.

In [19], Bressoud also gave an even modulus analogue of the Andrews–Gordon identities in (4.1), namely

$$\sum_{k\in\Lambda^{(r-1)}}\frac{q^{\sum_{j=1}^{r-1}k_j^2+\sum_{j=i}^{r-1}k_j}}{(q^2;q^2)_{k_{r-1}}\prod_{j=1}^{r-2}(q;q)_{k_j-k_{j+1}}}=\frac{(q^i,q^{2r-i},q^{2r};q^{2r})_\infty}{(q;q)_\infty}, \tag{4.3}$$

where $1\leq i\leq r$. The $(r,i)=(2,1)$ and $(r,i)=(2,2)$ cases of (4.3) are special cases of the q-binomial theorem.

Our multilateral summations in this section are obtained by a procedure completely analogous to the single series case. Our starting point is the following multisum transformation which follows from a result by Agarwal, Andrews and Bressoud [1, Theorem 3.1 with Equations (4.1) and (4.2)]:

Proposition 4.1. *Let r and i be integers with $r\geq 2$ and $1\leq i\leq r$. Further, let n be a nonnegative integer. Then, with $k_0:=n$ and $k_r:=0$, we have the following series transformation:*

$$\begin{aligned}
&\sum_{n\geq k_1\geq\cdots\geq k_{r-1}\geq 0}\Bigg(\prod_{j=1}^{r}\frac{(b_j,c_j;q)_{k_j}}{(q;q)_{k_{j-1}-k_j}}\prod_{j=1}^{i-1}\frac{(a/b_jc_j;q)_{k_{j-1}-k_j}}{(a/b_j,a/c_j;q)_{k_{j-1}}}\\
&\qquad\times\prod_{j=i}^{r}\frac{(aq/b_jc_j;q)_{k_{j-1}-k_j}}{(aq/b_j,aq/c_j;q)_{k_{j-1}}}\prod_{j=1}^{r-1}\left(\frac{a}{b_jc_j}\right)^{k_j}\cdot q^{\sum_{j=i}^{r-1}k_i}\Bigg)\\
&=\sum_{k=0}^{n}\Bigg(\frac{(a;q)_k(-1)^kq^{\binom{k}{2}+(r+1-i)k}}{(q;q)_k(q;q)_{n-k}(a;q)_{n+k}}\frac{a^{rk}}{\prod_{j=1}^{r}(b_jc_j)^k}\\
&\qquad\times\prod_{j=1}^{i-1}\frac{(b_j,c_j;q)_k}{(a/b_j,a/c_j;q)_k}\prod_{j=i}^{r}\frac{(b_j,c_j;q)_k}{(aq/b_j,aq/c_j;q)_k}\\
&\qquad\times\left[1+\frac{(1-q^k)aq^{k-1}}{(1-aq^{k-1})}\prod_{j=i}^{r}\frac{b_jc_j\,(1-aq^k/b_j)(1-aq^k/c_j)}{aq\,(1-b_jq^{k-1})(1-c_jq^{k-1})}\right]\Bigg).
\end{aligned}\tag{4.4}$$

This is (even in the $i=r$ case) different from the multivariate Watson transformation due to Andrews [5, Thm. 4] (which, if used as a starting point instead, would only serve to prove the extremal $i=1$ and $i=r$ cases of the multisum identities we are after).

By multilateralization, we now deduce the following transformation of multisums.

Corollary 4.2. *Assuming $k_0 := \infty$, we have for $r \geq 2$ and $1 \leq i \leq r$ the following transformation:*

$$
\begin{aligned}
&\sum_{k\in\Lambda^{(r-1)}} \left(\frac{\prod_{j=1}^{r-1}(b_j,c_j;q)_{k_j}}{\prod_{j=1}^{r-2}(q;q)_{k_j-k_{j+1}}} \prod_{j=1}^{i-1} \frac{(a/b_jc_j;q)_{k_{j-1}-k_j}}{(a/b_j,a/c_j;q)_{k_{j-1}}} \right.\\
&\qquad \left. \times \frac{\prod_{j=i}^{r}(aq/b_jc_j;q)_{k_{j-1}-k_j}}{\prod_{j=i}^{r}(aq/b_j,aq/c_j;q)_{k_{j-1}}} \prod_{j=1}^{r-1}\left(\frac{a}{b_jc_j}\right)^{k_j} \cdot q^{\sum_{j=i}^{r-1}k_j} \right)\\
&= \frac{(q/b_r,q/c_r;q)_\infty}{(a,q/a,aq/b_rc_r;q)_\infty}\\
&\times \sum_{k=-\infty}^{\infty} \left(q^{k^2+(r-i)k} \frac{a^{(r+1)k}}{\prod_{j=1}^{r}(b_jc_j)^k} \prod_{j=1}^{i-1} \frac{(b_j,c_j;q)_k}{(a/b_j,a/c_j;q)_k} \prod_{j=i}^{r} \frac{(b_j,c_j;q)_k}{(aq/b_j,aq/c_j;q)_k} \right.\\
&\qquad \left. \times \left[1 - \prod_{j=i}^{r} \frac{b_jc_j(1-aq^k/b_j)(1-aq^k/c_j)}{aq(1-b_jq^{k-1})(1-c_jq^{k-1})} \right] \right),
\end{aligned}
\tag{4.5}
$$

valid for $\left|q^{r-i}\prod_{j=1}^{r-1}(a/b_jc_j)\right| < 1$ *and* $\left|q^{r-i}\prod_{j=1}^{r-1}(a/b_{j+1}c_{j+1})\right| < 1$.

