ENUMERATION OF STANDARD BARELY SET-VALUED TABLEAUX OF SHIFTED SHAPES

JANG SOO KIM, MICHAEL J. SCHLOSSER, AND MEESUE YOO

Abstract. A standard barely set-valued tableau of shape λ is a filling of the Young diagram λ with integers $1, 2, \dots, |\lambda| + 1$ such that the integers are increasing in each row and column, and every cell contains one integer except one cell that contains two integers. Counting standard barely set-valued tableaux is closely related to the coincidental down-degree expectations (CDE) of lower intervals in Young's lattice. Using q-integral techniques we give a formula for the number of standard barely set-valued tableaux of arbitrary shifted shape. We show how it can be used to recover two formulas, originally conjectured by Reiner, Tenner and Yong, and proved by Hopkins, for numbers of standard barely set-valued tableaux of particular shifted-balanced shapes. We also prove a conjecture of Reiner, Tenner and Yong on the CDE property of the shifted shape $(n, n-2, n-4, \dots, n-2k+2)$. Finally, in the appendix we raise a conjecture on an $\mathsf{a};q$-analogue of the down-degree expectation with respect to the uniform distribution for a specific class of lower intervals in Young's lattice.

1. Introduction

Recently, Reiner, Tenner and Yong [17] introduced a property on posets, called the coincidental down-degree expectations (CDE). They showed that many interesting posets have this property. For example, disjoint unions of chains, Cartesian products of chains, weak Bruhat order on a finite Coxeter group, Tamari lattices on polygon triangulations, and connected minuscule posets have the CDE property. Another important family of posets included in their results is a family of lower intervals of Young's lattice. They also considered lower intervals of the shifted Young's lattice and proposed two conjectures on the CDE property of a certain family of lower intervals of the shifted Young's lattice. One of the two conjectures was proved by Hopkins [10]. The main goal of this paper is to prove the remaining conjecture.

If P is an interval of (shifted or usual) Young's lattice, the CDE property of P is closely related to standard barely set-valued tableaux, which are the main object of interest in this paper. Reiner, Tenner and Yong [17] found a formula for the number of barely set-valued tableaux of any partition shape using an "uncrowding algorithm", which is a special case of a map introduced by Buch [4], called "a column bumping algorithm".

In this paper we give an analogous formula for the number of standard barely set-valued tableaux of any shifted shape. In doing so, we use a modification of their uncrowding algorithm and also the q-integral techniques developed in [14].

We now give precise definitions needed to state our results. Let P be a finite poset. The *down-degree* $\operatorname{ddeg}(x)$ of an element $x \in P$ is the number of elements in P covered by x. Define X (resp. Y) to be the random variable computing the down-degree of $x \in P$ with respect to the uniform distribution (resp. the probability distribution proportional to the number of maximal chains containing x). We say that the poset P has the *coincidental down-degree expectations (CDE) property* if $\mathbb{E}(X) = \mathbb{E}(Y)$. For example if P is the poset in Figure 1, then $\operatorname{ddeg}(a) = 0$, $\operatorname{ddeg}(b) = 1$, $\operatorname{ddeg}(c) = 1$, $\operatorname{ddeg}(d) = 0$, $\operatorname{ddeg}(e) = 2$, $\operatorname{ddeg}(f) = 2$, and the number of maximal

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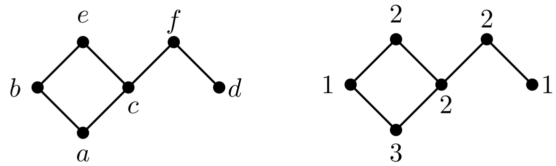


FIGURE 1. The poset $P = \{a, b, c, d, e, f\}$ on the left satisfies the CDE property with $\mathbb{E}(X) = \mathbb{E}(Y) = 1$. The number of maximal chains through each element is shown on the right.

FIGURE 2. The Young diagram (left) and the shifted Young diagram (right) of $\lambda = (4, 3, 1)$.

chains through a, b, c, d, e, f are $3, 1, 2, 1, 2, 2$, respectively. Thus

$$\mathbb{E}(X) = \frac{1}{6}(0 + 1 + 1 + 0 + 2 + 2) = 1,$$

$$\mathbb{E}(Y) = \frac{1}{3 + 1 + 2 + 1 + 2 + 2}(0 \cdot 3 + 1 \cdot 1 + 1 \cdot 2 + 0 \cdot 1 + 2 \cdot 2 + 2 \cdot 2) = 1,$$

and therefore this poset satisfies the CDE property.

A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers. Each integer λ_i is called a *part* of λ. The *size* $|\lambda|$ of λ is the sum of its parts and the *length* $\ell(\lambda)$ of λ is the number of its parts. If all parts in λ are distinct, we say that λ is *strict*. The set of all partitions is denoted by Par and the set of all partitions with at most n parts by Par_n. We also denote by Par^* the set of all strict partitions and by Par^*_n the set of all strict partitions with at most n parts.

If a partition λ has $\ell(\lambda) = n$ parts, then we will use the convention that $\lambda_i = 0$ for all $i > n$. For two partitions λ and μ, define

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots, \lambda_N + \mu_N),$$

where $N = \max(\ell(\lambda), \ell(\mu))$.

The *Young diagram* of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is the top-left justified array of unit squares (or cells) in which the ith row has λ_i squares. If λ is strict, the *shifted Young diagram* of λ is the array obtained from the Young diagram of λ by shifting the ith row to the right by $i - 1$ units for each $1 \le i \le \ell(\lambda)$. See Figure 2. We will identify a partition with its Young diagram (or its shifted Young diagram if it is strict).

Young's lattice is the poset in which the elements are all partitions and the relations are given by the inclusion of their Young diagrams, i.e., $\mu \le \lambda$ if and only if the Young diagram of μ is contained in that of λ. Similarly, the *shifted Young's lattice* is the poset in which the elements are all strict partitions and the relations are given by the inclusion of their shifted Young diagrams. In this paper we are interested in whether a given lower interval $[\emptyset, \lambda]$ in Young's lattice or in the shifted Young's lattice has the CDE property.

The CDE property of a poset was first observed by Chan, López Martín, Pflueger and Teixidor i Bigas [6, Remark 2.17]. They showed that the lower interval $[\emptyset, \lambda]$ in Young's lattice for a rectangular shape $\lambda = (a^b) = (a, a, \dots, a)$ with b rows of length a has the CDE property and the expectations are given by $\mathbb{E}(X) = \mathbb{E}(Y) = \frac{ab}{a+b}$. This result played an important role when they reproved a formula for the genera of Brill–Noether curves due to Eisenbud–Harris [7] and

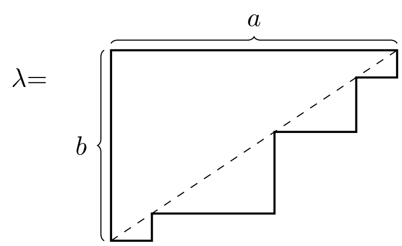


FIGURE 3. A typical shape of a balanced Young diagram of slope b/a.

Pirola [16]. Chan, Haddadan, Hopkins and Moci [5] generalized the CDE property of a rectangular shape to a much broader class of partitions, which we now describe.

Let μ be a Young diagram. An *addable corner* of μ is a cell $x \notin \mu$ such that $\mu \cup \{x\}$ is a Young diagram, and a *removable corner* is a cell in μ such that $\mu \setminus \{x\}$ is still a Young diagram. We say that μ is *balanced* if the upper left-hand corner cell of every addable corner of μ lies on the line connecting the top right corner of the last cell in the first row of μ and the bottom left corner of the last cell in the first column of μ. If in addition the line has slope m, then μ is *balanced of slope* m. See Figure 3 for an example of a balanced Young diagram.

Theorem 1.1. *[5, Corollary 3.8] Let λ be a balanced Young diagram of slope m. Then $[\emptyset, \lambda]$ has the CDE property and*

$$\mathbb{E}(X) = \mathbb{E}(Y) = \frac{\lambda_1}{1 + m^{-1}}.$$

We note that Chan, Haddadan, Hopkins and Moci [5] in fact proved the CDE property of more general balanced skew shapes. Related further results were achieved by Hopkins [11] who in particular proved a conjecture by Reiner, Tenner and Yong [17, Conjecture 1.2] on the CDE property for vexillary words of a particular shape.

Hopkins [10, Remark 3.7] suggested the following interesting conjecture, which is a converse of Theorem 1.1, and checked this conjecture for all partitions of size at most 30.

Conjecture 1.2. Let λ be a Young diagram. Then $[\emptyset, \lambda]$ has the CDE property if and only if λ is balanced (of any slope).

Reiner, Tenner and Yong [17] also considered the CDE property of a lower interval of the shifted Young's lattice. In what follows we describe their conjectures on certain shifted Young diagrams.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a strict partition with n parts. Define μ to be the Young diagram given by

$$\mu = \begin{cases} \lambda - \delta_{n+1} = (\lambda_1 - n, \lambda_2 - n + 1, \ldots, \lambda_n - 1), & \text{if } \lambda_n = 1, \\ \lambda - \delta_n = (\lambda_1 - n + 1, \lambda_2 - n + 2, \ldots, \lambda_n), & \text{if } \lambda_n > 1, \end{cases}$$

where $\delta_n = (n-1, n-2, \ldots, 1, 0)$. Note that if $\lambda_n = 1$, then $\ell(\mu) < n$. We say that the shifted Young diagram of λ is *balanced* if μ is a balanced Young diagram of slope 1. See Figure 4 for examples of balanced shifted Young diagrams.

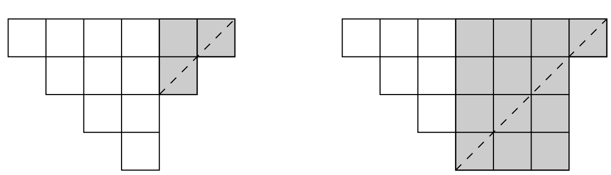


FIGURE 4. Examples of balanced shifted Young diagrams. On the left $\mu = \lambda - \delta_5 = (2, 1)$ and on right $\mu = \lambda - \delta_4 = (4, 3, 3, 3)$.

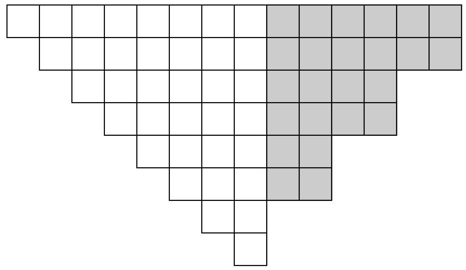


FIGURE 5. The shifted Young diagram of $\lambda = \delta_9 + \delta_4(2^2)$.

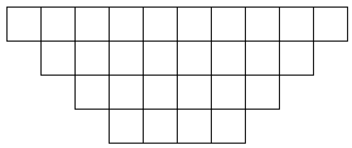


FIGURE 6. A trapezoidal shifted Young diagram with $n = 10$, $k = 4$.

Reiner, Tenner and Yong conjectured the following CDE property for a special class of strict partitions. Here, $\delta_e(a^a)$ is the Young diagram obtained from that of δ_e by replacing each cell by the $a \times a$ square (a^a), see Figure 5.

Theorem 1.3. *[17, Theorem 2.23] Let λ be the shifted shape $\delta_d + \delta_e(a^a)$ with $a, d, e \geq 1$ and $d > a(e-1)+1$. Then the interval $[\emptyset, \lambda]$ has the CDE property and*

$$\mathbb{E}(X) = \mathbb{E}(Y) = \frac{\lambda_1 + 1}{4}. \tag{1.1}$$

Hopkins proved the following generalization of Theorem 1.3.

