BILATERAL q-ULTRASPHERICAL FUNCTIONS

MICHAEL J. SCHLOSSER

Dedicated to the memory of Richard A. Askey

Abstract. We introduce a bilateral extension of the continuous q-ultraspherical polynomials which we call bilateral q-ultraspherical functions. These functions are given as specific bilateral basic hypergeometric ${}_2\psi_2$ series, they are analytic in a variable $x = \cos\theta$ and depend on two parameters β and γ and on a base q. For these bilateral q-ultraspherical functions we derive a bilateral generating function, find a three-term recurrence relation, explain how they behave under the action of the Askey–Wilson divided difference operator, and show that they satisfy a type of shifted orthogonality.

1. Introduction

Orthogonal and q-orthogonal polynomials (cf. [14] and [8] for the one-variable theory) are a central topic in the theory of special functions. The most fundamental ones are the families of *classical* orthogonal and q-orthogonal polynomials. These concern those that are of *hypergeometric* and *basic hypergeometric* type and are listed as the entries of the Askey and q-Askey schemes [10, 11]. An intriguing feature about the classical families of (q-)orthogonal polynomials is that they are connected to a vast amount of explicit identities for (basic) hypergeometric series. Many summation and transformation formulas are responsible for various fundamental properties that orthogonal polynomials possess. It is very natural to switch the perspective and ask, given a (basic) hypergeometric series identity, whether it has any relevance in the theory of orthogonal polynomials or, more generally, of orthogonal functions.

Richard Askey had a keen interest in orthogonal and q-orthogonal polynomials and made numerous valuable contributions to this rich subject. While his most influential contribution in this area arguably concerns his work with his student James Wilson on the discovery of (what today are called) the Askey–Wilson polynomials [3]—the top entry in the q-Askey scheme, another significant contribution concerns his work with Mourad Ismail on the continuous q-ultraspherical polynomials [2] (which are q-analogues of the Gegenbauer polynomials). The continuous q-ultraspherical polynomials are a family of q-orthogonal polynomials that were originally introduced by Rogers [13] in the late 19th century. Rogers devised some important properties for these polynomials but was apparently not aware of their orthogonality. Askey and Ismail [2] were the first to show that the continuous q-ultraspherical polynomials are indeed orthogonal with respect to a specific

2020 *Mathematics Subject Classification.* Primary 33D45; Secondary 33D15.

Key words and phrases. bilateral basic hypergeometric series, orthogonal functions.

The author's research was partly supported by FWF Austrian Science Fund grant 10.55776/P32305.

positive measure, and they worked out many properties for this extremely important family of special functions. The continuous q-ultraspherical polynomials are actually highly relevant to symmetric functions. In particular, the Macdonald polynomials [12, Ch. VI] (a well-investigated family of symmetric functions in the variables $x_1, \dots, x_n$ with coefficients that are rational functions in q and t) reduce to the continuous q-ultraspherical polynomials in the two-variable case. Further, the well-known Pieri formula for Macdonald polynomials [12, p. 331] reduces in the two-variable case to Rogers' linearization formula (see (3.7)). It is clear that a good knowledge about the continuous q-ultraspherical polynomials helps in the study of various extensions, including those to multivariables.

Our aim in this paper is to study a *bilateral series extension* of the continuous q-ultraspherical polynomials, which we refer to as *bilateral q-ultraspherical functions*. We (naively) define them as specific ${}_2\psi_2$ series and demonstrate that they satisfy some nice properties, similarly nice to those in the standard polynomial special case. In particular, we show that their bilateral generating function admits a representation as a closed form product. This result is used to find a three-term recurrence relation for the bilateral q-ultraspherical functions, and we also use it to deduce a simple formula for them under the action of the Askey–Wilson divided difference operator. Finally, in view of the nice properties we are able to extend here, what about orthogonality? Regrettably, so far we have *not* been able to establish full orthogonality for the here introduced bilateral q-ultraspherical functions. However, we do show that they satisfy a type of *shifted orthogonality*, a concept which requires working with full sequences, rather than just comparing two single functions.

Our paper is organized as follows: In Section 2 we recall some well-known material on basic hypergeometric series (with a focus on identities for ${}_2\psi_2$ series) that is useful for the current study. In Section 3 we review some material on the continuous q-ultraspherical polynomials. We turn to their new bilateral series extension in Section 4 and work out some properties, most of which nicely extend those for the continuous q-ultraspherical polynomials. We conclude with a result involving shifted orthogonality in Section 5.

Concerning our notation for fractions we follow the convention used in the text book by Gasper and Rahman [6] that all factors that appear after the slash symbol '/' are understood to be part of the denominator. For instance, 'dq/abz' (appearing in Equation (2.5)) is meant to stand for '$dq/(abz)$', etc.

2. Preliminaries on basic hypergeometric series

Let $\mathbb{Z}$ denote the set of integers. Throughout we assume q to be fixed with $0 < |q| < 1$. (Occasionally, when concerned with positive definiteness, we restrict q further to be real and satisfy $0 < q < 1$.) We refer to q as the "base". For a parameter $a \in \mathbb{C}$ and $k \in \mathbb{Z}$, the q-shifted factorial is defined by

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}, \qquad \text{where} \qquad (a;q)_\infty := \prod_{j\geq 0}(1-aq^j).$$

For brevity, we frequently use the shorthand notations

$$(te^{\pm \mathrm{i}\theta};q)_\infty = (te^{\mathrm{i}\theta}, te^{-\mathrm{i}\theta};q)_\infty,$$

$$(a_1,\dots,a_m;q)_k=(a_1;q)_k\dots(a_m;q)_k,\qquad k\in\mathbb{Z}\cup\infty.$$

Following Gasper and Rahman [6], basic hypergeometric ${}_r\phi_s$ series with r upper parameters $a_1,\dots,a_r$, s lower parameters $b_1,\dots,b_s$, base q and argument z are defined by

$$ {}_r\phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right]:=\sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(q,b_1,\dots,b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}z^k. \tag{2.1}$$

Such a series terminates if one of the upper parameters, say, a_r, is of the form q^{-n}, where n is a nonnegative integer. If the series does not terminate, then it converges for $r>s+1$, while for $r=s+1$ it converges for $|z|<1$.