Remark 4.3. Notice that for $i=r$ the expression in square brackets on the right-hand side of (4.5) simplifies to

$$
1 - \frac{bc(1-aq^k/b)(1-aq^k/c)}{aq(1-bq^{k-1})(1-cq^{k-1})} = \frac{(1-aq^{2k-1})(1-bc/aq)}{(1-bq^{k-1})(1-cq^{k-1})}
$$

(whereas the corresponding larger expression in square brackets on the right-hand side of (4.4) does not factorize for $i=r$), where we replaced (b_{r+1},c_{r+1}) by (b,c), and the transformation in (4.5) is then seen to be an $(r-1)$-dimensional generalization of the bilateral transformation in (3.2) (with a replaced by a/q) which alternatively could also be obtained by multilateralization of Andrews' formula [5, Thm. 4].

Proof of Corollary 4.2. To obtain (4.5) from (4.4), replace n by $2n$, shift the summation indices $k_1,\dots,k_{r-1}$ (on the left-hand side) and k (on the right-hand side) by n, perform the substitutions $a \mapsto aq^{-2n}$, $b_j \mapsto b_jq^{-n}$, $c_j \mapsto c_jq^{-n}$, for $j=1,\dots,r$, and let $n\to\infty$ while appealing to Tannery's theorem for taking termwise limits. $\square$

All the multisum identities of Rogers–Ramanujan type in this section are derived by means of the following lemma, which extends Equation (3.3):

Lemma 4.4. *We have for $r \geq 2$ and $1 \leq i \leq r$ the following transformation:*

$$\sum_{k\in\Lambda^{(r-1)}} \frac{(q^{1-k_1}/a;q)_{k_1}\, q^{\sum_{j=1}^{r-1}k_j^2+\sum_{j=i}^{r-1}k_j}}{(bq,cq;q)_{k_{r-1}}\prod_{j=1}^{r-2}(q;q)_{k_j-k_{j+1}}}(az)^{\sum_{j=1}^{r-1}k_j}$$
$$= \frac{(zq,b/az,c/az;q)_\infty}{(azq,1/az,bc/az;q)_\infty}$$
$$\times \sum_{k=-\infty}^{\infty}\Bigg(q^{(2r-1)\binom{k}{2}}\left(-a^{r-2}bcz^r q^{2r-i}\right)^k \frac{(a,azq/b,azq/c;q)_k}{(zq,bq,cq;q)_k}$$
$$\times\left[1-\frac{a^{i+1-r}z^{i+1-r}q^{2(i-r)k}(1-bq^k)(1-cq^k)}{b\,c\,(1-azq^k/b)(1-azq^k/c)}\right]\Bigg). \tag{4.6}$$

Proof. In Corollary 4.2 successively let $b_2,\dots,b_{r-1}\to\infty$ and $c_1,c_2,\dots,c_{r-1}\to\infty$, and perform the substitution $(a,b_1,b_r,c_r)\mapsto(azq,a,azq/b,azq/c)$. This establishes (together with some elementary manipulations of q-shifted factorials) the $i=2,\dots,r$ cases of the Lemma. The $i=1$ case can be established as follows: Start with the $i=1$ case of the right-hand side of (4.6) and split the sum according to the two terms in brackets. After shifting the summation index k by one in the second sum, the two sums can be combined and the resulting expression is seen to be equal to the $i=r$ and $z\mapsto zq$ case of the right-hand side of (4.6). (For $i=r$ the expression in brackets factorizes as we know from Remark 4.3.) Thus the sum equals the left-hand side of the $i=r$ and $z\mapsto zq$ case of (4.6) which is the same as its $i=1$ case with z left unchanged. □

For convenience, we state the $i=r$ case of Lemma 4.4 separately:

Lemma 4.5. *We have for $r\geq 2$ the following transformation:*

$$\sum_{k\in\Lambda^{(r-1)}} \frac{(q^{1-k_1}/a;q)_{k_1}\, q^{\sum_{j=1}^{r-1}k_j^2}(az)^{\sum_{j=1}^{r-1}k_j}}{(bq,cq;q)_{k_{r-1}}\prod_{j=1}^{r-2}(q;q)_{k_j-k_{j+1}}} = \frac{(zq,bq/az,cq/az;q)_\infty}{(azq,q/az,bcq/az;q)_\infty}$$
$$\times\sum_{k=-\infty}^{\infty}\frac{(1-azq^{2k})}{(1-az)}\frac{(a,az/b,az/c;q)_k}{(zq,bq,cq;q)_k}q^{(2r-1)\binom{k}{2}}\left(-a^{r-2}bcz^{r-1}q^r\right)^k. \tag{4.7}$$

From Lemma 4.4 (and its special case of Lemma 4.5) we readily deduce a number of multilateral identities of the Rogers–Ramanujan type.

We start with a multisum generalization of Theorem 2.1.