Theorem 1.4. *[10, Theorem 4.2] Let λ be a balanced shifted Young diagram. Then the interval $[\emptyset, \lambda]$ has the CDE property and*

$$\mathbb{E}(X) = \mathbb{E}(Y) = \frac{\lambda_1 + 1}{4}.$$

A shifted Young diagram is *trapezoidal* if it is of the form $(n, n-2, \dots, n-2k+2)$. See Figure 6 for an example. The main goal of this paper is to prove the following theorem conjectured by Reiner, Tenner and Yong.

Theorem 1.5. *[17, Conjecture 2.24] Let λ be a trapezoidal shifted Young diagram. Then the interval $[\emptyset, \lambda]$ has the CDE property and*

$$\mathbb{E}(X) = \mathbb{E}(Y) = \frac{|\lambda|}{\lambda_1 + 1}.$$

Interestingly, as before, it seems that the shifted Young diagrams described in Theorems 1.4 and 1.5 are the only ones having the CDE property. Hopkins [10, Remark 4.9] suggested the following conjecture and confirmed it for all shifted Young diagrams of size at most 18. Using our method of computing $\mathbb{E}(X)$ and $\mathbb{E}(Y)$, we have checked the conjecture for all shifted Young diagrams of size up to 50.

Conjecture 1.6. Let λ be a shifted Young diagram. Then the interval $[\emptyset, \lambda]$ has the CDE property if and only if λ is balanced or trapezoidal.

There are stronger properties on posets, called mCDE and tCDE. The mCDE property concerns multichains and the tCDE property concerns toggles on posets introduced by Striker and Williams [24]. If the poset is graded, then the mCDE property implies the CDE property. If the poset is

1	2	5	8
3	6	7	
4			

1	2	3	6
4	5 7	9	
8			

FIGURE 7. A standard Young tableau of shape $(4, 3, 1)$ (left) and a standard barely set-valued tableau of shape $(4, 3, 1)$ (right).

distributive, then the tCDE property implies both the mCDE and CDE properties. See [10] for more details. Hopkins [10] showed Theorem 1.4 by showing that the poset in the theorem has the tCDE property. However, his proof cannot apply to Theorem 1.5 since the poset in this theorem does not have the tCDE property, see [10, Section 4.2].

Recall that our main goal is to prove Theorem 1.5. Our approach is to compute $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ separately and show that they are equal. If λ is a (shifted) Young diagram, computing $\mathbb{E}(Y)$ for the lower interval $[\emptyset, \lambda]$ is closely related to standard barely set-valued tableaux of shape λ, which are the main object of focus in this paper.

Let λ be a partition. A *standard Young tableau* of shape λ is a filling of λ with integers $1, 2, \dots, |\lambda|$ in such a way that the integers are increasing in each row and each column. See Figure 7 for an example. The number of standard Young tableaux of shape λ is denoted by f^λ. The famous hook-length formula [8] states that if λ is a partition of length n, then

$$f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h_\lambda(u)} = \frac{|\lambda|!}{\prod_{i=1}^n (\lambda_i + n - i)!} \prod_{1\le i<j\le n} (\lambda_i - \lambda_j - i + j), \tag{1.2}$$

where for each cell $u = (i, j) \in \lambda$, the *hook length* $h_\lambda(u)$ is defined to be $\lambda_i + \lambda'_j - i - j + 1$ and λ'_j is the number of integers k such that $\lambda_k \ge j$.

A *standard barely set-valued tableau* of shape λ is a filling of the Young diagram λ with integers $1, 2, \dots, |\lambda| + 1$ such that the integers are increasing in each row and column, and every cell contains one integer except one cell that contains two integers. See Figure 7 for an example. Let $f^\lambda(+1)$ denote the number of standard barely set-valued tableaux of shape λ. The number $f^\lambda(+1)$ is also known to be equal to a certain coefficient of the Grothendieck polynomial due to Buch [4, Theorem 3.1].

Reiner, Tenner and Yong [17, Corollary 3.7] showed that if the poset is the lower interval $[\emptyset, \lambda]$ for a partition λ, then

$$\mathbb{E}(Y) = \frac{f^\lambda(+1)}{(|\lambda| + 1)f^\lambda}. \tag{1.3}$$

Since we already have a formula for f^λ, in order to evaluate $\mathbb{E}(Y)$ it suffices to find $f^\lambda(+1)$. Using what they call an uncrowding algorithm, Reiner, Tenner and Yong [17, Corollary 3.11 and Remark 3.13] showed that

$$f^\lambda(+1) = \sum_{k:\lambda_k<\lambda_{k-1}} \lambda_k f^{\lambda\cup\{(k,\lambda_k+1)\}}, \tag{1.4}$$

where we set $\lambda_0 = \infty$. They used (1.4) to evaluate $\mathbb{E}(Y)$ for the interval $[\emptyset, \lambda]$ when λ is the rectangular staircase shape $\delta_d(b^a)$, which is the Young diagram obtained from that of δ_d by replacing each cell by the $a \times b$ rectangle (b^a). Using (1.2), one can restate (1.4) as follows.

Theorem 1.7. *Let λ be a partition with n parts. Then*

$$f^\lambda(+1) = \sum_{k:\lambda_k<\lambda_{k-1}} \frac{\lambda_k(|\lambda| + 1)!}{(\lambda_k + n - k + 1)\prod_{i=1}^n (\lambda_i + n - i)!} \times \prod_{1\le i<j\le n} (\lambda_i - \lambda_j - i + j) \prod_{i\ne k} \frac{\lambda_k - \lambda_i + k - i - 1}{\lambda_k - \lambda_i + k - i},$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & 6 & 8 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline \multicolumn{1}{c|}{} & 3 & {}^{5}7 & 9 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

FIGURE 8. A standard Young tableau of shifted shape $(4,3,1)$ (left) and a standard barely set-valued tableau of shifted shape $(4,3,1)$ (right).

where $\lambda_0=\infty$*. Equivalently, the expectation* $\mathbb{E}(Y)$ *for the interval* $[\emptyset,\lambda]$ *is given by*

$$\mathbb{E}(Y)=\frac{f^\lambda(+1)}{(|\lambda|+1)f^\lambda}=\sum_{k:\lambda_k<\lambda_{k-1}}\frac{\lambda_k}{\lambda_k+n-k+1}\prod_{i\neq k}\frac{\lambda_k-\lambda_i+k-i-1}{\lambda_k-\lambda_i+k-i}. \tag{1.5}$$

Now let λ be a strict partition. A *standard Young tableau* and a *standard barely set-valued tableau* of shifted shape λ are defined similarly as fillings of the shifted Young diagram λ, see Figure 8. We denote by g^λ (resp. $g^\lambda(+1)$) the number of standard Young tableaux (resp. standard barely set-valued tableaux) of shifted shape λ.

For a strict partition λ with n parts, there is a shifted hook-length formula due to Thrall [25]:

$$g^\lambda=\frac{|\lambda|!}{\prod_{u\in\lambda}h_\lambda(u)}=\frac{|\lambda|!}{\prod_{i=1}^n\lambda_i!}\prod_{1\le i<j\le n}\frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}. \tag{1.6}$$

Since we do not need the shifted hook length $h_\lambda(u)$ in this paper, we refer the reader to [15, Fig. 7] for its definition.

If λ is a shifted Young diagram, the expectation $\mathbb{E}(Y)$ for the interval $[\emptyset,\lambda]$ can be similarly computed[1]:

$$\mathbb{E}(Y)=\frac{g^\lambda(+1)}{(|\lambda|+1)g^\lambda}. \tag{1.7}$$

In this paper, using a modification of the uncrowding algorithm and some q-integral techniques from [14], we express $g^\lambda(+1)$ as a sum of g^μ's in a similar fashion as (1.4). This leads us to the following explicit formula for $g^\lambda(+1)$, which is a shifted analogue of Theorem 1.7.

Theorem 1.8. *Let* λ *be a shifted Young diagram with* n *rows. Then*

$$\begin{aligned}g^\lambda(+1)&=\frac{(|\lambda|+1)!}{2\prod_{i=1}^n\lambda_i!}\prod_{1\le i<j\le n}\frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}\\&\quad\times\left(n+\sum_{k:\lambda_k\le\lambda_{k-1}-2}\frac{\lambda_k-2n+2k-1}{\lambda_k+1}\prod_{i\neq k}\frac{\lambda_k+\lambda_i}{\lambda_k-\lambda_i}\frac{\lambda_k-\lambda_i+1}{\lambda_k+\lambda_i+1}\right),\end{aligned}$$

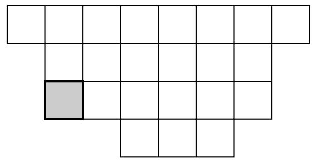
where $\lambda_0=\infty$*. Equivalently, the expectation* $\mathbb{E}(Y)$ *for the interval* $[\emptyset,\lambda]$ *is given by*

$$\mathbb{E}(Y)=\frac{g^\lambda(+1)}{(|\lambda|+1)g^\lambda}=\frac{1}{2}\left(n+\sum_{k:\lambda_k\le\lambda_{k-1}-2}\frac{\lambda_k-2n+2k-1}{\lambda_k+1}\prod_{i\neq k}\frac{\lambda_k+\lambda_i}{\lambda_k-\lambda_i}\frac{\lambda_k-\lambda_i+1}{\lambda_k+\lambda_i+1}\right). \tag{1.8}$$

The advantage of Theorem 1.8 is that it expresses, for any shifted Young diagram λ, the expectation $\mathbb{E}(Y)$ for the interval $[\emptyset,\lambda]$ as an explicit sum. In some fortunate cases, we are able to simplify the sum using the theory of hypergeometric series and express $\mathbb{E}(Y)$ as a product. We apply exactly this procedure to give new proofs of Theorems 1.3 and 1.4 and a first proof of Theorem 1.5.

The remainder of this paper is organized as follows. In Section 2 we give basic definitions and results on semistandard Young tableaux and q-integrals. In Section 3 we prove Theorem 1.8. In Section 4 we use Theorem 1.8 to evaluate $\mathbb{E}(Y)$ for the intervals described in Theorems 1.3, 1.4, and

[1]The proof is completely analogous to that of (1.3) for ordinary partitions by Reiner, Tenner and Yong [17, Corollary 3.7], so we omit the details.



0	0	0	0	1	1	1	2
	1	1	2	2	2	3	
	2	3	3	3	4	4	
			4	5	5		

FIGURE 9. An extended shifted Young diagram $\lambda \cup \{(3,2)\}$ for $\lambda = (8,6,5,3)$ on the left and a semistandard Young tableau of this shape on the right.

1.5. In Section 5 we compute $\mathbb{E}(X)$ for the interval described in Theorem 1.5, hence complete the proof of Theorem 1.5. Finally, in Appendix A we propose a q-analogue, with an extra parameter a, for the down-degree expectation $\mathbb{E}(X)$ for which we conjecture a product formula for lower intervals $[\emptyset, \lambda]$ of Young's lattice, in case λ is an ordinary partition of balanced shape of any slope.

2. Preliminaries

In this section, we define some notations used in hypergeometric series and basic hypergeometric series. Also we review some basic definitions and results relating q-integrals and semistandard Young tableaux.

For any complex number a, the *shifted factorial* is defined by

$$(a)_0 = 1,\ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)},\ n = 1,2,\dots,$$

and we write $(a_1, a_2, \dots, a_k)_n$ for the product $\prod_{i=1}^k (a_i)_n$. The *q-shifted factorial* is defined by

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1,2,\dots, \end{cases}$$

and we use $(a_1, a_2, \dots, a_k; q)_n$ to denote $\prod_{i=1}^k (a_i;q)_n$.