Bilateral basic hypergeometric ${}_r\psi_s$ series are defined by

$$ {}_r\psi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right]:=\sum_{k=-\infty}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{s-r}z^k. \tag{2.2}$$

Such a series terminates from above if one of the upper parameters, say, a_r, is of the form q^{-n}, and it terminates from below, if one of the lower parameters, say, b_s, is of the form q^{1+m}, where n and m are integers (with $n+m\geq 0$, in case both occur).

If the series does not terminate, then it converges for $r<s$ and diverges for $r>s$, while for $r=s+1$ it converges (if it does not terminate from above) for $|z|<1$ and (if it does not terminate from below) for $|b_1\cdots b_r/a_1\cdots a_rz|<1$.

The most fundamental result in the theory of basic hypergeometric series is the nonterminating q-binomial theorem (cf. [6, Equation (II.3)]),

$$ {}_1\phi_0\left[\begin{matrix}a\\ -\end{matrix};q,z\right]=\frac{(az;q)_\infty}{(z;q)_\infty},\qquad |z|<1. \tag{2.3}$$

The following bilateral extension of (2.4) is due to Ramanujan (cf. [6, Equation (II.29)]),

$$ {}_1\psi_1\left[\begin{matrix}a\\ b\end{matrix};q,z\right]=\frac{(q,az,q/az,b/a;q)_\infty}{(b,z,b/az,q/a;q)_\infty},\qquad |b/a|<|z|<1. \tag{2.4}$$

A very simple and fundamental proof of (2.4) was delivered by Ismail [7] which makes use of the analyticity of the variable b around the origin, together with the identity theorem for analytic functions. For an excellent survey of Ramanujan's ${}_1\psi_1$ summation, see [16].

Among the numerous existing identities for basic hypergeometric series, we list some for ${}_2\psi_2$ series, because of their relevance to the current study. In [4, Eq. (2.3)] Bailey derived the transformation

$$ {}_2\psi_2\left[\begin{matrix}a,b\\ c,d\end{matrix};q,z\right]=\frac{(az,d/a,c/b,dq/abz;q)_\infty}{(z,d,q/b,cd/abz;q)_\infty}\,{}_2\psi_2\left[\begin{matrix}a,abz/d\\ az,c\end{matrix};q,\frac{d}{a}\right], \tag{2.5}$$

where $\max(|z|,|cd/abz|,|d/a|,|c/b|)<1$, which he iterated to obtain [4, Eq. (2.4)]

$$ {}_2\psi_2\left[\begin{matrix}a,b\\ c,d\end{matrix};q,z\right]=\frac{(az,bz,cq/abz,dq/abz;q)_\infty}{(q/a,q/b,c,d;q)_\infty}\,{}_2\psi_2\left[\begin{matrix}abz/c,abz/d\\ az,bz\end{matrix};q,\frac{cd}{abz}\right], \tag{2.6}$$

where $\max(|z|,|cd/abz|)<1$. Another important identity is Bailey's transformation of a general ${}_2\psi_2$ series to a multiple of a very-well-poised ${}_6\psi_8$ series (cf. [4, Eq. (3.2)]; see also

[6, Exercise 5.11, second identity]):

$$
\begin{aligned}
{}_2\psi_2\left[\begin{matrix} e, f \\ aq/c, aq/d \end{matrix}; q, \frac{aq}{ef}\right] &= \frac{(q/c, q/d, aq/e, aq/f; q)_\infty}{(aq, q/a, aq/cd, aq/ef; q)_\infty} \\
&\times {}_6\psi_8\left[\begin{matrix} qa^{\frac12}, -qa^{\frac12}, c, d, e, f \\ a^{\frac12}, -a^{\frac12}, aq/c, aq/d, aq/e, aq/f, 0, 0 \end{matrix}; q, \frac{a^3q^2}{cdef}\right],
\end{aligned}
\tag{2.7}
$$

valid for $|aq/cd| < 1$ and $|aq/ef| < 1$. Notice that (2.5) and (2.6) can be derived from (2.7), by exploiting the symmetry of the ${}_6\psi_8$ series.

For further material on basic hypergeometric series and more generally, to special functions, we refer to the text books by Gasper and Rahman [6], and by Andrews, Askey and Roy [1], respectively. In particular, in our our computations we implicitly make heavy use of elementary manipulations of q-shifted factorials (see [6, Appendix I]).

3. The continuous q-ultraspherical polynomials

We consider functions in $x = \cos\theta = (e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta})/2$ (where θ need not be real).

The continuous q-ultraspherical polynomials, which depend on a parameter β and the base q, are given by [8, Sec. 13.2]

$$
C_n(x;\beta\,|\,q) = \sum_{k=0}^{n} \frac{(\beta;q)_k\,(\beta;q)_{n-k}}{(q;q)_k\,(q;q)_{n-k}} e^{\mathrm{i}(n-2k)\theta}, \qquad x = \cos\theta. \tag{3.1}
$$

They were originally considered by Rogers [13] in 1884 (not aware of their orthogonality) in the pursuit of (what now are called) the Rogers–Ramanujan identities.