Theorem 4.6. *We have for $r\geq 2$ and $1\leq i\leq r$ the following multilateral summations:*

$$\sum_{k\in\Lambda^{(r-1)}}\frac{q^{\sum_{j=1}^{r-1}k_j^2+\sum_{j=i}^{r-1}k_j}\, z^{2\sum_{j=1}^{r-1}k_j}}{(zq;q)_{k_{r-1}}\prod_{j=1}^{r-2}(q;q)_{k_j-k_{j+1}}} = \frac{(1/z;q)_\infty}{(1/z^2,z^2q;q)_\infty}(q^{2r+1};q^{2r+1})_\infty$$
$$\times\left[(z^{2r+1}q^{2r+1-i},z^{-2r-1}q^i;q^{2r+1})_\infty + z^{2i-1-2r}(z^{2r+1}q^i,z^{-2r-1}q^{2r+1-i};q^{2r+1})_\infty\right], \tag{4.8}$$

for complex z such that $z \notin \{q^{-1}, q^{-2}, \dots\}$.

Proof. In Lemma 4.4, first let $c \to 0$, then perform the substitution $(b, z) \mapsto (z, z^2/a)$ and let $a \to \infty$. After two applications of (3.1) the identity (4.8) is obtained. $\square$

The $z \to 1$ limit of (4.8) reduces to Andrews–Gordon identities in (4.1). As an immediate consequence of Theorem 4.6 we obtain the following multisum generalization of Corollary 2.4:

Corollary 4.7. *We have for $r \geq 2$ and $1 \leq i \leq r$ the following multilateral summations:*

$$\sum_{k\in\Lambda^{(r-1)}} \frac{q^{(2r+1)\sum_{j=1}^{r-1}k_j^2-2i\sum_{j=1}^{r-1}k_j+(2r+1)\sum_{j=i}^{r-1}k_j}}{(q^{2r+1-i};q^{2r+1})_{k_{r-1}}\prod_{j=1}^{r-2}(q^{2r+1};q^{2r+1})_{k_j-k_{j+1}}}$$
$$= \frac{(q^i;q^{2r+1})_\infty(q^{2i(2r+1)},q^{(2r+1-2i)(2r+3)},q^{(2r+1)^2};q^{(2r+1)^2})_\infty}{(q^{2i},q^{2r+1-2i};q^{2r+1})_\infty}, \tag{4.9a}$$

and

$$q^{(i-1)(2r+1-2i)}\sum_{k\in\Lambda^{(r-1)}} \frac{q^{(2r+1)\sum_{j=1}^{r-1}k_j^2-2(2r+1-i)\sum_{j=1}^{r-1}k_j+(2r+1)\sum_{j=i}^{r-1}k_j}}{(q^{i};q^{2r+1})_{k_{r-1}}\prod_{j=1}^{r-2}(q^{2r+1};q^{2r+1})_{k_j-k_{j+1}}}$$
$$= \frac{(q^{2r+1-i};q^{2r+1})_\infty(q^{2i(2r+1)},q^{(2r+1-2i)(2r+1)},q^{(2r+1)^2};q^{(2r+1)^2})_\infty}{(q^{2i},q^{2r+1-2i};q^{2r+1})_\infty}. \tag{4.9b}$$

Proof. First replace q by q^{2r+1} in (4.6). Then the special case $z = q^{-i}$ gives (4.9a), while the special case $z = q^i$, after some elementary manipulations (including a simultaneous shift of the summation indices by -1), gives (4.9b). $\square$

Next we give a multisum generalization of Theorem 2.6.

Theorem 4.8. *We have for $r \geq 2$ and $1 \leq i \leq r$ the following multilateral summations:*

$$\sum_{k\in\Lambda^{(r-1)}} \frac{(-q^{1-2k_1}/z;q^2)_{k_1}\,q^{2\sum_{j=1}^{r-1}k_j^2+2\sum_{j=i}^{r-1}k_j}}{(zq^2;q^2)_{k_{r-1}}\prod_{j=1}^{r-2}(q^2;q^2)_{k_j-k_{j+1}}}z^{2\sum_{j=1}^{r-1}k_j} = \frac{(-zq,1/z;q^2)_\infty}{(z^2q^2,1/z^2;q^2)_\infty}(q^{4r};q^{4r})_\infty$$
$$\times\Big[(z^{2r}q^{4r+1-2i},z^{-2r}q^{2i-1};q^{4r})_\infty + z^{2i-1-2r}(z^{2r}q^{2i-1},z^{-2r}q^{4r+1-2i};q^{4r})_\infty\Big], \tag{4.10}$$

for complex z such that $z \notin \{q^{-2}, q^{-4}, q^{-6}, \dots\} \cup \{-q, -q^3, -q^5 \dots\}$.

Proof. In Lemma 4.4, first let $c \to 0$, replace q by q^2 and set $(a, b, z) \mapsto (-zq, z, -zq^{-1})$. After two applications of (3.1) the identity (4.10) is obtained. $\square$

The $z \to 1$ limit of (4.10) reduces to the Andrews–Bressoud generalization of the Göllnitz–Gordon identities in (4.2). As an immediate consequence of Theorem 4.8 we obtain the following multisum generalization of Corollary 2.7:

Corollary 4.9. *We have for $r \geq 2$ and $1 \leq i \leq r$ the following multilateral summations:*

$$\sum_{k\in\Lambda^{(r-1)}} \frac{(-q^{2r-1+2i-4rk_1};q^{4r})_{k_1}\, q^{4r\sum_{j=1}^{r-1}k_j^2-2(2i-1)\sum_{j=1}^{r-1}k_j+4r\sum_{j=i}^{r-1}k_j}}{(q^{4r+1-2i};q^{4r})_{k_r}\prod_{j=1}^{r-2}(q^{4r};q^{4r})_{k_j-k_{j+1}}}$$
$$= \frac{(q^{2i-1};q^{4r})_\infty(q^{4r(2i-1)},q^{4r(2r+1-2i)},q^{8r^2};q^{8r^2})_\infty}{(q^{2(2i-1)},q^{2r+1-2i};q^{4r})_\infty(q^{2(4r+1-2i)};q^{8r})_\infty}, \tag{4.11a}$$

and

$$q^{2(2i-1)(r-i)}\sum_{k\in\Lambda^{(r-1)}} \frac{(-q^{2r+1-2i-4rk_1};q^{4r})_{k_1} q^{4r\sum_{j=1}^{r-1}k_j^2+2(2i-1)\sum_{j=1}^{r-1}k_j+4r\sum_{j=i}^{r-1}k_j}}{(q^{2i-1};q^{4r})_{1+k_{r-1}}\prod_{j=1}^{r-2}(q^{4r};q^{4r})_{k_j-k_{j+1}}}$$
$$= \frac{(q^{4r+1-2i};q^{4r})_\infty(q^{4r(2i-1)},q^{4r(2r+1-2i)},q^{8r^2};q^{8r^2})_\infty}{(q^{2r-1+2i},q^{2(2r+1-2i)};q^{4r})_\infty(q^{2(2i-1)};q^{8r})_\infty}. \tag{4.11b}$$

Proof. First replace q by q^{2r} in (4.8). Then the special case $z = q^{1-2i}$ gives (4.11a), while the special case $z = q^{2i-1}$, after some elementary manipulations, gives (4.11b). □

Next we give a multilateral extension of the extremal $i = r$ case of Bressoud's even modulus analogue of the Andrews–Gordon identities (4.3). (For $1 \leq i \leq r-1$ the corresponding multilateral series do not absolutely converge. This problem already occurs in the $r = 2$ case.)

Theorem 4.10. *We have for $r \geq 2$ the following multilateral summation:*

$$\sum_{k\in\Lambda^{(r-1)}} \frac{q^{\sum_{j=1}^{r-1}k_j^2}\, z^{\sum_{j=1}^{r-1}k_j}}{(zq^2;q^2)_{k_{r-1}}\prod_{j=1}^{r-2}(q;q)_{k_j-k_{j+1}}} = \frac{(q;q^2)_\infty(z^rq^r,z^{-r}q^r,q^{2r};q^{2r})_\infty}{(zq;q)_\infty(q/z;q^2)_\infty}, \tag{4.12}$$

for complex z such that $z \notin \{q^{-2},q^{-4},q^{-6},\dots\}$.

Proof. In Lemma 4.5, perform the substitution $(b,c,z) \mapsto (z^{\frac12},-z^{\frac12},z/a)$, let $a \to \infty$ and apply (3.1). □

For $z = 1$ (4.12) reduces to the $i = r$ case of (4.3). For $r = 2$ (4.12) reduces to a special case of Ramanujan's ${}_1\psi_1$ summation (1.3).

Next we give a multilateral generalization of Theorem 2.8. (Again, for reasons of absolute convergence, we are only able to apply Lemma 4.5, i.e. the $i = r$ case of Lemma 4.4. The $c = 1$ cases of the latter could be applied to obtain multisum identities which would be naturally bounded from below, such as the original Andrews–Gordon identities. However, in this work we are after *multilateral* identities.)

Theorem 4.11. *We have for $r \geq 2$ the following multilateral summations:*

$$\sum_{k\in\Lambda^{(r-1)}} \frac{(-q^{1-k_1}/z;q)_{k_1}\, q^{\sum_{j=1}^{r-1}k_j(k_j-1)}}{(z^2q;q^2)_{k_{r-1}}\prod_{j=1}^{r-2}(q;q)_{k_j-k_{j+1}}} z^{2\sum_{j=1}^{r-1}k_j}$$

$$= \frac{(-z;q)_\infty (q;q^2)_\infty (z^{2r-1}, z^{1-2r}q^{2r-1}, q^{2r-1}; q^{2r-1})_\infty}{(z^2;q)_\infty (q^2/z^2;q^2)_\infty}, \tag{4.13a}$$

$$\sum_{k\in\Lambda^{(r-1)}} \frac{(-q^{2-2k_1}/z;q^2)_{k_1}\, q^{2\sum_{j=1}^{r-1}k_j^2}}{(zq;q)_{2k_{r-1}} \prod_{j=1}^{r-2}(q^2;q^2)_{k_j-k_{j+1}}} z^{2\sum_{j=1}^{r-1}k_j}$$
$$= \frac{(q/z;q)_\infty (-zq^2;q^2)_\infty (-z^{2r-1}q^{2r-1}, -z^{1-2r}q^{2r-1}, q^{4r-2}; q^{4r-2})_\infty}{(z^2q^2, q^2/z^2, q; q^2)_\infty}, \tag{4.13b}$$

$$\sum_{k\in\Lambda^{(r-1)}} \frac{(-q^{2-2k_1}/z;q^2)_{k_1}\, q^{2\sum_{j=1}^{r-1}k_j(k_j-1)}}{(z;q)_{2k_{r-1}} \prod_{j=1}^{r-2}(q^2;q^2)_{k_j-k_{j+1}}} z^{2\sum_{j=1}^{r-1}k_j} = \frac{(q/z;q)_\infty(-z;q^2)_\infty}{(z^2,q^2/z^2,q;q^2)_\infty}(q^{4r-2};q^{4r-2})_\infty$$
$$\times \left[(-z^{2r-1}q, -z^{1-2r}q^{4r-3}; q^{4r-2})_\infty + z^{2r-2}(-z^{2r-1}q^{4r-3}, -z^{1-2r}q; q^{4r-2})_\infty\right], \tag{4.13c}$$