The ${}_rF_s$ *(generalized) hypergeometric series* with r numerator parameters $a_1, \dots, a_r$, s denominator parameters $b_1, \dots, b_s$, and argument z is defined by

$${}_rF_s(a_1, a_2, \dots, a_r; b_1, \dots, b_s; z) \equiv {}_rF_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z\right] = \sum_{n=0}^\infty \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{n!(b_1)_n \cdots (b_s)_n} z^n.$$

We also require the ${}_r\phi_s$ *basic hypergeometric series* which is defined by

$$\begin{aligned} {}_r\phi_s(a_1, a_2, \dots, a_r; b_1, \dots, b_s; q, z) &\equiv {}_r\phi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z\right] \\ &= \sum_{n=0}^\infty \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \end{aligned}$$

For more information, see [9].

Let μ be a partition and λ a strict partition. One can naturally identify the Young diagram of μ with the set $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} : 1 \le i \le l(\mu),\ 1 \le j \le \mu_i\}$ and the shifted Young diagram of λ with the set $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} : 1 \le i \le l(\lambda),\ i \le j \le \lambda_i + i - 1\}$. In this section and the next we consider another family of diagrams defined as follows. An *extended shifted Young diagram* is a diagram obtained from a shifted Young diagram λ by adding a cell below the main diagonal, i.e., a cell in row i and column $i-1$ for some $i \ge 1$. In this case we denote the extended shifted Young diagram by $\lambda \cup \{(i, i-1)\}$. See Figure 9 for an example.

From now on, a diagram means a Young diagram, a shifted Young diagram, or an extended shifted Young diagram.

Let π be a diagram. A *standard Young tableau* of shape π is a filling of the cells in π with integers $1, 2, \dots, |\pi|$ such that the integers are increasing in each row and in each column and each integer $1 \le i \le |\pi|$ is used exactly once. A *semistandard Young tableau* of shape π is a filling of π with nonnegative integers such that the integers are weakly increasing in each row and strictly increasing in each column. See Figure 9 for an example. The set of standard (resp. semistandard) Young tableaux of shape π is denoted by $\mathrm{SYT}(\pi)$ (resp. $\mathrm{SSYT}(\pi)$). For $T \in \mathrm{SSYT}(\pi)$, let $|T|$ be the sum of integers in T.

The following proposition can be obtained from a well known result in the P-partition theory [22, Theorem 3.15.7]. Note that if π is a Young diagram, then $|\mathrm{SYT}(\pi)| = f^\pi$, and if π is a shifted Young diagram, then $|\mathrm{SYT}(\pi)| = g^\pi$.

Proposition 2.1. *For any diagram π,*

$$|\mathrm{SYT}(\pi)| = \lim_{q\to 1} (q;q)_{|\pi|} \sum_{T\in \mathrm{SSYT}(\pi)} q^{|T|}.$$

For $\lambda \in \mathrm{Par}_n$ and a sequence $x = (x_1, \dots, x_n)$ of variables, let

$$a_\lambda(x) = \det(x_j^{\lambda_i})_{i,j=1}^n, \qquad \overline{a}_\lambda(x) = (-1)^{\binom{n}{2}} a_\lambda(x).$$

For an n-variable function $f(x) = f(x_1, \dots, x_n)$, we write $f(q^\lambda)$ to mean $f(q^{\lambda_1}, \dots, q^{\lambda_n})$.

The following result will be used, see [14, Theorem 8.7] or [15, Corollary 2.3].

Proposition 2.2. *Let λ be a shifted Young diagram with n parts and let ν be a partition with at most n parts. Then*

$$\sum_{\substack{T\in \mathrm{SSYT}(\lambda)\\ \mathrm{rdiag}(T)=\nu}} q^{|T|} = \frac{q^{|\nu|}}{\prod_{j=1}^n (q;q)_{\lambda_j - 1}} \overline{a}_{\lambda-(1^n)}(q^\nu),$$

where $\mathrm{rdiag}(T)$ *is the partition obtained by reading the main diagonal entries in T in reversed order.*

The *q-integral* of $f(x)$ over $[a,b]$ is defined by

$$\int_a^b f(x) d_q x = (1-q) \sum_{i\ge 0} (f(bq^i)bq^i - f(aq^i)aq^i),$$

where $0 < q < 1$ and the sum is assumed to absolutely converge. If q approaches 1, the q-integral converges to the usual integral:

$$\lim_{q\to 1} \int_a^b f(x) d_q x = \int_a^b f(x) dx. \tag{2.1}$$

In this paper the following multivariate q-integral will be considered:

$$\int_{0\le x_1\le\cdots\le x_n\le 1} f(x_1,\dots,x_n) d_q x = \int_0^1 \int_0^{x_n} \int_0^{x_{n-1}} \cdots \int_0^{x_2} f(x_1,\dots,x_n) d_q x_1 \cdots d_q x_n.$$

Note that by (2.1) we have

$$\lim_{q\to 1} \int_{0\le x_1\le\cdots\le x_n\le 1} f(x_1,\dots,x_n) d_q x = \int_{0\le x_1\le\cdots\le x_n\le 1} f(x_1,\dots,x_n) dx.$$

Lemma 2.3. *[14, Lemma 4.3] For a function $f(x_1,\dots,x_n)$ satisfying $f(x_1,\dots,x_n) = 0$ if $x_i = x_j$ for any $i \ne j$,*

$$\sum_{\nu\in \mathrm{Par}_n} q^{|\nu|} f(q^\nu) = \frac{1}{(1-q)^n} \int_{0\le x_1\le\cdots\le x_n\le 1} f(x_1,\dots,x_n) d_q x.$$

The following two propositions express the number of standard Young tableaux of shape λ as an integral, when λ is a shifted Young diagram and extended shifted Young diagram, respectively.

Proposition 2.4. *Let λ be a strict partition with n parts. Then*

$$g^\lambda = \frac{|\lambda|!}{\prod_{j=1}^n (\lambda_j - 1)!} \int_{0\le x_1\le\cdots\le x_n\le 1} \overline{a}_{\lambda-(1^n)}(x) dx.$$

Proof. By Proposition 2.2,

$$\sum_{T\in\mathrm{SSYT}(\lambda)} q^{|T|} = \sum_{\nu\in\mathrm{Par}_n}\sum_{\substack{T\in\mathrm{SSYT}(\lambda)\\ \mathrm{rdiag}(T)=\nu}} q^{|T|} = \sum_{\nu\in\mathrm{Par}_n} \frac{q^{|\nu|}}{\prod_{j=1}^n (q;q)_{\lambda_j-1}} \overline{a}_{\lambda-(1^n)}(q^\nu).$$

Then by Lemma 2.3,

$$\sum_{T\in\mathrm{SSYT}(\lambda)} q^{|T|} = \frac{(1-q)^{-n}}{\prod_{j=1}^n (q;q)_{\lambda_j-1}} \int_{0\le x_1\le\cdots\le x_n\le 1} \overline{a}_{\lambda-(1^n)}(x) d_q x.$$

Multiplying both sides of this equation by $(q;q)_{|\lambda|}$ and taking the $q\to 1$ limit gives the result by Proposition 2.1. $\square$

Proposition 2.5. *Let λ be a strict partition with n parts. Then for $1\le i\le n$,*

$$g^{\lambda\cup\{(n-i+1,n-i)\}} = \frac{(|\lambda|+1)!}{\prod_{j=1}^n(\lambda_j-1)!}\int_{0\le x_1\le\cdots\le x_n\le 1} (x_{i+1}-x_i)\overline{a}_{\lambda-(1^n)}(x)dx,$$

where $x_{n+1}=1$.

Proof. By the same argument as in the proof of Proposition 2.4, it suffices to prove the following identity:

$$\sum_{T\in\mathrm{SSYT}(\lambda\cup\{(n-i+1,n-i)\})} q^{|T|} = \frac{q(1-q)^{-n-1}}{\prod_{j=1}^n (q;q)_{\lambda_j-1}} \int_{0\le y_1\le\cdots\le y_n\le 1} (y_{i+1}-y_i)\overline{a}_{\lambda-(1^n)}(y) d_q y, \tag{2.2}$$

where $y_{n+1}=q^{-1}$. First, observe that

$$\sum_{T\in\mathrm{SSYT}(\lambda\cup\{(n-i+1,n-i)\})} q^{|T|} = \sum_{\nu\in\mathrm{Par}_n}\sum_{\substack{T\in\mathrm{SSYT}(\lambda\cup\{(n-i+1,n-i)\})\\ \mathrm{rdiag}(T)=\nu}} q^{|T|}.$$

Let $\nu\in\mathrm{Par}_n$ and $T\in\mathrm{SSYT}(\lambda\cup\{(n-i+1,n-i)\})$ with $\mathrm{rdiag}(T)=\nu$. If s is the $(n-i+1,n-i)$-entry of T, then s can be any integer satisfying $\nu_{i+1}<s\le\nu_i$, where $\nu_{n+1}=-1$. Therefore,

$$\sum_{\nu\in\mathrm{Par}_n}\sum_{\substack{T\in\mathrm{SSYT}(\lambda\cup\{(n-i+1,n-i)\})\\ \mathrm{rdiag}(T)=\nu}} q^{|T|} = \sum_{\nu\in\mathrm{Par}_n}\sum_{\substack{T\in\mathrm{SSYT}(\lambda)\\ \mathrm{rdiag}(T)=\nu}} q^{|T|}(q^{\nu_{i+1}+1}+q^{\nu_{i+1}+2}+\cdots+q^{\nu_i}),$$

which is, by Proposition 2.2, equal to

$$\sum_{\nu\in\mathrm{Par}_n}\frac{q^{|\nu|}}{\prod_{j=1}^n (q;q)_{\lambda_j-1}}\overline{a}_{\lambda-(1^n)}(q^\nu)\frac{q(q^{\nu_{i+1}}-q^{\nu_i})}{1-q}.$$

Applying Lemma 2.3 to the above sum gives the right hand side of (2.2) which completes the proof. $\square$

3. A formula for the number of shifted SBTs

In this section we prove Theorem 1.8 in the introduction, which is a formula for the number of standard barely set-valued tableaux of shifted shape.

Let λ be a shifted Young diagram with n rows. For a standard barely set-valued tableau (in short, SBT) T of shape λ, there is a unique cell that contains two integers. We call this cell the *double cell* of T. Denote by $g_k^\lambda(+1)$ the number of standard barely set-valued tableaux of shifted shape λ with the double cell in column k. We also define $g_{i,i}^\lambda(+1)$ to be the number of standard barely set-valued tableaux of shape λ with double cell in column i and in row i. Then, by definition,

$$g^\lambda(+1) = \sum_{i=1}^{\lambda_1} g_i^\lambda(+1). \tag{3.1}$$

A *northeast corner* of $\lambda\in\mathrm{Par}_n^*$ is a cell $(i,i+\lambda_i)$ such that $1\le i\le n$ and $\lambda_i\le\lambda_{i-1}-2$, where $\lambda_0=\infty$. In other words, (i,j) is a northeast corner of λ if and only if $1\le i\le n$, $(i,j)\notin\lambda$ and $\lambda\cup\{(i,j)\}$ is a shifted Young diagram. The set of northeast corners of λ is denoted by $\mathrm{NE}(\lambda)$. See

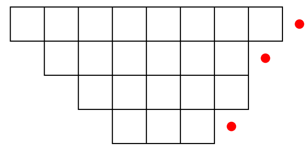


FIGURE 10. The set $\mathrm{NE}(\lambda)$ of northeast corners of $\lambda = (8, 6, 5, 3)$.