These functions, which can be written as

$$
C_n(x;\beta\,|\,q) = \frac{(\beta;q)_n}{(q;q)_n} e^{\mathrm{i}n\theta}\,{}_2\phi_1\left[\begin{matrix} \beta, q^{-n} \\ q^{1-n}/\beta \end{matrix}; q, qe^{-2\mathrm{i}\theta}\beta\right], \tag{3.2}
$$

are polynomials in x of degree n. They have the generating function

$$
\sum_{n=0}^{\infty} C_n(x;\beta\,|\,q)t^n = \frac{(\beta te^{\pm\mathrm{i}\theta};q)_\infty}{(te^{\pm\mathrm{i}\theta};q)_\infty}, \tag{3.3}
$$

which readily follows from the definition (3.1) and the q-binomial theorem in (2.3). From this generating function one easily deduces their recurrence relation, which is [8, Eq. (13.2.12)]

$$
\begin{aligned}
&2x(1-\beta q^n)\,C_n(x;\beta\,|\,q) \\
&= (1-q^{n+1})\,C_{n+1}(x;\beta\,|\,q) + (1-\beta^2q^{n-1})\,C_{n-1}(x;\beta\,|\,q),
\end{aligned}
$$

with the initial conditions

$$
C_{-1}(x;\beta\,|\,q) = 0 \quad \text{and} \quad C_0(x;\beta\,|\,q) = 1.
$$

By induction it follows that the $C_{2n+1}(x;\beta\,|\,q)$ are odd functions, the $C_{2n}(x;\beta\,|\,q)$ are even. Their constant terms are [8, Eq. (13.2.19)]

$$
C_{2n+1}(0;\beta\,|\,q) = 0 \quad \text{and} \quad C_{2n}(0;\beta\,|\,q) = (-1)^n \frac{(\beta^2;q^2)_n}{(q^2;q^2)_n},
$$

for all integer n.

Let $\mathcal{D}_q$ denote the *Askey–Wilson operator*, defined on functions in $x = (z + 1/z)/2$ (recall that $x = \cos\theta$, so $z = e^{\mathrm{i}\theta}$) by [8, Eq. (12.1.10)]

$$\begin{aligned}\mathcal{D}_q f(x) &= \mathcal{D}_q f((z+1/z)/2)\\ &:= \frac{f((q^{\frac12}z + q^{-\frac12}/z)/2) - f((q^{-\frac12}z + q^{\frac12}/z)/2)}{((q^{\frac12}z + q^{-\frac12}/z)/2) - ((q^{-\frac12}z + q^{\frac12}/z)/2)}\\ &= \frac{f((q^{\frac12}z + q^{-\frac12}/z)/2) - f((q^{-\frac12}z + q^{\frac12}/z)/2)}{(q^{\frac12} - q^{-\frac12})(z - 1/z)/2}.\end{aligned} \tag{3.4}$$

The action of $\mathcal{D}_q$ on the continuous q-ultraspherical polynomials is [8, Eq. (13.2.23)]

$$\mathcal{D}_q C_n(x;\beta\,|q) = \frac{2(1-\beta)}{(1-q)} q^{\frac{1-n}{2}} C_{n-1}(x;q\beta\,|q).$$

As was established by Askey and Ismail [2] (see also [8, Thm. 13.2.1]), the continuous q-ultraspherical polynomials satisfy the orthogonality relation

$$\begin{aligned}\frac{1}{2\pi}\int_{-1}^{1} C_m(x;\beta\,|\,q)\, C_n(x;\beta\,|\,q)\, \frac{(e^{\pm 2\mathrm{i}\theta};q)_\infty}{(\beta e^{\pm 2\mathrm{i}\theta};q)_\infty} \frac{\mathrm{d}x}{\sqrt{1-x^2}}&\\ = \frac{(\beta, q\beta;q)_\infty}{(q,\beta^2;q)_\infty} \frac{(\beta^2;q)_n}{(q;q)_n} \frac{(1-\beta)}{(1-\beta q^n)}\,\delta_{m,n}&.\end{aligned} \tag{3.5}$$

Askey and Ismail [2, p. 6] noted that the required positive definiteness in (3.5) is equivalent to

$$\frac{(1-\beta^2 q^{n-1})(1-q^n)}{(1-\beta q^{n-1})(1-\beta q^n)} > 0, \qquad \text{for } n = 1, 2, \ldots,$$

from which one determines the following restrictions on the real parameters q and β:

$$\begin{aligned}
&-1 < \beta < q^{-\frac12} && \text{for } 0 < q < 1,\\
&-1 < \beta < -q^{-1} && \text{for } -1 < q < 0,\\
&\beta > -1 && \text{for } q = 0,\\
&\beta = q^\lambda \quad \text{with } \lambda > -\frac12 && \text{for } q \to 1,\\
&\beta = |q|^\lambda \quad \text{with } \lambda > -1 &&\\
\textit{or}\quad &\beta = -|q|^\lambda \quad \text{with } \lambda > 0 && \text{for } q \to -1.
\end{aligned}$$

Ismail [8, Thm. 13.2.2] further observed that the orthogonality relation in (3.5) is equivalent to the following integral evaluation.

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{(\beta t_1 e^{\pm\mathrm{i}\theta}, \beta t_2 e^{\pm\mathrm{i}\theta}, e^{\pm 2\mathrm{i}\theta};q)_\infty}{(t_1 e^{\pm\mathrm{i}\theta}, t_2 e^{\pm\mathrm{i}\theta}, \beta e^{\pm 2\mathrm{i}\theta};q)_\infty} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{(\beta, q\beta;q)_\infty}{(q,\beta^2;q)_\infty}\, {}_2\phi_1\left[\begin{matrix}\beta^2, \beta\\ q\beta\end{matrix}; q, t_1 t_2\right], \tag{3.6}$$

where $|t_1| < 1$ and $|t_2| < 1$. We will make use of (3.6) in Section 5.