$$\sum_{k\in\Lambda^{(r-1)}} \frac{q^{2\sum_{j=1}^{r-1}k_j^2} z^{2\sum_{j=1}^{r-1}k_j}}{(z;q)_{1+2k_{r-1}} \prod_{j=1}^{r-2}(q^2;q^2)_{k_j-k_{j+1}}} = \frac{(q/z;q)_\infty (q^{4r};q^{4r})_\infty}{(z^2,q^2/z^2,q;q^2)_\infty}$$
$$\times \left[(-z^{2r}q^{2r-1}, -z^{-2r}q^{2r+1}; q^{4r})_\infty + z(-z^{2r}q^{2r+1}, -z^{-2r}q^{2r-1}; q^{4r})_\infty\right], \tag{4.13d}$$

for complex z such that the series on the left-hand sides have no poles.

Proof. To prove the respective identities, apply Lemma 4.5, perform a specific substitution of variables (as specified below), occasionally combined with taking a limit, and then apply one or two instances of Jacobi's triple product identity (3.1). For (4.13a), take $(a,b,c,z) \mapsto (-z, zq^{-\frac{1}{2}}, -zq^{-\frac{1}{2}}, -z)$. For (4.13b), take $(a,b,c,z,q) \mapsto (-z, z, -zq^{-1}, -z, q^2)$. For (4.13c), take $(a,b,c,z,q) \mapsto (-z, zq^{-1}, -zq^{-2}, -zq^{-2}, q^2)$. Finally, for (4.13d), take $(b,c,z,q) \mapsto (z, zq^{-1}, z^2/a, q^2)$, divide both sides by $(1-z)$ and subsequently let $a \to \infty$. $\square$

All identities from Theorem 4.11 reduce to multisum generalizations of corresponding unilateral identities discussed after Theorem 2.8. For instance, we have

$$\sum_{k\in\Lambda^{(r-1)}} \frac{q^{2\sum_{j=1}^{r-1}k_j(k_j+\delta)}}{(q;q)_{\delta+2k_{r-1}} \prod_{j=1}^{r-2}(q^2;q^2)_{k_j-k_{j+1}}} = \frac{(-q^{2r(1+\delta)-1}, -q^{2r(1-\delta)+1}, q^{4r}; q^{4r})_\infty}{(q^2;q^2)_\infty}, \tag{4.14}$$

where $\delta = 0, 1$, which generalizes (2.15) and (2.16), respectively. The $\delta = 0$ case is obtained by multiplying both sides of (4.13d) by $(1-z)$ and letting $z \to 1$, while the $\delta = 1$ case is obtained from (4.13d) by letting $z \to q$. We leave other specializations of identities from Theorem 4.11 which generalize classical unilateral summations to the reader.

When we replace q by q^{2r-1} in (4.13c) and set $z = -q^{-1}$ or $z = -q$, we obtain the following two multilateral summations generalizing Corollary 2.9.

Corollary 4.12. *For $r \geq 2$ we have the following multilateral summations:*

$$\sum_{k\in\Lambda^{(r-1)}} \frac{(q^{1-2(2r-1)k_1};q^{2(2r-1)})_{k_1}\, q^{2(2r-1)\sum_{j=1}^{r-1}k_j^2+4(r-1)\sum_{j=1}^{r-1}k_j}}{(-q^{2r-2};q^{2r-1})_{1+2k_{r-1}}\prod_{j=1}^{r-2}(q^{2(2r-1)};q^{2(2r-1)})_{k_j-k_{j+1}}}$$
$$= \frac{(q^{4r-3},q^{2(2r-1)})_\infty (q^{2(2r-1)},q^{4(r-1)(2r-1)},q^{2(2r-1)^2};q^{2(2r-1)^2})_\infty}{(q;q^{2r-1})_\infty (q^{2r-1},q^{4(r-1)};q^{2(2r-1)})_\infty}, \tag{4.15a}$$

$$\sum_{k\in\Lambda^{(r-1)}} \frac{(q^{4r-3-2(2r-1)k_1};q^{2(2r-1)})_{k_1}\, q^{2(2r-1)\sum_{j=1}^{r-1}k_j^2-4(r-1)\sum_{j=1}^{r-1}k_j}}{(-q;q^{2r-1})_{2k_{r-1}}\prod_{j=1}^{r-2}(q^{2(2r-1)};q^{2(2r-1)})_{k_j-k_{j+1}}}$$
$$= \frac{(q,q^{2(2r-1)})_\infty (q^{2(2r-1)},q^{4(r-1)(2r-1)},q^{2(2r-1)^2};q^{2(2r-1)^2})_\infty}{(q^{2(r-1)};q^{2r-1})_\infty (q^2,q^{2r-1};q^{2(2r-1)})_\infty}. \tag{4.15b}$$

Finally, we have the following generalization of Corollary 2.10:

Corollary 4.13. *For $r \geq 2$ we have the following multilateral summations:*

$$\sum_{k\in\Lambda^{(r-1)}} \frac{q^{4r\sum_{j=1}^{r-1}k_j^2-2(2r-1)\sum_{j=1}^{r-1}k_j}}{(q;q^{4r})_{k_{r-1}}(-q^{2r+1};q^{4r})_{k_{r-1}}\prod_{j=1}^{r-2}(q^{4r};q^{4r})_{k_j-k_{j+1}}}$$
$$= \frac{(q^{2r},q^{4r-1};q^{4r})_\infty (q^{4r},q^{4r(2r-1)},q^{8r^2};q^{8r^2})_\infty}{(q^2,q^{2r-1};q^{4r})_\infty (q^{4r},q^{2(4r-1)};q^{8r})_\infty}, \tag{4.16a}$$

$$\sum_{k\in\Lambda^{(r-1)}} \frac{q^{4r\sum_{j=1}^{r-1}k_j^2+2(2r-1)\sum_{j=1}^{r-1}k_j}}{(q^{2r-1};q^{4r})_{1+k_{r-1}}(-q^{4r-1};q^{4r})_{k_{r-1}}\prod_{j=1}^{r-2}(q^{4r};q^{4r})_{k_j-k_{j+1}}}$$
$$= \frac{(q^{2r},q^{2r+1};q^{4r})_\infty (q^{4r},q^{4r(2r-1)},q^{8r^2};q^{8r^2})_\infty}{(q,q^{2(2r-1)};q^{4r})_\infty (q^{4r},q^{2(2r+1)};q^{8r})_\infty}, \tag{4.16b}$$

$$\sum_{k\in\Lambda^{(r-1)}} \frac{q^{4r\sum_{j=1}^{r-1}k_j^2-2(2r-1)\sum_{j=1}^{r-1}k_j}}{(q^{2r+1};q^{4r})_{k_{r-1}}(-q;q^{4r})_{k_{r-1}}\prod_{j=1}^{r-2}(q^{4r};q^{4r})_{k_j-k_{j+1}}}$$
$$= \frac{(q^{2r-1},q^{2r};q^{4r})_\infty (q^{4r},q^{4r(2r-1)},q^{8r^2};q^{8r^2})_\infty}{(q^2,q^{4r-1};q^{4r})_\infty (q^{2(2r-1)},q^{4r};q^{8r})_\infty}, \tag{4.16c}$$

$$\sum_{k\in\Lambda^{(r-1)}} \frac{q^{4r\sum_{j=1}^{r-1}k_j^2+2(2r-1)\sum_{j=1}^{r-1}k_j}}{(q^{4r-1};q^{4r})_{k_{r-1}}(-q^{2r-1};q^{4r})_{1+k_{r-1}}\prod_{j=1}^{r-2}(q^{4r};q^{4r})_{k_j-k_{j+1}}}$$
$$= \frac{(q,q^{2r};q^{4r})_\infty (q^{4r},q^{4r(2r-1)},q^{8r^2};q^{8r^2})_\infty}{(q^{2r+1},q^{2(2r-1)};q^{4r})_\infty (q^2,q^{4r};q^{8r})_\infty}. \tag{4.16d}$$

To deduce the multilateral identities in Corollary 4.13, first replace q by $-q^{2r}$ in (4.13d) and then put $z = q^{2r-1}$ to deduce (4.16b) or $z = q^{1-2r}$ to deduce (4.16c). The identities in (4.16a) and (4.16d) follow by replacing q by $-q$ in (4.16c) and (4.16b), respectively.

5. Concluding remarks

In this paper, we derived a number of bilateral and multilateral identities of the Rogers–Ramanujan type by analytic means. The closed form bilateral summations exhibited here appear to be the very first of their kind. We expect that more identities of this kind can be found. (In fact, after an earlier version of this paper appeared as a preprint, some results were established in [57] which are closely related to Theorem 2.1 and the results described in Remark 2.2.) Their very compact form and beauty suggests that these objects merit further study.

In view of the well-established connections of the classical Rogers–Ramanujan identities to various areas in mathematics and in physics (including combinatorics, number theory, orthogonal polynomials, probability theory, statistical mechanics, representations of Lie algebras, vertex operator algebras, knot theory and conformal field theory) and in view of the connections to modularity which in Remark 2.5 were made explicit (at least for the series appearing in Corollary 2.4), we speculate that similar connections to other areas can be established for some of the bilateral identities presented in this paper.

The deeper reasons for the existence of the here presented bilateral and multilateral identities are currently unclear. One might ask if there is a yet to be explained link with representation theory and wonder whether similar results hold for Rogers–Ramanujan identities for more complicated representations, such as for those appearing in [10, 25, 42, 53, 55]. A principal candidate to obtain multilateral extensions of those identities would be the constructive basic hypergeometric method which was successfully applied in this paper. Unfortunately, we have no idea which basic hypergeometric transformation one would have to start with to successfully apply this method in those more complicated settings. Another natural candidate to derive further multilateral extensions of Rogers–Ramanujan type identities would be the functional equation method employed in in Remark 2.3. Unfortunately, the author was unable to find such identities, that would extend those in [10, 25, 42, 53, 55], by this method (or by other means, so far). It is feasible that the product sides of the sought multilateral identities would contain more than two terms (whereas the product sides of the bilateral and multilateral identities in this paper all contain at most two terms) and would thus be rather difficult to find.

We note that we are able to give pure combinatorial interpretations of the left- and right-hand sides of several bilateral identities (such as those appearing in Corollaries 2.4 and 2.7) in the spirit of the well-known partition-theoretic interpretations of the classical Rogers–Ramanujan identities (1.1) provided by MacMahon [46] and Schur [50]. However, having such interpretations of the left- and right-hand sides of the respective identities alone does not automatically bring forward the bijective proofs one would like to have. Already in the classical case the problem of finding a direct, completely bijective, proof of (1.1a) or of (1.1b) has shown to be rather intractable. Since the formulations of our combinatorial interpretations of the new bilateral identities are rather lengthy (and they do not help us to prove the identities), we defer the details to another paper focused on those combinatorial interpretations.