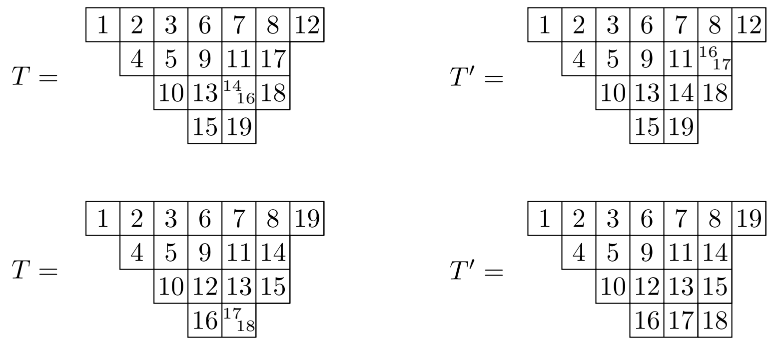


FIGURE 11. Two examples of the map $T \mapsto T'$ in the proof of Lemma 3.1.

Figure 10 for an example. Note that in our definition $(n+1, n+1) \notin \mathrm{NE}(\lambda)$ even when $\lambda_n \geq 2$ and $\lambda \cup \{(n+1, n+1)\}$ is a shifted Young diagram.

Since we have the shifted hook length formula for g^μ, the idea of proof of Theorem 1.8 is to express $g^\lambda(+1)$ as a sum of g^μ's (Proposition 3.6). To this end, we need a sequence of lemmas. The methods used in the proofs of the first two lemmas are similar to the "uncrowding" algorithm in [17].

Lemma 3.1. *For $\lambda \in \mathrm{Par}_n^*$ and $n \leq k \leq \lambda_1$,*

$$g_k^\lambda(+1) = \sum_{(i,j)\in \mathrm{NE}(\lambda),\, j>k} g^{\lambda\cup\{(i,j)\}}.$$

Proof. It is enough to show that

$$g_k^\lambda(+1) = g_{k+1}^\lambda(+1) + g^{\lambda\cup\{(i+1,k+1)\}}, \tag{3.2}$$

where i is the length of the kth column of λ. We define $g_{k+1}^\lambda(+1) = 0$ if $k+1 > \lambda_1$, and $g^{\lambda\cup\{(i+1,k+1)\}} = 0$ if $(i+1, k+1) \notin \mathrm{NE}(\lambda)$.

We prove (3.2) by constructing a bijection from A to $B \cup C$, where A is the set of SBT of shape λ with double cell in column k, B is the set of SBTs of shape λ with double cell in column $k+1$, and C is the set of SYTs of shape $\lambda \cup \{(i+1, k+1)\}$.

Suppose $T \in A$. Let $a < b$ be the integers in the double cell in T. Let c be the smallest integer larger than b in the $(k+1)$st column of T. Define T' to be the SBT obtained from T by moving b to the cell containing c. If there is no such integer c, then define T' to be the SYT of shape $\lambda \cup \{(i+1, k+1)\}$ obtained from T by moving b to the new cell $(i+1, k+1)$. See Figure 11 for examples of this map.

It is easy to see that the map $T \mapsto T'$ is a bijection from A to $B \cup C$, and the proof follows. $\square$

Lemma 3.2. *For $\lambda \in \mathrm{Par}_n^*$ and $1 \leq k \leq n$,*

$$g_k^\lambda(+1) = \sum_{i=1}^{k} g_{i,i}^\lambda(+1).$$

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 8 & 10 & 11 \\ \hline \multicolumn{1}{c|}{} & 4 & 6 & {}^{9}{}_{12} & 15 & 16 \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & 13 & 17 & 19 \\ \cline{3-6} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 14 & 18 \\ \cline{4-5} \end{array} \qquad T' = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 8 & 10 & 11 \\ \hline \multicolumn{1}{c|}{} & 4 & 6 & 12 & 15 & 16 \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & {}^{7}{}_{9} & 13 & 17 & 19 \\ \cline{3-6} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 14 & 18 \\ \cline{4-5} \end{array}$$

FIGURE 12. An example of the map $T \mapsto T'$ in the proof of Lemma 3.2.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline \multicolumn{1}{c|}{} & 3 & \mathbf{6} & 9 \\ \cline{2-4} \multicolumn{1}{c|}{} & \mathbf{5} & 8 \\ \cline{2-3} \end{array} \longleftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline \multicolumn{1}{c|}{} & {}^{3}\mathbf{5} & 6 & 9 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 \\ \cline{3-3} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline \multicolumn{1}{c|}{} & 3 & \mathbf{5} & 9 \\ \cline{2-4} \multicolumn{1}{c|}{} & \mathbf{6} & 8 \\ \cline{2-3} \end{array} \longleftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & 9 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \mathbf{6}_{8} \\ \cline{3-3} \end{array}$$

FIGURE 13. Two examples of the map $T \mapsto T'$ in the proof of Lemma 3.3.

Proof. The proof is similar to that of Lemma 3.1. It is enough to show that

$$g_k^\lambda(+1) = g_{k,k}^\lambda(+1) + g_{k-1}^\lambda(+1), \tag{3.3}$$

where $g_0^\lambda(+1) = 0$. We prove (3.3) by constructing a bijection from A to B, where A is the set of SBT of shape λ with double cell in column k but not in the diagonal cell (k,k) and B is the set of SBT of shape λ with double cell in column $k-1$.

Suppose $T \in A$. Let $a < b$ be the integers in the double cell in T. Let c be the largest integer smaller than b in the $(k-1)$th column of T. Define T' to be the SBT obtained from T by moving a to the cell containing c. See Figure 12 for an example of this map.

It is easy to see that the map $T \mapsto T'$ is a bijection from A to B, and the proof follows. $\square$

Lemma 3.3. *For $\lambda \in \mathrm{Par}_n^*$ and $1 \le i \le n-1$,*

$$g^{\lambda\cup\{(i+1,i)\}} = g_{i,i}^\lambda(+1) + g_{i+1,i+1}^\lambda(+1).$$

Proof. It suffices to find a bijection from A to $B \cup C$, where A is the set of SYTs of shape $\lambda \cup \{(i+1,i)\}$, B is the set of SBTs of shape λ with double cell (i,i), and C is the set of SBTs of shape λ with double cell $(i+1,i+1)$.

Let $T \in A$. Let a and b be the integers in cell $(i+1,i)$ and $(i,i+1)$ of T, respectively. Define T' to be the SBT obtained from T by removing cell $(i+1,i)$ and putting a into cell (i,i) if $a < b$ and into cell $(i+1,i+1)$ if $a > b$. See Figure 13 for examples of this map.

It is easy to see that the map $T \mapsto T'$ is a desired bijection. $\square$

Lemma 3.4. *For $\lambda \in \mathrm{Par}_n^*$,*

$$\sum_{i=1}^n g_i^\lambda(+1) = \frac{1}{2} \sum_{(i,j)\in \mathrm{NE}(\lambda)} g^{\lambda\cup\{(i,j)\}} + \frac{1}{2}\sum_{i=0}^{n-1}(n-i) g^{\lambda\cup\{(i+1,i)\}}.$$

Proof. Let L be the left hand side of the identity. Then by Lemma 3.2,

$$\begin{aligned} 2L - g_n^\lambda(+1) &= \sum_{i=1}^n (2n-2i+1) g_{i,i}^\lambda(+1) \\ &= \sum_{i=1}^n (n-i) g_{i,i}^\lambda(+1) + \sum_{i=0}^{n-1}(n-i) g_{i+1,i+1}^\lambda(+1) \\ &= n g_{1,1}^\lambda(+1) + \sum_{i=1}^{n-1}(n-i)(g_{i,i}^\lambda(+1) + g_{i+1,i+1}^\lambda(+1)) \end{aligned}$$

$$= ng^{\lambda}_{1,1}(+1) + \sum_{i=1}^{n-1}(n-i)g^{\lambda\cup\{(i+1,i)\}},$$

where the last equality follows from Lemma 3.3. Since $g^{\lambda}_{1,1}(+1) = g^{\lambda\cup\{(1,0)\}}$ and, by Lemma 3.1,

$$g^{\lambda}_{n}(+1) = \sum_{(i,j)\in \mathrm{NE}(\lambda)} g^{\lambda\cup\{(i,j)\}},$$

hence we obtain the desired formula for L. □

Lemma 3.5. *For $\lambda \in \mathrm{Par}^*_n$,*

$$\sum_{i=0}^{n-1}(n-i)g^{\lambda\cup\{(i+1,i)\}} = n(|\lambda|+1)g^{\lambda} - \sum_{(i,j)\in \mathrm{NE}(\lambda)} \lambda_i\, g^{\lambda\cup\{(i,j)\}}.$$

Proof. The left hand side can be rewritten as

$$\sum_{i=1}^{n} i\cdot g^{\lambda\cup\{(n-i+1,n-i)\}}.$$

By Proposition 2.5, this is equal to

$$\frac{(|\lambda|+1)!}{\prod_{k=1}^{n}(\lambda_k-1)!}\sum_{i=1}^{n}\int_{0\le x_1\le\cdots\le x_n\le 1}(ix_{i+1}-ix_i)\overline{a}_{\lambda-(1^n)}(x)dx$$

$$= \frac{(|\lambda|+1)!}{\prod_{k=1}^{n}(\lambda_k-1)!}\int_{0\le x_1\le\cdots\le x_n\le 1}(n-x_1-\cdots-x_n)\overline{a}_{\lambda-(1^n)}(x)dx$$

$$= n(|\lambda|+1)g^{\lambda} - \frac{(|\lambda|+1)!}{\prod_{k=1}^{n}(\lambda_k-1)!}\int_{0\le x_1\le\cdots\le x_n\le 1}e_1(x)\overline{a}_{\lambda-(1^n)}(x)dx, \tag{3.4}$$

where the last equality follows from Proposition 2.4 and $e_1(x) = x_1+\cdots+x_n$. Let $\lambda = \delta_{n+1}+\mu$, or equivalently, $\lambda-(1^n) = \delta_n+\mu$, where we consider μ as a partition shape. Then the integrand of the q-integral in (3.4) is

$$e_1(x)\overline{a}_{\lambda-(1^n)}(x) = e_1(x)\overline{a}_{\mu+\delta_n}(x) = e_1(x)s_\mu(x)\overline{a}_{\delta_n}(x),$$

where $s_\mu(x) = a_{\mu+\delta_n}(x)/a_{\delta_n}(x)$ is the Schur function.

By the Pieri rule,

$$e_1(x)s_\mu(x) = \sum_{(i,j)\in \mathrm{NE}(\mu)} s_{\mu\cup\{(i,j)\}}(x).$$

Notice that since $\mu\in \mathrm{Par}_n$, there is no $(n+1,j)\in \mathrm{NE}(\mu)$. Therefore,

$$e_1(x)s_\mu(x)\overline{a}_{\delta_n}(x) = \sum_{(i,j)\in \mathrm{NE}(\mu),\, i\le n} s_{\mu\cup\{(i,j)\}}(x)\overline{a}_{\delta_n}(x) = \sum_{(i,j)\in \mathrm{NE}(\mu),\, i\le n}\overline{a}_{\mu\cup\{(i,j)\}+\delta_n}(x),$$

and the second term of (3.4) (without the minus sign) is equal to

$$\sum_{(i,j)\in \mathrm{NE}(\mu),\, i\le n}\frac{(|\lambda|+1)!}{\prod_{k=1}^{n}(\lambda_k-1)!}\int_{0\le x_1\le\cdots\le x_n\le 1}\overline{a}_{\delta_{n+1}+(\mu\cup\{(i,j)\})-(1^n)}(x)dx.$$

By Proposition 2.4, this is equal to

$$\sum_{(i,j)\in \mathrm{NE}(\mu),\, i\le n}\lambda_i g^{\delta_{n+1}+(\mu\cup\{(i,j)\})}.$$

Recall that λ is a shifted shape and μ is a partition shape. Since $(i,j)\in \mathrm{NE}(\mu)$ and $i\le n$ if and only if $(i,n+j)\in \mathrm{NE}(\lambda)$, and

$$\delta_{n+1}+(\mu\cup\{(i,j)\}) = \lambda\cup\{(i,n+j)\},$$

the proof is complete. □

The following proposition allows us to express the number of SBTs of any shifted shape as a sum of the numbers of SYTs of shifted shapes.