What is especially remarkable about the continuous q-ultraspherical polynomials is that they possess a linearization formula whose coefficients completely factorize. This result is due to Rogers [13] (cf. also [8, Thm. 13.3.2]) who obtained the following formula:

$$\begin{aligned}&C_m(x;\beta\,|\,q)\,C_n(x;\beta\,|\,q)\\&=\sum_{k=0}^{\min(m,n)}\bigg(\frac{(q;q)_{m+n-2k}(\beta;q)_{m-k}(\beta;q)_{n-k}(\beta;q)_k(\beta^2;q)_{m+n-k}}{(\beta^2;q)_{m+n-2k}(q;q)_{m-k}(q;q)_{n-k}(q;q)_k(q\beta;q)_{m+n-k}}\\&\qquad\times\frac{(1-\beta q^{m+n-2k})}{(1-\beta)}\,C_{m+n-2k}(x;\beta\,|\,q)\bigg).\end{aligned}\tag{3.7}$$

4. The bilateral q-ultraspherical functions

We now define for two parameters β,γ, base q and variable $x=\cos\theta$ the *bilateral q-ultraspherical functions* by

$$C_n(x;\beta,\gamma\,|\,q):=\sum_{k=-\infty}^{\infty}\frac{(\beta\gamma;q)_k(\beta\gamma;q)_{n-k}}{(q\gamma;q)_k(q\gamma;q)_{n-k}}e^{\mathrm{i}(n-2k)\theta},\tag{4.1}$$

for $|qe^{\pm2\mathrm{i}\theta}/\beta|<1$. (The parameters can be further restricted for a positive measure when needed.) It is clear that for $\gamma=1$, the bilateral q-ultraspherical function $C_n(x;\beta,\gamma\,|\,q)$ reduces to the continuous q-ultraspherical polynomial $C_n(x;\beta\,|\,q)$ in (3.1). Also, it is clear (by shifting the summation index $k\mapsto k+n$) that the symmetry

$$C_n(x;\beta,\gamma\,|\,q)=\left(\frac{\beta}{q}\right)^n C_{-n}(x;\beta,1/\beta\gamma\,|\,q)\tag{4.2}$$

holds, for all $n\in\mathbb{Z}$.

The function $C_n(x;\beta,\gamma\,|\,q)$ can be written as

$$C_n(x;\beta,\gamma\,|\,q)=\frac{(\beta\gamma;q)_n}{(q\gamma;q)_n}e^{\mathrm{i}n\theta}\,{}_2\psi_2\left[\begin{matrix}\beta\gamma,q^{-n}/\gamma\\ q\gamma,q^{1-n}/\beta\gamma\end{matrix};q,qe^{-2\mathrm{i}\theta}/\beta\right]\tag{4.3}$$

(notice that the ${}_2\psi_2$ series is *well-poised*; see [6, p. 39 and p. 138] for this terminology in the setting of basic hypergeometric and bilateral basic hypergeometric series). The function $C_n(x;\beta,\gamma\,|\,q)$ has different series representations, due to Bailey's ${}_2\psi_2$ transformations in (2.5) and (2.6), and the transformation in (2.7). We omit displaying details.

We have the following bilateral generating function for the bilateral q-ultraspherical functions:

Theorem 1. *We have*

$$\sum_{n=-\infty}^{\infty}C_n(x;\beta,\gamma\,|\,q)t^n=\frac{(q,q/\beta;q)_\infty^2}{(q\gamma,q/\beta\gamma;q)_\infty^2}\frac{(\beta\gamma te^{\pm\mathrm{i}\theta},qe^{\pm\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(te^{\pm\mathrm{i}\theta},qe^{\pm\mathrm{i}\theta}/\beta t;q)_\infty},\tag{4.4}$$

where $|q/\beta|<|te^{\pm\mathrm{i}\theta}|<1$.

Proof. We have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} C_n(x;\beta,\gamma\,|\,q)t^n &= \sum_{n,\,k=-\infty}^{\infty} \frac{(\beta\gamma;q)_k}{(q\gamma;q)_k}\frac{(\beta\gamma;q)_{n-k}}{(q\gamma;q)_{n-k}} e^{\mathrm{i}(-k+(n-k))\theta} t^{k+(n-k)}\\
&= \sum_{k=-\infty}^{\infty} \frac{(\beta\gamma;q)_k}{(q\gamma;q)_k} e^{-\mathrm{i}k\theta}t^k \sum_{n=-\infty}^{\infty} \frac{(\beta\gamma;q)_n}{(q\gamma;q)_n} e^{\mathrm{i}n\theta}t^n\\
&= \frac{(q,\beta\gamma te^{-\mathrm{i}\theta},qe^{\mathrm{i}\theta}/\beta\gamma t,q/\beta;q)_\infty}{(q\gamma,te^{-\mathrm{i}\theta},qe^{\mathrm{i}\theta}/\beta t,q/\beta\gamma;q)_\infty}\frac{(q,\beta\gamma te^{\mathrm{i}\theta},qe^{-\mathrm{i}\theta}/\beta\gamma t,q/\beta;q)_\infty}{(q\gamma,te^{\mathrm{i}\theta},qe^{-\mathrm{i}\theta}/\beta t,q/\beta\gamma;q)_\infty}.\end{aligned}$$

by two applications of Ramanujan's ${}_1\psi_1$ summation (2.4). □

The $t\mapsto qt$ case of (4.4) is

$$\begin{aligned}\sum_{n=-\infty}^{\infty} C_n(x;\beta,\gamma\,|\,q)q^nt^n &= \frac{(q,q/\beta;q)_\infty^2}{(q\gamma,q/\beta\gamma;q)_\infty^2}\frac{(\beta\gamma qte^{\pm\mathrm{i}\theta},e^{\pm\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(qte^{\pm\mathrm{i}\theta},e^{\pm\mathrm{i}\theta}/\beta t;q)_\infty}\\
&= \frac{(1-te^{\mathrm{i}\theta})(1-te^{-\mathrm{i}\theta})(1-e^{\mathrm{i}\theta}/\beta\gamma t)(1-e^{-\mathrm{i}\theta}/\beta\gamma t)}{(1-\beta\gamma te^{\mathrm{i}\theta})(1-\beta\gamma te^{-\mathrm{i}\theta})(1-e^{\mathrm{i}\theta}/\beta t)(1-e^{-\mathrm{i}\theta}/\beta t)}\\
&\quad\times \frac{(q,q/\beta;q)_\infty^2}{(q\gamma,q/\beta\gamma;q)_\infty^2}\frac{(\beta\gamma te^{\pm\mathrm{i}\theta},qe^{\pm\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(te^{\pm\mathrm{i}\theta},qe^{\pm\mathrm{i}\theta}/\beta t;q)_\infty}.\end{aligned}$$