On the conceptual level the question arises whether the work in this paper tells us anything new about the classical case. On one hand it is interesting to observe that one bilateral identity may contain different unilateral identities of interest. Examples are the bilateral identities in (2.10b) and (2.10c) which each include three different unilateral identities, as made explicit in the discussion after Theorem 2.8. On the other hand we would like to emphasize that the derivations of our bilateral identities of the Rogers–Ramanujan type do not require the combination of two unilateral sums into a bilateral sum (such as by replacing the summation index k in the second sum by $-1-k$), which one usually requires, before applying Jacobi's triple product identity in order to obtain the respective summations. In this respect, our derivations are very natural and straightforward while furnishing more general results than in the classical unilateral cases.

Acknowledgments. I would like to thank Ole Warnaar for stimulating discussions that helped to improve the paper. Further, I would like to thank George Andrews for making me aware of his results in [2]. I am also indebted to Tim Huber for sharing his insight related to Remark 2.5. Finally, I would like to thank Christian Krattenthaler and Wadim Zudilin for their interest and comments. This research was partly supported by FWF Austrian Science Fund grants F50-08 and P32305.

References

[1] A. K. Agarwal, G. E. Andrews, and D. M. Bressoud, *The Bailey lattice*, J. Indian Math. Soc. **51** (1987), 57–73.

[2] G. E. Andrews, *On a transformation of bilateral series with applications*, Proc. Amer. Math. Soc. **25** (1970), 554–558.

[3] G. E. Andrews, *On the general Rogers–Ramanujan theorem*, Mem. Amer. Math. Soc. **152** (1974), 86 pp.

[4] G. E. Andrews, *An analytic generalization of the Rogers–Ramanujan identities for odd moduli*, Proc. Nat. Acad. Sci. USA **71** (10) (1974), 4082–4085.

[5] G. E. Andrews, *Problems and prospects for basic hypergeometric series*, Theory and application of special functions, Math. Res. Center, Univ. Wisconsin, Publ. no. 35, Academic Press, New York, 1975; pp. 191–224.

[6] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics And Its Applications 2, Addison–Wesley Publishing Co., Reading, Massachussetts, 1976.

[7] G. E. Andrews, *Multiple series Rogers–Ramanujan type identities*, Pacific J. Math. **114** (2) (1984), 267–283.

[8] G. E. Andrews, *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, CBMS Regional Conference Series in Mathematics **66**, AMS, Providence, RI, 1986.

[9] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part II*, Springer, New York, 2008.

[10] G. E. Andrews, A. Schilling, and S. O. Warnaar, *An A_2 Bailey lemma and Rogers–Ramanujan type identities*, J. Amer. Math. Soc. **12**(3) (1999), 677–702.

[11] C. Armond and O. T. Dasbach, *Rogers–Ramanujan type identities and the head and tail of the colored Jones polynomial*, preprint arXiv:1106.3948, 2011.

[12] W. N. Bailey, *Some identities in combinatory analysis*, Proc. London Math. Soc. (2) **49** (1947), 421–435.

[13] W. N. Bailey, *Identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) **50** (1948), 1–10.

[14] W. N. Bailey, *On the basic bilateral hypergeometric series* ${}_2\psi_2$, Quart. J. Math. Oxford (2) **1** (1950), 194–198.

[15] R. J. Baxter, *Rogers–Ramanujan identities in the hard hexagon model*, J. Stat. Phys. **26** (3) (1981), 427–452.

[16] A. Berkovich and B. M. McCoy, *Rogers–Ramanujan identities: a century of progress from mathematics to physics*, Doc. Math., Extra Vol. ICM III (1998), 163–172.

[17] G. Bhatnagar, *How to discover the Rogers–Ramanujan identities*, Resonance **20** (5) (2015), 416–430.

[18] C. Boulet and I. Pak, *A combinatorial proof of the Rogers–Ramanujan and Schur identities*, J. Combin. Theory Ser. A **113** (6) (2006), 1019–1030.

[19] D. M. Bressoud, *Analytic and combinatorial generalizations of the Rogers–Ramanujan identities*, Mem. Amer. Math. Soc. **227** (1980), 54 pp.

[20] D. M. Bressoud, *Lattice paths and the Rogers–Ramanujan identities*, in: K. Alladi (ed.) *Number Theory*, Madras, India, 1987. Lec. Notes in Math. **1395**, Springer, Berlin, Heidelberg, 1989; pp. 140–172.

[21] D. M. Bressoud and D. Zeilberger, *A short Rogers–Ramanujan bijection*, Discrete Math. **38** (2–3) (1982), 313–315

[22] A. L. Cauchy, *Mémoire sur les fonctions dont plusieurs valeurs. . .* , C. R. Acad. Sci. Paris **17** (1843), 523; reprinted in *Oeuvres de Cauchy*, Ser. 1 (8), Gauthier-Villars, Paris (1893), pp. 42–50.

[23] W. Chu and W. Zhang, *Bilateral Bailey lemma and Rogers–Ramanujan identities*, Adv. Appl. Math. **42** (3) (2009), 358–391.