Proposition 3.6. *For $\lambda \in \mathrm{Par}_n^*$,*

$$g^\lambda(+1) = \frac{n(|\lambda|+1)}{2} g^\lambda + \frac{1}{2} \sum_{(i,j)\in \mathrm{NE}(\lambda)} (2j-2n-1-\lambda_i) g^{\lambda\cup\{(i,j)\}}.$$

Proof. Considering the column containing the extra cell, we have

$$g^\lambda(+1) = \sum_{i=1}^{n} g_i^\lambda(+1) + \sum_{i=n+1}^{\lambda_1} g_i^\lambda(+1).$$

By Lemmas 3.4 and 3.5,

$$\sum_{i=1}^{n} g_i^\lambda(+1) = \frac{n(|\lambda|+1)}{2} g^\lambda + \frac{1}{2} \sum_{(i,j)\in \mathrm{NE}(\lambda)} (1-\lambda_i) g^{\lambda\cup\{(i,j)\}}.$$

By Lemma 3.1,

$$\sum_{i=n+1}^{\lambda_1} g_i^\lambda = \sum_{(i,j)\in \mathrm{NE}(\lambda)} (j-n-1) g^{\lambda\cup\{(i,j)\}}.$$

The proof follows by adding the above two equations. □

Now we can prove Theorem 1.8 easily.

Proof of Theorem 1.8. Proposition 3.6 can be restated as

$$g^\lambda(+1) = \frac{n(|\lambda|+1)}{2} g^\lambda + \frac{1}{2} \sum_{k:\lambda_k \le \lambda_{k-1}-2} (\lambda_k - 2n + 2k - 1) g^{\lambda\cup\{(k,k+\lambda_k)\}}.$$

Using the formula (1.6), we obtain the result. □

4. Enumeration of special classes of shifted SBTs

In this section we compute the expectation $\mathbb{E}(Y)$ for the lower intervals $[\emptyset, \lambda]$ described in Theorems 1.3, 1.4, and 1.5: the cases that λ is $\delta_d + \delta_e(a^a)$, λ is a balanced shifted shape, and λ is the trapezoidal shape $(m+2n, m+2n-2, \dots, m+2)$.

Recall that since

$$\mathbb{E}(Y) = \frac{g^\lambda(+1)}{(|\lambda|+1)g^\lambda},$$

and g^λ is known, computing $\mathbb{E}(Y)$ is equivalent to computing $g^\lambda(+1)$.

The following theorem computes the expectation $\mathbb{E}(Y)$ for the lower interval $[\emptyset, \lambda]$ when λ is $\delta_d + \delta_e(a^a)$. This shows the formula for $\mathbb{E}(Y)$ in Theorem 1.3.

Theorem 4.1. *Let $\lambda = \delta_d + \delta_e(a^a)$ with $a, d, e \ge 1$ and $d > a(e-1)+1$. Then*

$$g^\lambda(+1) = (|\lambda|+1)\frac{d+a(e-1)}{4} g^\lambda.$$

Equivalently, the expectation $\mathbb{E}(Y)$ for the lower interval $[\emptyset, \lambda]$ is equal to

$$\mathbb{E}(Y) = \frac{d+a(e-1)}{4}.$$

Proof. We want to show that

$$\frac{g^\lambda(+1)}{(|\lambda|+1)g^\lambda} = \frac{d+a(e-1)}{4}.$$

By Proposition 3.6 we have (using $n = d-1$)

$$\frac{g^\lambda(+1)}{(|\lambda|+1)g^\lambda} = \frac{d-1}{2} + \frac{1}{2}(2-d+a(e-1))\, G,$$

where

$$G=\sum_{(i,j)\in \mathrm{NE}(\lambda)} \frac{g^{\lambda\cup\{(i,i+\lambda_i)\}}}{(|\lambda|+1)g^{\lambda}}.$$

(Notice that the factor $2-d+a(e-1)$ was pulled out of the sum. The shifted-balanced shape of the partition λ made this possible!) Since

$$\frac{d+a(e-1)}{4}-\frac{d-1}{2}=\frac{2-d+a(e-1)}{4},$$

we thus need to show that $G=1/2$. Now, to compute G, we observe that for $\lambda=\delta_d+\delta_e(a^a)$ we have $\mathrm{NE}(\lambda)=\{(i,i+\lambda_i)\mid i=ja+1 \text{ for } 0\le j\le e-1\}$, where $\lambda_i=d-1+a(e-1-2j)$. By the shifted hook-length formula (1.6), and a rewriting of the product in terms of Pochhammer symbols (cf. [1]) we have

$$\frac{g^{\lambda\cup\{(i,i+\lambda_i)\}}}{(|\lambda|+1)g^{\lambda}}=\frac{(2j)!}{2^{4e-4}(j!)^2}\frac{(2e-2-2j)!}{((e-j-1)!)^2}\frac{\left(\frac{2d-1}{a}+1-2j\right)_{2e-2}}{\left(\left(\frac{2d-1}{2a}+1-j\right)_{e-1}\right)^2}\frac{(2d-1+2a(e-1-2j))}{(2d-1-2ja)},$$

for each $i=ja+1$, $0\le j\le e-1$. The identity that we need to show to establish the theorem is

$$2^{4e-5}=\sum_{j=0}^{e-1}\frac{(2j)!}{(j!)^2}\frac{(2e-2-2j)!}{((e-j-1)!)^2}\frac{\left(\frac{2d-1}{a}+1-2j\right)_{2e-2}}{\left(\left(\frac{2d-1}{2a}+1-j\right)_{e-1}\right)^2}\frac{(2d-1+2a(e-1-2j))}{(2d-1-2ja)}. \tag{4.1}$$

The right hand side of (4.1) can be rewritten as

$$\begin{aligned}&\frac{(2e-2)!}{2((e-1)!)^2}\frac{\left(\frac{2d-1}{a}+1\right)_{2e-2}}{\left(\left(\frac{2d-1}{2a}+1\right)_{e-1}\right)^2}\frac{(1-2d+2a(1-e))}{(1-2d)}\\ &\times\sum_{j=0}^{e-1}\frac{\left(\frac{1-2d}{2a}+1-e,\frac{1-2d}{4a}+\frac32-\frac e2,\frac12,1-e,\frac{1-2d}{2a}+\frac12\right)_j}{\left(1,\frac{1-2d}{4a}+\frac12-\frac e2,\frac32-e+\frac{1-2d}{2a},\frac{1-2d}{2a}+1,\frac32-e\right)_j}.\end{aligned} \tag{4.2}$$

To evaluate the terminating series in (4.2), we apply Dougall's summation formula [9, (2.1.7)]

$$\begin{aligned}&{}_5F_4\left[\begin{matrix}\mathsf{a},1+\frac12\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\\ \frac12\mathsf{a},1+\mathsf{a}-\mathsf{b},1+\mathsf{a}-\mathsf{c},1+\mathsf{a}-\mathsf{d}\end{matrix};1\right]\\ &=\frac{\Gamma(1+\mathsf{a}-\mathsf{b})\Gamma(1+\mathsf{a}-\mathsf{c})\Gamma(1+\mathsf{a}-\mathsf{d})\Gamma(1+\mathsf{a}-\mathsf{b}-\mathsf{c}-\mathsf{d})}{\Gamma(1+\mathsf{a})\Gamma(1+\mathsf{a}-\mathsf{b}-\mathsf{c})\Gamma(1+\mathsf{a}-\mathsf{b}-\mathsf{d})\Gamma(1+\mathsf{a}-\mathsf{c}-\mathsf{d})},\end{aligned}$$

with $(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d})\mapsto\left(\frac{1-2d}{2a}+1-e,\frac12,1-e,\frac{1-2d}{2a}+\frac12\right)$. As a result, we obtain

$$\sum_{j=0}^{e-1}\frac{\left(\frac{1-2d}{2a}+1-e,\frac{1-2d}{4a}+\frac32-\frac e2,\frac12,1-e,\frac{1-2d}{2a}+\frac12\right)_j}{\left(1,\frac{1-2d}{4a}+\frac12-\frac e2,\frac32-e+\frac{1-2d}{2a},\frac{1-2d}{2a}+1,\frac32-e\right)_j}=\frac{\left(\frac{1-2d}{2a}+2-e\right)_{e-1}(1-e)_{e-1}}{\left(\frac{1-2d}{2a}+\frac32-e\right)_{e-1}\left(\frac32-e\right)_{e-1}}.$$

Hence the identity that we need to verify becomes

$$\frac{(2e-2)!}{((e-1)!)^2}\frac{\left(\frac{2d-1}{a}+1\right)_{2e-2}}{\left(\left(\frac{2d-1}{2a}+1\right)_{e-1}\right)^2}\frac{(1-2d+2a(1-e))}{(1-2d)}\frac{\left(\frac{1-2d}{2a}+2-e\right)_{e-1}(1-e)_{e-1}}{\left(\frac{1-2d}{2a}+\frac32-e\right)_{e-1}\left(\frac32-e\right)_{e-1}}=2^{4e-4}. \tag{4.3}$$

Note the following simple reformulations

$$\begin{aligned}\left(\frac{2d-1}{a}+1\right)_{2e-2}&=2^{2e-2}\left(\frac{2d-1}{2a}+\frac12\right)_{e-1}\left(\frac{2d-1}{2a}+1\right)_{e-1},\\ \left(\frac{1-2d}{2a}+2-e\right)_{e-1}&=(-1)^{e-1}\left(\frac{2d-1}{2a}\right)_{e-1},\\ (1-e)_{e-1}&=(-1)^{e-1}(e-1)!,\\ \left(\frac{1-2d}{2a}+\frac32-e\right)_{e-1}&=(-1)^{e-1}\left(\frac{2d-1}{2a}+\frac12\right)_{e-1},\end{aligned}$$

$$\left(\frac{3}{2}-e\right)_{e-1} = (-1)^{e-1}2^{2-2e}\frac{(2e-2)!}{(e-1)!}.$$

By applying these reformulated terms, we obtain (4.3), which completes the proof. $\square$

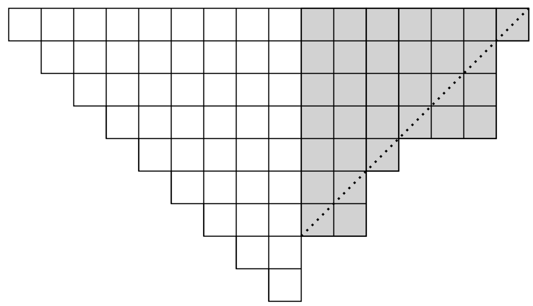


FIGURE 14. $\lambda = \delta_{10} + \nu$ with $k = 7$

The following theorem computes the expectation $\mathbb{E}(Y)$ for the lower interval $[\emptyset, \lambda]$ when λ is a balanced shifted shape. This shows the formula for $\mathbb{E}(Y)$ in Theorem 1.4.