Now since

$$\frac{(1-e^{\mathrm{i}\theta}/\beta\gamma t)(1-e^{-\mathrm{i}\theta}/\beta\gamma t)}{(1-\beta\gamma te^{\mathrm{i}\theta})(1-\beta\gamma te^{-\mathrm{i}\theta})} = \frac{1}{\beta^2\gamma^2t^2},$$

the right-hand side simplifies and (recalling $x=(e^{\mathrm{i}\theta}+e^{-\mathrm{i}\theta})/2$) we deduce

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} C_n(x;\beta,\gamma\,|\,q)q^nt^n\\
&= \frac{(1-te^{\mathrm{i}\theta})(1-te^{-\mathrm{i}\theta})}{(1-e^{\mathrm{i}\theta}/\beta t)(1-e^{-\mathrm{i}\theta}/\beta t)\beta^2\gamma^2t^2}\,\frac{(q,q/\beta;q)_\infty^2}{(q\gamma,q/\beta\gamma;q)_\infty^2}\frac{(\beta\gamma te^{\pm\mathrm{i}\theta},qe^{\pm\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(te^{\pm\mathrm{i}\theta},qe^{\pm\mathrm{i}\theta}/\beta t;q)_\infty}\\
&= \frac{1-2tx+t^2}{\gamma^2-2\beta\gamma^2xt+\beta^2\gamma^2t^2}\,\frac{(q,q/\beta;q)_\infty^2}{(q\gamma,q/\beta\gamma;q)_\infty^2}\frac{(\beta\gamma te^{\pm\mathrm{i}\theta},qe^{\pm\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(te^{\pm\mathrm{i}\theta},qe^{\pm\mathrm{i}\theta}/\beta t;q)_\infty}\\
&= \frac{1-2tx+t^2}{\gamma^2-2\beta\gamma^2xt+\beta^2\gamma^2t^2}\sum_{n=-\infty}^{\infty} C_n(x;\beta,\gamma\,|\,q)t^n. \qquad (4.5)\end{aligned}$$

Multiplying the left- and right-hand sides of (4.5) by $(\gamma^2-2\beta\gamma^2xt+\beta^2\gamma^2t^2)$ and comparing coefficients of t^n, we arrive at the following result:

Theorem 2. *The bilateral q-ultraspherical functions satisfy for all $n\in\mathbb{Z}$ the following recurrence relation:*

$$\begin{aligned}&2x(1-\beta\gamma^2q^n)\,C_n(x;\beta,\gamma\,|\,q)\\
&= (1-\gamma^2q^{n+1})\,C_{n+1}(x;\beta,\gamma\,|\,q) + (1-\beta^2\gamma^2q^{n-1})\,C_{n-1}(x;\beta,\gamma\,|\,q). \qquad (4.6)\end{aligned}$$

Moreover, they are uniquely determined by (4.6) *and the two initial conditions*

$$C_0(x;\beta,\gamma\,|\,q) = {}_2\psi_2\left[\begin{matrix}\beta\gamma, 1/\gamma\\ q\gamma, q/\beta\gamma\end{matrix};q, qe^{-2\mathrm{i}\theta}/\beta\right], \tag{4.7a}$$

$$C_{-1}(x;\beta,\gamma\,|\,q) = \frac{1-\gamma}{1-\beta\gamma/q}e^{-\mathrm{i}\theta}\,{}_2\psi_2\left[\begin{matrix}\beta\gamma, q/\gamma\\ q\gamma, q^2/\beta\gamma\end{matrix};q, qe^{-2\mathrm{i}\theta}/\beta\right]. \tag{4.7b}$$

The bilateral q-ultraspherical functions $C_0(x)$ and $C_{-1}(x)$ in general do not evaluate in closed form, however from the $b=\beta\gamma$, $c=1/\gamma$, $d=q^{\frac{1}{2}}$ case of

$$ {}_3\psi_3\left[\begin{matrix}b,c,d\\ q/b,q/c,q/d\end{matrix};q,\frac{q}{bcd}\right] = \frac{(q,q/bc,q/bd,q/cd;q)_\infty}{(q/b,q/c,q/d,q/bcd;q)_\infty}, \tag{4.8a}$$

and the $b=\beta\gamma$, $c=q/\gamma$, $d=q$ case of

$$ {}_3\psi_3\left[\begin{matrix}b,c,d\\ q^2/b,q^2/c,q^2/d\end{matrix};q,\frac{q^2}{bcd}\right] = \frac{(q,q^2/bc,q^2/bd,q^2/cd;q)_\infty}{(q^2/b,q^2/c,q^2/d,q^2/bcd;q)_\infty}, \tag{4.8b}$$

which are formulas by Bailey for specific well-poised ${}_3\psi_3$ series (cf. [6, Ex. 5.18 (i) and (ii)]), we can deduce the following evaluations for special x:

$$C_0\big((q^{\frac{1}{4}}+q^{-\frac{1}{4}})/2;\beta,\gamma\,|\,q\big) = \frac{(q,q/\beta,q^{\frac{1}{2}}/\beta\gamma,q^{\frac{1}{2}}\gamma;q)_\infty}{(q/\beta\gamma,q\gamma,q^{\frac{1}{2}},q^{\frac{1}{2}}/\beta;q)_\infty}, \tag{4.9a}$$

$$C_{-1}\big((q^{\frac{1}{2}}+q^{-\frac{1}{2}})/2;\beta,\gamma\,|\,q\big) = \frac{q^{\frac{1}{2}}(1-\gamma)^2}{\gamma(1-\beta)}. \tag{4.9b}$$