[24] S. Corteel, *Rogers–Ramanujan identities and the Robinson–Shensted–Knuth correspondence*, Proc. Amer. Math. Soc. **145** (5) (2017), 2011–2022.

[25] S. Corteel, J. Dehousse, and A. K. Uncu, *Cylindric partitions and some new A_2 Rogers–Ramanujan identities*, Proc. Amer. Math. Soc. **150**(2) (2022), 481–497.

[26] A. El-Guindy and M. E. H. Ismail, *On certain generalizations of Rogers–Ramanujan type identities*, preprint arXiv:1602.00316, 2016.

[27] J. Fulman, *A probabilistic proof of the Rogers–Ramanujan identities*, Bull. London Math. Soc. **33** (4) (2001), 397–407.

[28] K. Garrett, M. E. H. Ismail, and D. Stanton, *Variants of the Rogers–Ramanujan identities*, Adv. Appl. Math. **23** (3) (1999), 274–299.

[29] A. M. Garsia and S. C. Milne, *A Rogers–Ramanujan bijection*, J. Combin. Theory Ser. A **31** (3) (1981), 289–339.

[30] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Second Edition, Encyclopedia of Mathematics and its Applications 96, Cambridge University Press, Cambridge, 2004.

[31] H. Göllnitz, *Einfache Partitionen*, Diplomarbeit, Univ. Göttingen, 1960.

[32] H. Göllnitz, *Partitionen mit Differenzenbedingungen*, J. reine angew. Math. **225** (1967), 154–190.

[33] B. Gordon, *A combinatorial generalization of the Rogers–Ramanujan identities*, Amer. J. Math. **83** (1961), 393–399.

[34] B. Gordon, *Some continued fractions of the Rogers–Ramanujan type*, Duke Math. J. **32** (1965), 741–748.

[35] M. J. Griffin, K. Ono, and S. O. Warnaar, *A framework of Rogers–Ramanujan identities and their arithmetic properties*, Duke Math. J. **165** (2016), 1475–1527.

[36] G. H. Hardy, *The Indian mathematician Ramanujan*, Amer. Math. Monthly **44** (1937), 135–155.

[37] G. H. Hardy, *Ramanujan*, Cambridge University Press, Cambridge, 1940; reprinted by AMS Chelsea Publishing, New York, 1970.

[38] G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson, *Collected Papers of Srinivasa Ramanujan*, Cambridge University Press, Cambridge, 1927; reprinted by AMS Chelsea Publishing, New York, 1962.

[39] T. Huber, *A theory of theta functions to the quintic base*, J. Number Th. **134** (2014), 49–92.

[40] M. E. H. Ismail and D. Stanton, *Tribasic integrals and identities of Rogers–Ramanujan type*, Trans. Amer. Math. Soc. **355** (10) (2003), 4061–4091.

[41] M. E. H. Ismail and R. Zhang, *q-Bessel functions and Rogers–Ramanujan type identities*, Proc. Amer. Math. Soc. **146**(9) (2018), 3633–3646.

[42] S. Kanade and M. Russell, *Completing the A_2 Andrews–Schilling–Warnaar identities*, preprint `arXiv:2203.05690`, 2022.

[43] F. H. Jackson, *Examples of a generalization of Euler's transformation for power series*, Mess. Math. **57** (1928), 169–187.

[44] J. Lepowsky and S. C. Milne, *Lie algebraic approaches to classical partition identities*, Adv. Math. **29** (1) (1978), 15–59.

[45] J. Lepowsky and R. L. Wilson, *The Rogers–Ramanujan identities: Lie theoretic interpretation and proof*, Proc. Nat. Acad. Sci. USA **78** (1981), no. 2, part 1, 699–701.

[46] P. A. MacMahon, *Combinatory Analysis, vol. II*, Cambridge University Press, 1918.

[47] J. McLaughlin, A. V. Sills, and P. Zimmer, *Rogers–Ramanujan–Slater type identities*, Electron. J. Combin. **15** (2008), #DS15.

[48] M. Primc, *Vertex algebras and combinatorial identities*, Acta Appl. Math. **73** (1-2) (2002), 221–238.

[49] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.

[50] I. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, Sitzungsberichte der Berliner Akademie (1917), 302–321.

[51] A. V. Sills, *An Invitation to the Rogers–Ramanujan Identities*, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2018.

[52] L. J. Slater, *Further identities of the Rogers–Ramanujan type*, Proc. London Math. Soc., Ser. 2 **54** (1952), 147–167.

[53] M. Takigiku and S. Tsuchioka, *Andrews–Gordon type series for the level 5 and 7 standard modules of the affine Lie algebra $A_2^{(2)}$*, Proc. Amer. Math. Soc. **149**(7) (2021), 2763–2776.

[54] S. O. Warnaar, *The Andrews–Gordon identities and q-multinomial coefficients*, Commun. Math. Phys. **184** (1997), 203–232.

[55] S. O. Warnaar, *The A_2 Andrews–Gordon identities and cylindric partitions*, to appear in Trans. Amer. Math. Soc., Ser. B.; preprint `arXiv:2111.07550`, 2021.

[56] G. N. Watson, *A new proof of the Rogers–Ramanujan identities*, J. London Math. Soc. **4** (1929), 4–9.

[57] C. Wei, Y. Yu, and Q. Hu, *Two generalizations of Jacobi's triple product identity and their applications*, Ramanujan J. (2022), `https://doi.org/10.1007/s11139-022-00640-x`.

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

Email address: michael.schlosser@univie.ac.at