Theorem 4.2. *Let $0 \le k < n$ and let ν be a partition which as a straight shape is balanced with height and width both equal to k.*

(1) *Let $\lambda = \delta_{n+1} + \nu$. Then*

$$g^\lambda(+1) = (|\lambda|+1)\frac{n+1+k}{4}g^\lambda.$$

Equivalently, the expectation $\mathbb{E}(Y)$ for the lower interval $[\emptyset, \lambda]$ is equal to

$$\mathbb{E}(Y) = \frac{n+1+k}{4}.$$

(2) *Let $\lambda = \delta_{n+1} + (n-1-k)^n + \nu$. Then*

$$g^\lambda(+1) = (|\lambda|+1)\frac{n}{2}g^\lambda.$$

Equivalently, the expectation $\mathbb{E}(Y)$ for the lower interval $[\emptyset, \lambda]$ is equal to

$$\mathbb{E}(Y) = \frac{n}{2}.$$

Proof. Since ν is balanced, we can write $\nu = ((a_1+a_2+\cdots+a_\ell)^{a_1}, (a_2+\cdots+a_\ell)^{a_2}, \ldots, (a_\ell)^{a_\ell})$, for a_i's such that $a_i \ge 1$ for all $1 \le i \le \ell$ and $a_1+\cdots+a_\ell = k$. Then $\mathrm{NE}(\lambda) = \{(r_i, c_i) \mid r_i = a_1+\cdots+a_i+1,\ c_i = r_i + \lambda_{r_i},\ \text{for } i = 0,\ldots,\ell\}$.

In the case when $\lambda = \delta_{n+1} + \nu$, using the fact that $\lambda_{a_1+\cdots+a_i+1} = n+k-2(a_1+\cdots+a_i)$, for $i = 0,\ldots,\ell$, by expressing $g^{\lambda\cup\{(r_i,c_i)\}}$ in terms of g^λ and comparing the terms in equation (1.8), the identity that we need to prove becomes

$$\begin{aligned}(4.4)\quad 1 = \sum_{i=0}^{\ell} &\frac{\frac12 a_i}{a_i}\frac{(a_i+\frac12 a_{i-1})}{(a_i+a_{i-1})}\frac{(a_i+a_{i-1}+\frac12 a_{i-2})}{(a_i+a_{i-1}+a_{i-2})}\cdots\frac{(a_i+a_{i-1}+\cdots+a_2+\frac12 a_1)}{(a_i+a_{i-1}+\cdots+a_2+a_1)}\\ &\times\frac{\frac12 a_{i+1}}{a_{i+1}}\frac{(a_{i+1}+\frac12 a_{i+2})}{(a_{i+1}+a_{i+2})}\frac{(a_{i+1}+a_{i+2}+\frac12 a_{i+3})}{(a_{i+1}+a_{i+2}+a_{i+3})}\cdots\frac{(a_{i+1}+a_{i+2}+\cdots+a_{\ell-1}+\frac12 a_\ell)}{(a_{i+1}+a_{i+2}+\cdots+a_{\ell-1}+a_\ell)}\\ &\times\frac{(n+\frac12-r_i+a_\ell+a_{\ell-1}+\cdots+a_{i+1})}{(n+\frac12-r_i)}\\ \times&\frac{(n+\frac12-r_i+\frac12 a_\ell)}{(n+\frac12-r_i+a_\ell)}\frac{(n+\frac12-r_i+a_\ell+\frac12 a_{\ell-1})}{(n+\frac12-r_i+a_\ell+a_{\ell-1})}\cdots\frac{(n+\frac12-r_i+a_\ell+a_{\ell-1}+\cdots+a_2+\frac12 a_1)}{(n+\frac12-r_i+a_\ell+a_{\ell-1}+\cdots+a_2+a_1)},\end{aligned}$$

where $r_i = a_1 + \cdots + a_i + 1$. This identity can be proved by partial fraction decomposition, or by Lagrange interpolation. The latter says that if $f(x)$ is a polynomial of degree ℓ, then

$$f(x) = \sum_{i=0}^{\ell} f(\mathsf{b}_i) \prod_{0\le j\le \ell,\, j\ne i} \frac{x-\mathsf{b}_j}{\mathsf{b}_i-\mathsf{b}_j},$$

where the b_i's are all distinct. Dividing both sides of this identity by $f(x)$, choosing $f(x) = \prod_{j=0}^{\ell-1}(x-\mathsf{c}_j)$ and letting $x\to\infty$ we obtain the well-known formula

$$1 = \sum_{i=0}^{\ell} \frac{\prod_{0\le j\le \ell-1}(\mathsf{b}_i-\mathsf{c}_j)}{\prod_{0\le j\le \ell,\, j\ne i}(\mathsf{b}_i-\mathsf{b}_j)}.$$

By making the substitutions $\mathsf{b}_i \mapsto (b_i-u/2)^2$ and $\mathsf{c}_j \mapsto (c_j-u/2)^2$, for $0\le i\le \ell$ and $0\le j\le \ell-1$, we obtain the equivalent identity

$$1 = \sum_{i=0}^{\ell} \frac{\prod_{0\le j\le \ell-1}(b_i-c_j)(u-b_i-c_j)}{\prod_{0\le j\le \ell,\, j\ne i}(b_i-b_j)(u-b_i-b_j)}. \tag{4.5}$$

Now, the sum in (4.4) can be evaluated by specializing the parameters in (4.5) as $b_i \mapsto a_1+\cdots+a_i$, for $0\le i\le \ell$, $c_j \mapsto (a_1+\cdots+a_j)+a_{j+1}/2$, for $0\le j\le \ell-1$, and $u\mapsto a_1+\cdots+a_l+n-1/2$. This establishes the first part (1) of the theorem.

For the second part (2) of the theorem, we instead set $\lambda = \delta_{n+1}+(n-1-k)^n+\nu$, then $\lambda_{r_i} = 2n+1-2r_i$ where $r_i = a_1+\cdots+a_i+1$, for $i=0,\dots,\ell$. By using this fact in Proposition 3.6, we obtain

$$g^\lambda(+1) = (|\lambda|+1)\frac{n}{2}g^\lambda$$

which completes the proof. □

Remark 4.3. In fact, the shifted partition $\lambda = \delta_d + \delta_e(a^a)$ considered in Theorem 4.1 is a special case of shifted partitions considered in Theorem 4.2. However, we separated two cases as different theorems, since we applied Dougall's summation formula to prove Theorem 4.1 as opposed to the Lagrange interpolation being used in the proof of Theorem 4.2.

Finally we consider the case that λ is the trapezoidal shape $(m+2n, m+2n-2, \dots, m+2)$ whose typical diagram is as shown in Figure 15.

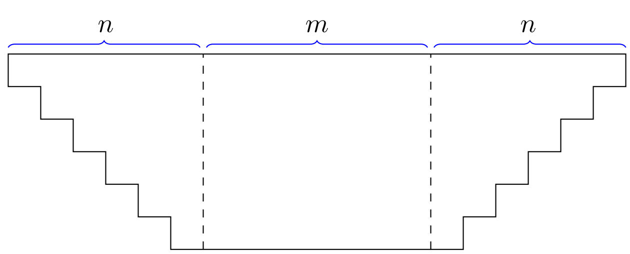


FIGURE 15. $\lambda = (m+2n, m+2n-2, \dots, m+2)$

Note that in this case, by (1.6),

$$g^\lambda = \frac{(nm+n^2+n)!}{\prod_{i=1}^{n}(m+2i)!}\prod_{i=1}^{n-1}\frac{i!(m+1+i)!}{(m+1+2i)!}. \tag{4.6}$$

The following theorem shows the formula for $\mathbb{E}(Y)$ in Theorem 1.5.

Theorem 4.4. *Let* $\lambda = (m+2n, m+2n-2, \dots, m+2)$*. Then*

$$g^\lambda(+1) = (|\lambda|+1)\frac{|\lambda|}{\lambda_1+1}g^\lambda = \frac{(mn+n^2+n)(mn+n^2+n+1)!}{(m+2n+1)\prod_{i=1}^{n}(m+2i)!}\prod_{i=1}^{n-1}\frac{i!(m+1+i)!}{(m+1+2i)!}.$$

Equivalently, the expectation $\mathbb{E}(Y)$ *for the lower interval* $[\emptyset,\lambda]$ *is equal to*

$$\mathbb{E}(Y)=\frac{g^{\lambda}(+1)}{(|\lambda|+1)g^{\lambda}}=\frac{|\lambda|}{\lambda_1+1}.$$

Proof. Note that $\mathrm{NE}(\lambda)=\{(i+1,\lambda_{i+1}+1)\mid 0\le i\le n-1\}$, where $\lambda_{i+1}=m+2n-2i$, for $0\le i\le n-1$. By applying

$$\begin{aligned}(4.7)\quad &\frac{g^{\lambda\cup\{(i+1,\lambda_{i+1}+1)\}}}{g^{\lambda}(|\lambda|+1)}\\ &=2\frac{(2i)!}{(i!)^2}\frac{(2n-2i-1)!}{((n-i-1)!)^2}\frac{((m+2n-i)!)^2}{((m+n-i)!)^2}\frac{(2m+2n-2i+1)!}{(2m+4n-2i+1)!}\frac{(2m+4n-4i+1)}{(m+2n-2i)(m+2n-2i+1)},\end{aligned}$$

for $0\le i\le n-1$, in equation (1.8), the identity that we need to prove is

$$\begin{aligned}(4.8)\quad &\frac{n}{2(m+2n+1)}=\\ &\sum_{i=0}^{n-1}\frac{(2i)!}{(i!)^2}\frac{(2n-2i-1)!}{((n-i-1)!)^2}\frac{((m+2n-i)!)^2}{((m+n-i)!)^2}\frac{(2m+2n-2i+1)!}{(2m+4n-2i+1)!}\frac{(2m+4n-4i+1)}{(m+2n-2i)(m+2n-2i+1)}.\end{aligned}$$

Note that the right hand side of (4.8) can be rewritten as

$$\begin{aligned}(4.9)\qquad &\frac{(2n-1)!((m+2n)!)^2(2m+2n+1)!}{((n-1)!)^2((m+n)!)^2(2m+4n)!(m+2n)(m+2n+1)}\\ &\times\sum_{i=0}^{n-1}\frac{\left(-m-2n-\frac12,-\frac{m}{2}-n+\frac34,\frac12,-m-n,-\frac{m}{2}-n-\frac12,-\frac{m}{2}-n,1-n\right)_i}{\left(1,-\frac{m}{2}-n-\frac14,-m-2n,\frac12-n,-\frac{m}{2}-n+1,-\frac{m}{2}-n+\frac12,-m-n-\frac12\right)_i}.\end{aligned}$$

Now we compute the hypergeometric sum in (4.9) and express it in a product form. To do that, we observe that it can also be obtained from the $q\to1$ limit of the $(a,b,c,d,e,n)\mapsto(q^{-m-2n-\frac12},q^{\frac12},q^{-m-n},q^{-\frac{m}{2}-n-\frac12},q^{-\frac{m}{2}-n},n-1)$ special case of the basic hypergeometric sum

$$(4.10)\qquad {}_8\phi_7\left[\begin{matrix}a,\ qa^{\frac12},-qa^{\frac12},b,c,d,e,q^{-n}\\ a^{\frac12},-a^{\frac12},aq/b,aq/c,aq/d,aq/e,aq^{n+1}\end{matrix};q,\frac{a^2q^{n+2}}{bcde}\right].$$

Now we analyze the ${}_8\phi_7$ series in (4.10). First we use the well-known Watson transformation (cf. [9, Appendix (III.18)]):

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}a,\ qa^{\frac12},-qa^{\frac12},b,c,d,e,q^{-n}\\ a^{\frac12},-a^{\frac12},aq/b,aq/c,aq/d,aq/e,aq^{n+1}\end{matrix};q,\frac{a^2q^{n+2}}{bcde}\right]\\ (4.11)\qquad &=\frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix}aq/bc,d,e,q^{-n}\\ aq/b,aq/c,deq^{-n}/a\end{matrix};q,q\right].\end{aligned}$$

We also use the following summation (cf. [9, Exercise 3.34])

$$(4.12)\qquad {}_4\phi_3\left[\begin{matrix}q^{-2n},c^2,a,aq\\ a^2q^2,cq^{-n},cq^{1-n}\end{matrix};q^2,q^2\right]=\frac{(-q,qa/c;q)_n}{(-aq,q/c;q)_n}.$$