The $C_{2n+1}(x;\beta,\gamma\,|\,q)$ are odd functions, the $C_{2n}(x;\beta,\gamma\,|\,q)$ are even.
Their constant terms are

$$C_{2n+1}(0;\beta,\gamma\,|\,q)=0,$$

and

$$C_{2n}(0;\beta,\gamma\,|\,q) = (-1)^n\frac{(\beta^2\gamma^2;q^2)_n}{(q^2\gamma^2;q^2)_n}\,\frac{(q,q/\beta,-q\gamma,-q/\beta\gamma;q)_\infty}{(-q,-q/\beta,q\gamma,q/\beta\gamma;q)_\infty},$$

for all $n\in\mathbb{Z}$, by virtue of an instance of the bilateral q-Kummer summation [6, Eq. (II.30)], namely

$$ {}_2\psi_2\left[\begin{matrix}b,c\\ aq/b,aq/c\end{matrix};q,-\frac{aq}{bc}\right] = \frac{(aq/bc;q)_\infty(q^2,aq,q/a,aq^2/b^2,aq^2/c^2;q^2)_\infty}{(aq/b,aq/c,q/b,q/c,-aq/bc;q)_\infty}, \tag{4.10}$$

valid for $|aq/bc|<1$.

The action of $\mathcal{D}_q$ on the bilateral q-ultraspherical functions is

$$\mathcal{D}_qC_n(x;\beta,\gamma\,|\,q) = \frac{2(1-\beta\gamma)^2}{(1-q)(1-\beta)\gamma}q^{\frac{1-n}{2}}C_{n-1}(x;q\beta,\gamma\,|\,q). \tag{4.11}$$

This can be easily shown with the help of the explicit generating function in Theorem 1. We have

$$
\begin{aligned}
\mathcal{D}_q \sum_{n=-\infty}^{\infty} C_n(x;\beta,\gamma\,|\,q)t^n &= \frac{(q,q/\beta;q)_\infty^2}{(q\gamma,q/\beta\gamma;q)_\infty^2}\mathcal{D}_q \frac{(\beta\gamma t e^{\pm\mathrm{i}\theta}, qe^{\pm\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(te^{\pm\mathrm{i}\theta}, qe^{\pm\mathrm{i}\theta}/\beta t;q)_\infty} \\
&= \frac{2\,(q,q/\beta;q)_\infty^2}{(q^{\frac12}-q^{-\frac12})(e^{\mathrm{i}\theta}-e^{-\mathrm{i}\theta})(q\gamma,q/\beta\gamma;q)_\infty^2} \\
&\quad\times\Bigg(\frac{(\beta\gamma t q^{\frac12}e^{\mathrm{i}\theta},\beta\gamma t q^{-\frac12}e^{-\mathrm{i}\theta},q^{\frac32}e^{\mathrm{i}\theta}/\beta\gamma t,q^{\frac12}e^{-\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(tq^{\frac12}e^{\mathrm{i}\theta},tq^{-\frac12}e^{-\mathrm{i}\theta},q^{\frac32}e^{\mathrm{i}\theta}/\beta t,q^{\frac12}e^{-\mathrm{i}\theta}/\beta t;q)_\infty} \\
&\qquad - \frac{(\beta\gamma t q^{-\frac12}e^{\mathrm{i}\theta},\beta\gamma t q^{\frac12}e^{-\mathrm{i}\theta},q^{\frac12}e^{\mathrm{i}\theta}/\beta\gamma t,q^{\frac32}e^{-\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(tq^{-\frac12}e^{\mathrm{i}\theta},tq^{\frac12}e^{-\mathrm{i}\theta},q^{\frac12}e^{\mathrm{i}\theta}/\beta t,q^{\frac32}e^{-\mathrm{i}\theta}/\beta t;q)_\infty}\Bigg) \\
&= \frac{2\,(q,q/\beta;q)_\infty^2}{(q^{\frac12}-q^{-\frac12})(e^{\mathrm{i}\theta}-e^{-\mathrm{i}\theta})(q\gamma,q/\beta\gamma;q)_\infty^2}\,\frac{(\beta\gamma t q^{\frac12}e^{\pm\mathrm{i}\theta},q^{\frac32}e^{\pm\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(tq^{-\frac12}e^{\pm\mathrm{i}\theta},q^{\frac12}e^{\pm\mathrm{i}\theta}/\beta t;q)_\infty} \\
&\quad\times\big((1-\beta\gamma t q^{-\frac12}e^{-\mathrm{i}\theta})(1-q^{\frac12}e^{-\mathrm{i}\theta}/\beta\gamma t)(1-tq^{-\frac12}e^{\mathrm{i}\theta})(1-q^{\frac12}e^{\mathrm{i}\theta}/\beta t) \\
&\qquad -(1-\beta\gamma t q^{-\frac12}e^{\mathrm{i}\theta})(1-q^{\frac12}e^{\mathrm{i}\theta}/\beta\gamma t)(1-tq^{-\frac12}e^{-\mathrm{i}\theta})(1-q^{\frac12}e^{-\mathrm{i}\theta}/\beta t)\big) \\
&= \frac{2\,(q,q/\beta;q)_\infty^2}{(q^{\frac12}-q^{-\frac12})(e^{\mathrm{i}\theta}-e^{-\mathrm{i}\theta})(q\gamma,q/\beta\gamma;q)_\infty^2}\,\frac{(\beta\gamma t q^{\frac12}e^{\pm\mathrm{i}\theta},q^{\frac32}e^{\pm\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(tq^{-\frac12}e^{\pm\mathrm{i}\theta},q^{\frac12}e^{\pm\mathrm{i}\theta}/\beta t;q)_\infty} \\
&\quad\times \gamma t q^{-\frac12}e^{-\mathrm{i}\theta}(1-\beta)(1-e^{2\mathrm{i}\theta})(1-q^{\frac12}e^{\mathrm{i}\theta}/\beta\gamma t)(1-q^{\frac12}e^{-\mathrm{i}\theta}/\beta\gamma t) \\
&= \frac{2\gamma t(1-\beta)(q,q/\beta;q)_\infty^2}{(1-q)(q\gamma,q/\beta\gamma;q)_\infty^2}\,\frac{(\beta\gamma t q^{\frac12}e^{\pm\mathrm{i}\theta},q^{\frac12}e^{\pm\mathrm{i}\theta}/\beta\gamma t;q)_\infty}{(tq^{-\frac12}e^{\pm\mathrm{i}\theta},q^{\frac12}e^{\pm\mathrm{i}\theta}/\beta t;q)_\infty} \\
&= \frac{2t(1-\beta\gamma)^2}{(1-q)(1-\beta)\gamma}\sum_{n=-\infty}^{\infty} C_n(x;q\beta,\gamma\,|\,q)\big(tq^{-\frac12}\big)^n \\
&= \frac{2(1-\beta\gamma)^2}{(1-q)(1-\beta)\gamma}\sum_{n=-\infty}^{\infty} q^{\frac{1-n}{2}}C_{n-1}(x;q\beta,\gamma\,|\,q)t^n,
\end{aligned}
$$