If we take (4.11), replace q by q^2, and specialize $b=a/d^2$, $c=a^2q^{2+2n}/d^2$, the ${}_4\phi_3$ series on the right-hand side can be simplified by the $(a,c)\mapsto(d,d^2q^{-n/a})$ case of (4.12). As a consequence, we obtain the following summation (which is of interest by itself):

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}a,\ q^2a^{\frac12},-q^2a^{\frac12},a/d^2,a^2q^{2+2n}/d^2,d,dq,q^{-2n}\\ a^{\frac12},-a^{\frac12},d^2q^2,d^2q^{-2n}/a,aq^2/d,aq/d,aq^{2n+2}\end{matrix};q^2,\frac{d^2q}{a}\right]\\ (4.13)\qquad &=\frac{(-q,aq/d^2;q)_n(aq^2;q^2)_n}{(-dq,aq/d;q)_n(aq^2/d^2;q^2)_n}.\end{aligned}$$

We now apply the substitution $(a, d, n) \mapsto (q^{-1-2m-4n}, q^{-1-m-2n}, n-1)$ to (4.13) and take the limit $q \to 1$ to obtain

$$\sum_{i=0}^{n-1} \frac{\left(-m-2n-\frac{1}{2}, -\frac{m}{2}-n+\frac{3}{4}, \frac{1}{2}, -m-n, -\frac{m}{2}-n-\frac{1}{2}, -\frac{m}{2}-n, 1-n\right)_i}{\left(1, -\frac{m}{2}-n-\frac{1}{4}, -m-2n, \frac{1}{2}-n, -\frac{m}{2}-n+1, -\frac{m}{2}-n+\frac{1}{2}, -m-n-\frac{1}{2}\right)_i}$$
$$= \frac{(n-1)!n!((m+n)!)^2(2m+4n-1)!}{(2n-1)!((m+2n-1)!)^2(2m+2n+1)!}.$$

Using this result in (4.9) and simplifying the expression proves the identity (4.8). $\square$

5. Down degrees for shifted Young diagrams

In this section we compute the expectation $\mathbb{E}(X)$ for the interval $[\emptyset, \lambda]$ when λ is a trapezoidal shifted Young diagram. This completes our proof of Theorem 1.5. We first give a general way to express $\mathbb{E}(X)$ in terms of the number of shifted Young diagrams contained in a given shifted Young diagram.

Let λ be a shifted Young diagram. We denote by $R(\lambda)$ the number of shifted Young diagrams $\mu \subseteq \lambda$. Define $R^{(+1)}(\lambda)$ to be the sum of $\mathrm{ddeg}(\mu)$ for all shifted Young diagrams $\mu \subseteq \lambda$. Equivalently, $R^{(+1)}(\lambda)$ is the number of pairs $(\mu, (i,j))$ of $\mu \subseteq \lambda$ and a removable corner (i,j) of μ. By definition, the expectation $\mathbb{E}(X)$ for the interval $[\emptyset, \lambda]$ is given by

$$\mathbb{E}(X) = \frac{R^{(+1)}(\lambda)}{R(\lambda)}.$$

The *border* $\mathcal{B}(\lambda)$ of λ is the set of cells (i,j) in λ such that $(i+1, j+1)$ is not in λ. See Figure 16 for an example.

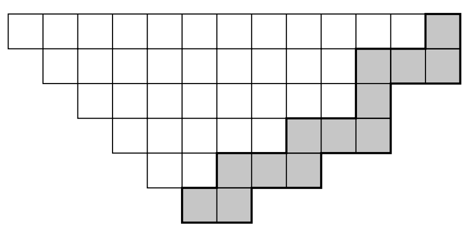


Figure 16. The border $\mathcal{B}(\lambda)$ for a shifted Young diagram λ.

Suppose that $\ell(\lambda) = n$. For $x = (i,j) \in \mathcal{B}(\lambda)$, define

$$\lambda(x) = \begin{cases} (\lambda_1 - 1, \dots, \lambda_{n-1} - 1), & \text{if } (i,j) = (n,n), \\ (\lambda_1 - 2, \dots, \lambda_{i-1} - 2, \widetilde{\lambda}_{i+1}, \dots, \widetilde{\lambda}_n), & \text{otherwise,} \end{cases}$$

where

$$\widetilde{\lambda}_t = \begin{cases} \lambda_t - 1, & \text{if } \lambda_t + t - 1 = j, \\ \lambda_t, & \text{otherwise.} \end{cases}$$

Pictorially, if $x \in \mathcal{B}(\lambda) \setminus \{(n,n)\}$, then $\lambda(x)$ is obtained from λ by removing the shaded region and attaching the remaining two connected regions as shown in Figure 17.

The following proposition allows us to write $R^{(+1)}(\lambda)$ as a sum of $R(\mu)$'s.

Proposition 5.1. *Let λ be a shifted Young diagram. Then*

$$R^{(+1)}(\lambda) = \sum_{x \in \mathcal{B}(\lambda)} R(\lambda(x)).$$

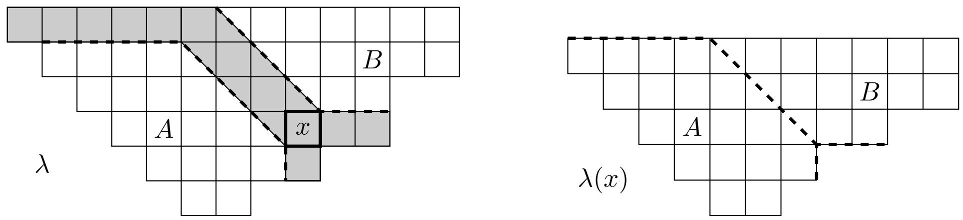


FIGURE 17. The shifted Young diagram λ and $x \in \mathcal{B}(\lambda)$ on the left and the diagram $\lambda(x)$ on the right.

Proof. By definition $R^{(+1)}(\lambda)$ is the number of pairs $(\mu,(i,j))$ of a shifted Young diagram $\mu \subseteq \lambda$ and a removable corner $c=(i,j)$ of μ. Let x be the cell such that $x=(i+t,j+t)\in\lambda$ for some $t\geq 0$ with $(i+t+1,j+t+1)\notin\lambda$. Then $x\in\mathcal{B}(\lambda)$. Define

$$\nu = \begin{cases} (\mu_1-1,\mu_2-1,\dots,\mu_{i-1}-1), & \text{if } x=(n,n),\\ (\mu_1-2,\mu_2-2,\dots,\mu_{i-1}-2,\mu_{i+1},\mu_{i+2},\dots,\mu_k), & \text{if } x\neq(n,n),\end{cases}$$

where $k=\ell(\mu)$. See Figure 18 and Figure 19 for an example of ν for the cases $x\neq(n,n)$ and $x=(n,n)$, respectively. By the construction we have $\nu\in R(\lambda(x))$. It is not hard to see that the map $(\mu,(i,j))\mapsto(x,\nu)$ is a bijection from $R^{(+1)}(\lambda)$ to $\cup_{x\in\mathcal{B}(\lambda)}R(\lambda(x))$. This proves the desired identity. $\square$

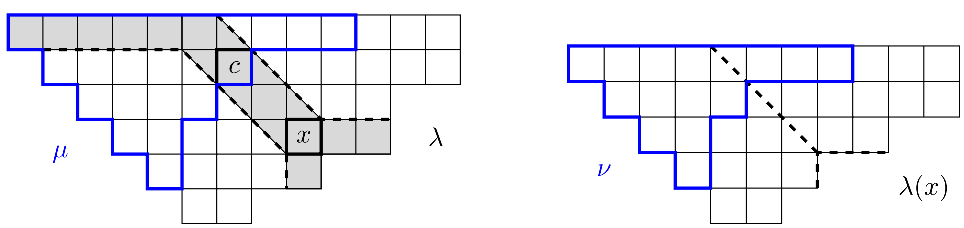


FIGURE 18. The diagram ν for $x\neq(n,n)$.

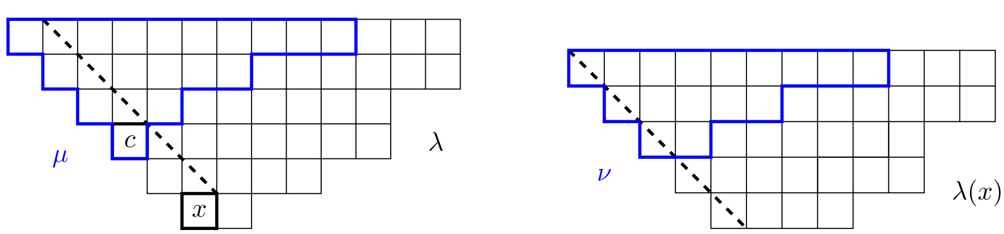


FIGURE 19. The diagram ν for $x=(n,n)$.

In the next two lemmas we find simple formulas for $R(\lambda)$ and $R^{(+1)}(\lambda)$ for a trapezoidal shifted Young diagram λ.

Lemma 5.2. *Let $\lambda=(N,N-2,\dots,N-2n+2)$. Then*

$$R(\lambda)=\binom{N+1}{n}.$$

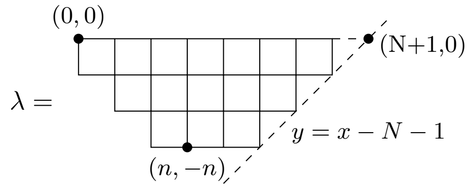


FIGURE 20. Embedding λ in $\mathbb{Z}^2$.

Proof. Let us embed the shifted Young diagram of λ in $\mathbb{Z}^2$ so that each cell is a unit square and the top left corner of λ is at $(0,0)$ as shown in Figure 20.

Each shifted Young diagram $\mu \subseteq \lambda$ can be identified with a lattice path from $(j,-j)$ to $(N+1,0)$ for some $0 \le j \le n$ consisting of north and east steps that never goes below the line $y = x - N - 1$. For example, if $N = 7, n = 3$ so that $\lambda = (7,5,3)$, then the shifted shape $\mu = (7,3) \subseteq \lambda$ is identified with the path shown in Figure 21.

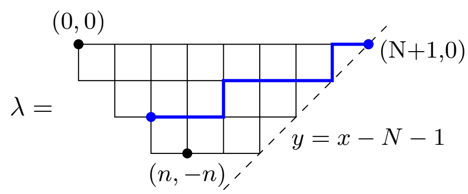


FIGURE 21. The lattice path representing $\mu = (7,3)$.

By the standard reflection method, one can see that, for a fixed $0 \le j \le n$, the number of such paths equals $\binom{N+1}{j} - \binom{N+1}{j-1}$. Therefore the total number of shifted Young diagrams μ contained in λ is

$$\sum_{j=0}^{n} \left(\binom{N+1}{j} - \binom{N+1}{j-1} \right) = \binom{N+1}{n}$$

as the sum telescopes. $\square$

Lemma 5.3. *Let* $\lambda = (N, N-2, \dots, N-2n+2)$. *Then*

$$R^{(+1)}(\lambda) = \frac{|\lambda|}{N+1} \binom{N+1}{n}.$$

Proof. Let $T_{a,b}$ denote the shifted Young diagram $(a, a-2, \dots, a-2b+2)$. By Proposition 5.1,

$$R^{(+1)}(T_{N,n}) = \sum_{x \in \mathcal{B}(T_{N,n})} R(T_{N,n}(x)). \tag{5.1}$$

Let $x \in \mathcal{B}(T_{N,n})$. It is straightforward to check that $R(T_{N,n}(x))$ is given as follows.