from which the claimed result follows from taking coefficients of t^n.

It may seem a bit unnatural to consider bilateral sequences of functions $(C_n(x))_{n\in\mathbb{Z}}$ where not even $C_0(x)$ (which in the ordinary, polynomial, case is just a constant) simplifies, as in (4.7a). Nevertheless, we show that $C_0(x;\beta,\gamma\,|\,q)$ satisfies the following integral evaluation in the special case $(\beta,\gamma)\mapsto(q^2\beta^2,1/\beta)$:

$$
\frac{1}{2\pi}\int_{-1}^{1} C_n(x;q^2\beta^2,1/\beta\,|\,q)\,\frac{(e^{\pm2\mathrm{i}\theta};q)_\infty}{(\beta e^{\pm2\mathrm{i}\theta};q)_\infty}\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{(q;q)_\infty^2(\beta,q/\beta^2;q)_\infty}{(q/\beta;q)_\infty^3(\beta^2;q)_\infty}\,\delta_{n,0}. \tag{4.12}
$$

One can prove (4.12) by employing the integral evaluation in (3.6). Indeed, we have

$$
\frac{1}{2\pi}\int_{-1}^{1}\sum_{n=-\infty}^{\infty} C_n(x;q^2\beta^2,1/\beta\,|\,q)t^n\,\frac{(e^{\pm2\mathrm{i}\theta};q)_\infty}{(\beta e^{\pm2\mathrm{i}\theta};q)_\infty}\frac{\mathrm{d}x}{\sqrt{1-x^2}}
$$

$$= \frac{1}{2\pi}\int_{-1}^{1} \frac{(q, q/\beta^2; q)_\infty^2}{(q/\beta; q)_\infty^4} \frac{(\beta t e^{\pm \mathrm{i}\theta}, q e^{\pm \mathrm{i}\theta}/\beta t, e^{\pm 2\mathrm{i}\theta}; q)_\infty}{(t e^{\pm \mathrm{i}\theta}, q e^{\pm \mathrm{i}\theta}/\beta^2 t, \beta e^{\pm 2\mathrm{i}\theta}; q)_\infty} \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$

$$= \frac{(q, q/\beta^2; q)_\infty^2}{(q/\beta; q)_\infty^4} \frac{(\beta, q\beta; q)_\infty}{(q, \beta^2; q)_\infty} {}_2\phi_1\left[\begin{matrix}\beta^2, \beta\\ q\beta\end{matrix}; q, q/\beta^2\right]$$

$$= \frac{(q, q/\beta^2; q)_\infty^2}{(q/\beta; q)_\infty^4} \frac{(\beta, q\beta; q)_\infty}{(q, \beta^2; q)_\infty} \frac{(q/\beta, q; q)_\infty}{(q\beta, q/\beta^2; q)_\infty}$$

$$= \frac{(q;q)_\infty^2 (\beta, q/\beta^2; q)_\infty}{(q/\beta; q)_\infty^3 (\beta^2; q)_\infty}. \tag{4.13}$$

In the first equality we used (4.4), in the second equality (3.6), while in the third we used the q-Gauß summation (cf. [6, Equation (II.8)]),

$${}_2\phi_1\left[\begin{matrix}a, b\\ c\end{matrix}; q, \frac{c}{ab}\right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \qquad |c/ab| < 1. \tag{4.14}$$

Comparison of coefficients of t^n on the left- and right-hand sides of (4.13) establishes (4.12).

5. Shifted orthogonality of the bilateral q-ultraspherical functions

For the bilateral q-ultraspherical function, we were unfortunately not able to establish their full orthogonality with respect to any positive measure. However, we came to something quite close to orthogonality which we call *shifted orthogonality*. We remark that in wavelet theory there is a similar concept of shifted orthogonality, for so-called *wavelet matrices* (cf. [5, 9, 15]). This concept involves the *full* sequence $(F_k(x))_{k\in\mathbb{Z}}$ of functions to be tested for shifted orthogonality.

We have the following result:

Theorem 3 (Shifted orthogonality of the bilateral q-ultraspherical functions)**.** *Let the positive definite weight function be given as*

$$w(x \,|\, \beta) = \frac{(e^{2\mathrm{i}\theta}, e^{-2\mathrm{i}\theta}; q)_\infty}{(\beta e^{2\mathrm{i}\theta}, \beta e^{-2\mathrm{i}\theta}; q)_\infty} \frac{1}{\sqrt{1-x^2}}.$$

Then the family of bilateral q-ultraspherical functions $(C_k(x;\beta,\gamma \,|\, q))_{k\in\mathbb{Z}}$ satisfy the following shifted orthogonality relation:

$$\frac{1}{2\pi}\int_{-1}^{1} \sum_{k=-\infty}^{\infty} C_{m+k}(x;\beta,\gamma \,|\, q)\, C_{n+k}(x;\beta,\gamma \,|\, q) \left(\frac{q}{\beta^2\gamma}\right)^k w(x \,|\, \beta)\,\mathrm{d}x$$

$$= \frac{(q;q)_\infty^3 (q/\beta; q)_\infty^4 (\beta, q\beta; q)_\infty}{(q\gamma, q/\beta\gamma; q)_\infty^4 (\beta^2; q)_\infty} {}_2\phi_1\left[\begin{matrix}\beta^2, \beta\\ q\beta\end{matrix}; q, \frac{q}{\beta^2\gamma}\right] \left(\frac{\beta^2\gamma}{q}\right)^n \delta_{m,n}, \tag{5.1}$$

where $|q/\beta^2\gamma| < |t| < |1/\beta\gamma|$.

Proof. Consider the double bilateral generating function

$$\sum_{m,k=-\infty}^{\infty} C_m(x;\beta,\gamma \,|\, q)\, C_k(x;\beta,\gamma \,|\, q)\; t_1^m\, t_2^k$$

$$= \frac{(q,q/\beta;q)_\infty^4}{(q\gamma,q/\beta\gamma;q)_\infty^4}\frac{(\beta\gamma t_1e^{\pm i\theta},qe^{\pm i\theta}/\beta\gamma t_1,\beta\gamma t_2e^{\pm i\theta},qe^{\pm i\theta}/\beta\gamma t_2;q)_\infty}{(t_1e^{\pm i\theta},qe^{\pm i\theta}/\beta t_1,t_2e^{\pm i\theta},qe^{\pm i\theta}/\beta t_2;q)_\infty},$$

and take $(t_1,t_2)=(t,q/\beta^2\gamma t)$. This gives

$$\sum_{m,k=-\infty}^{\infty} C_m(x;\beta,\gamma\,|\,q)\,C_k(x;\beta,\gamma\,|\,q)\;t^{m-k}\left(\frac{q}{\beta^2\gamma}\right)^k$$
$$= \frac{(q,q/\beta;q)_\infty^4}{(q\gamma,q/\beta\gamma;q)_\infty^4}\frac{(qe^{\pm i\theta}/\beta\gamma t,\beta tqe^{\pm i\theta};q)_\infty}{(te^{\pm i\theta},qe^{\pm i\theta}/\beta^2\gamma t;q)_\infty}.$$

Integration over x from -1 to 1 with respect to the q-ultraspherical weight function divided by 2π gives, by the integral evaluation in (3.6),

$$\frac{(q,q/\beta;q)_\infty^4}{(q\gamma,q/\beta\gamma;q)_\infty^4}\frac{(\beta,q\beta;q)_\infty}{(q,\beta^2;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}\beta^2,\beta\\ q\beta\end{matrix};q,\frac{q}{\beta^2\gamma}\right]$$

which is independent of t. Now shift indices (m,k) by $(m+k,n+k)$ and compare coefficients of t. $\square$

References

[1] G.E. Andrews, R.A. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, Cambridge, 1999.

[2] R.A. Askey and M.E.H. Ismail, "A generalization of the ultraspherical polynomials", in *Studies in Pure Mathematics*, Birkhäuser, Basel, 1983; pp. 55–78.

[3] R.A. Askey and J. Wilson, "Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials", Mem. Amer. Math. Soc. **54** (319), 1985.

[4] W. N. Bailey, "On the basic bilateral hypergeometric series ${}_2\psi_2$", Quart. J. Math. Oxford (2) **1** (1950), 194–198.

[5] L. Ephremidze and E. Lagvilava, "On compact wavelet matrices of rank m and of order and degree N", J. Fourier Anal. Appl. **20** (2014), 401–420.

[6] G. Gasper and M. Rahman, *Basic hypergeometric series*, second edition, Encyclopedia of Mathematics and Its Applications **96**, Cambridge University Press, Cambridge, 2004.

[7] M.E.H. Ismail, "A simple proof of Ramanujan's ${}_1\psi_1$ sum", Proc. Amer. Math. Soc. **63** (1977), 185–186.

[8] M.E.H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Encyclopedia of Mathematics and its Applications **98**, Cambridge University Press, Cambridge, 2005.

[9] J. Kautsky and R. Turcajová, "Pollen product factorization and construction of higher multiplicity wavelets", Linear Algebra Appl. **222** (1995), 241–260.

[10] R. Koekoek, P.A. Lesky, and R.F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer Monographs in Math., Springer, Berlin/New York, 2010.

[11] R. Koekoek and R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Reports of the Faculty of Technical Mathematics and Informatics, vol. **98**–17, Delft University of Technology, 1998.

[12] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, second edition, Oxford, 1995.

[13] L.J. Rogers, "Third memoir on the expansion of certain infinite products", Proc. London Math. Soc. **26** (1894), 15–32.

[14] G. Szegő, *Orthogonal polynomials*, 4th edition, Coll. Publ. **23**, Amer. Math. Soc., Providence, RI, 1975.

[15] R. Turcajová and J. Kautsky, "Shift products and factorizations of wavelet matrices", Numer. Algorithms **8**(1) (1994), 27–45.

[16] S.O. Warnaar, "Ramanujan's ${}_1\psi_1$ summation", in K. Alladi (ed.), *Srinivasa Ramanujan: going strong at 125, part II*, Notices Amer. Math. Soc. **60**, no. 1 (2013), 10–23.

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

Email address: michael.schlosser@univie.ac.at

URL: http://www.mat.univie.ac.at/~schlosse