- If $x = (n,n)$,
$$R(T_{N,n}(x)) = T_{N-1,n-1}.$$
- If $x \neq (n,n)$ and x is in the nth row,
$$R(T_{N,n}(x)) = T_{N-2,n-1}.$$
- If x is the rightmost cell in the ith row $(1 \le i \le n-1)$,
$$R(T_{N,n}(x)) = T_{N-2,n-1}.$$
- If x is the second rightmost cell in the ith row $(1 \le i \le n-1)$,
$$R(T_{N,n}(x)) = T_{N-2,n-1} \setminus \{(i, N-2i)\}.$$

Therefore, by (5.1),

$$R^{(+1)}(T_{N,n}(x)) = R(T_{N-1,n-1}) + ((N-2n+1)+(n-1))R(T_{N-2,n-1}) + \sum_{i=1}^{n-1} R(T_{N-2,n-1} \setminus \{(i, N-2i)\}). \tag{5.2}$$

Let $1 \le i \le n-1$. Then

$$R(T_{N-2,n-1} \setminus \{(i, N-2i)\}) = R(T_{N-2,n-1}) - t(i), \tag{5.3}$$

where $t(i)$ is the number of shifted Young diagrams $\mu \subseteq T_{N-2,n-1}$ containing the cell $(i, N-2i)$. Suppose μ is such a shifted Young diagram. Then μ is determined by the two sub-diagrams α and β of μ, where α is the set of cells of μ in columns $N-2, N-3, \dots, N-i$ and β is the set of cells of μ in rows $i+1, i+2, \dots, n-1$, see Figure 22.

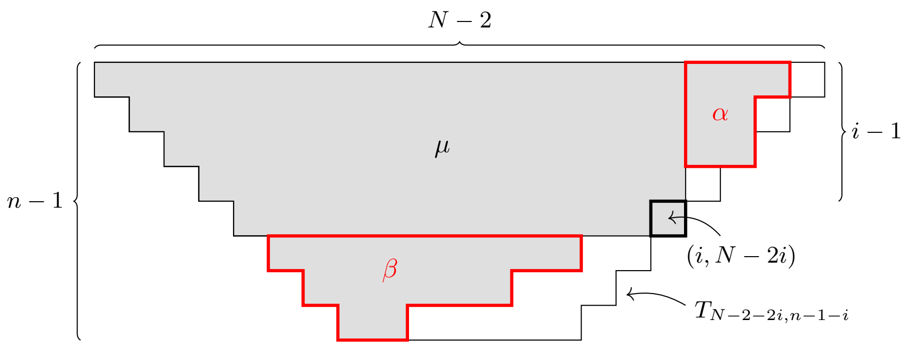


FIGURE 22. The diagram μ is determined by α and β.

Then one can regard α as a Young diagram contained in the Young diagram $(i-1, i-2, \dots, 1)$ and β as a shifted Young diagram contained in $T_{N-2-2i,n-1-i}$. It is well known that the number of such α is given by the Catalan number $\frac{1}{i+1}\binom{2i}{i}$. This argument shows that

$$t(i) = \frac{1}{i+1}\binom{2i}{i} R(T_{N-2-2i,n-1-i}). \tag{5.4}$$

By (5.2), (5.3), (5.4), and Lemma 5.2, we have

$$\begin{aligned} R^{(+1)}(T_{N,n}(x)) &= R(T_{N-1,n-1}) + (N-1)R(T_{N-2,n-1}) - \sum_{i=1}^{n-1} \frac{1}{i+1}\binom{2i}{i} R(T_{N-2-2i,n-1-i}), \\ &= \binom{N}{n-1} + (N-1)\binom{N-1}{n-1} - \sum_{i=1}^{n-1} \frac{1}{i+1}\binom{2i}{i}\binom{N-1-2i}{n-1-i}. \end{aligned}$$

Therefore the identity we need to show is

$$\frac{n(N-n+1)}{N+1}\binom{N+1}{n} = \binom{N}{n-1} + N\binom{N-1}{n-1} - \sum_{i=0}^{n-1} \frac{1}{i+1}\binom{2i}{i}\binom{N-1-2i}{n-1-i} \tag{5.5}$$

which can be simplified to

$$\frac{N}{N-n+1} = \sum_{i=0}^{n-1} \frac{\left(\frac{1}{2}, 1, n-N, 1-n\right)_i}{\left(1, 2, \frac{1-N}{2}, 1-\frac{N}{2}\right)_i}. \tag{5.6}$$

Equation (5.6) can be proved by applying the Bailey formula [2, p. 512, (c)]

$${}_4F_3\left[\begin{matrix} \frac{\mathsf{a}}{2}, \frac{\mathsf{a}+1}{2}, \mathsf{b}+\mathsf{n}, -\mathsf{n} \\ \frac{\mathsf{b}}{2}, \frac{\mathsf{b}+1}{2}, \mathsf{a}+1 \end{matrix}; 1\right] = \frac{(\mathsf{b}-\mathsf{a})_\mathsf{n}}{(\mathsf{b})_\mathsf{n}}$$

with $\mathsf{a} \mapsto 1$, $\mathsf{b} \mapsto 1 - N$ and $\mathsf{n} \mapsto n - 1$. □

Finally we can evaluate $\mathbb{E}(X)$ for the lower interval of the shifted Young poset below a trapezoidal shifted Young diagram.

Theorem 5.4. *Let λ be a trapezoidal shifted Young diagram. Then the expectation $\mathbb{E}(X)$ for the interval $[\emptyset, \lambda]$ is equal to*

$$\mathbb{E}(X) = \frac{|\lambda|}{\lambda_1 + 1}.$$

Proof. Since λ is trapezoidal, we can write $\lambda = (N, N-2, \dots, N-2n+2)$ for some nonnegative integers N, n. Then we have

$$\mathbb{E}(X) = \frac{R^{(+1)}(\lambda)}{R(\lambda)} = \frac{|\lambda|}{N+1} = \frac{|\lambda|}{\lambda_1 + 1},$$

by Lemmas 5.2 and 5.3. □

Theorem 1.5 in the introduction then follows from Theorems 4.4 and 5.4.

Remark 5.5. We note that Theorem 5.4 can also be proved by combining known results in the literature as follows. This proof is due to Sam Hopkins (personal communication).

Suppose L is a distributive lattice. Then L can be written as the poset $J(P)$ of order ideals of some poset P, see [22, 3.4.1 Theorem]. One can show that the down-degree expectation $\mathbb{E}_L(X)$ for L with respect to the uniform distribution is the average size of an antichain in P, see [12, Section 3.1].

Now let L be the lower interval $[\emptyset, \lambda]$ for a trapezoidal shifted Young diagram λ. Then $L = J(P_\lambda)$, where P_λ denotes the poset whose elements are the cells in λ and two elements $x, y \in P_\lambda$ satisfy $x < y$ if x is weakly to the southeast of y in λ. Then by the fact in the above paragraph, $\mathbb{E}_L(X)$ is the average size of an antichain in P_λ. Stembridge [23, Corollary 2.4] showed that if λ is the trapezoidal shape $(m+n-1, m+n-3, \dots, m+n-(2m-1))$ and μ is the rectangle (n^m), then, for each k, the number of antichains of size k is equal to $\binom{m}{k}\binom{n}{k}$ for both P_λ and P_μ. In particular, the average size of an antichain in P_λ is equal to that in P_μ. This shows that $\mathbb{E}_L(X) = \mathbb{E}_{L'}(X)$, where $L' = J(P_\mu)$. Then L' is isomorphic to the lower interval $[\emptyset, \mu]$ for $\mu = (n^m)$. Therefore, by Theorem 1.1,

$$\mathbb{E}_L(X) = \mathbb{E}_{L'}(X) = \frac{mn}{m+n} = \frac{|\lambda|}{\lambda_1 + 1},$$

which is Theorem 5.4.

Appendix A. An $\mathsf{a};q$-analogue of the expectation $\mathbb{E}(X)$

Our results in Section 2 make use of q-integrals and in our applications we either utilize identities for basic hypergeometric series (see in particular the proof of Theorem 4.4) or use summations which have q-analogues (see e.g. the proofs of Theorems 4.1 and 4.2). It is thus natural to ask whether q-analogues of the results proved in this paper exist.

Reiner, Tenner, and Yong [17, Proposition 1.5] considered the q-analogue $R(\lambda, q) = \sum_{\mu \subseteq \lambda} q^{|\mu|}$ of the number $R(\lambda)$ of Young diagrams contained in λ and found a recurrence satisfied by $R(\lambda, q)$. Hopkins [12, Section 3.3] considered certain q-analogues of down-degree generating functions for P-partitions, which contain the lower interval $[\emptyset, \lambda]$ as a special case.

Various possible q-analogues of $\mathbb{E}(X)$ or $\mathbb{E}(Y)$ are feasible. A good q-analogue may also be accompanied by nice results, such as product formulas extending those for ordinary enumeration.

It seems rather difficult to find a q-analogue of the down-degree expectation (with respect to the maximal chain cardinality distribution) $\mathbb{E}(Y)$ that would allow closed form results in cases of interest. We propose an $\mathsf{a};q$-analogue, where a is an extra free parameter, of the simpler expectation (with respect to the uniform distribution) $\mathbb{E}(X)$ which conjecturally can be expressed as a product for a suitably restricted class of lower intervals of Young's lattice.

We work with so-called $\mathsf{a};q$-weights. These are important special cases of the elliptic weights that were originally introduced by the second author in [18] in a closely related context and made further appearance in a series of papers devoted to elliptic combinatorics (see [3, 19, 21] and references therein). In some instances the enumeration with respect to elliptic weights does not yield closed formulas, but the specialization to $\mathsf{a};q$-weights does. See in particular, [20], which shows how the $\mathsf{a};q$-enumeration of rook configurations can be used to obtain (or recover) summations for basic hypergeometric series. A similar feature seems to apply here as well.

Let a and q be indeterminates. For a non-negative integer n, define the $\mathsf{a};q$-weight by

$$W_{\mathsf{a};q}(n) = \frac{1-\mathsf{a}q^{1+2n}}{1-\mathsf{a}q}q^{-n}, \tag{A.1a}$$

and the $\mathsf{a};q$-number by

$$[n]_{\mathsf{a};q} := \frac{(1-q^n)(1-\mathsf{a}q^n)}{(1-q)(1-\mathsf{a}q)}q^{1-n}. \tag{A.1b}$$

Clearly, $W_{\mathsf{a};q}(0)=1$. It is easy to see that the sum of the $\mathsf{a};q$-weights telescope to the $\mathsf{a};q$-numbers:

$$\sum_{k=0}^{n-1} W_{\mathsf{a};q}(k) = [n]_{\mathsf{a};q}.$$

For $\mathsf{a}\to\infty$, the $\mathsf{a};q$-weight in (A.1a) and the $\mathsf{a};q$-number in (A.1b) reduce to the q-weight q^n and to the standard q-number $[n]_q$, respectively. (Further, for $\mathsf{a}\to -1$ the $\mathsf{a};q$-number in (A.1b) reduces to the "quantum number" $(q^n-q^{-n})/(q-q^{-1})$, which is a q-analogue of n that satisfies the symmetry $q\leftrightarrow q^{-1}$.)

We now define an $\mathsf{a};q$-analogue of the down-degree expectation $\mathbb{E}(X)$. Recall that for each partition x, the number of cells of x is denoted by $|x|$.

Definition A.1. Let λ be a fixed Young diagram. Write $w=\lambda_1$ and $\ell=\ell(\lambda)$ for the width and length of λ, respectively, and let $d=\gcd(w,\ell)$. For $x,y\in[\emptyset,\lambda]$ write $y\lessdot x$ if y is covered by x in this lower interval. If y is obtained from x by deleting a cell in row s, define the weight $\mathrm{wt}(x,y)$ by

$$\mathrm{wt}(x,y) = W_{\mathsf{a}q^{\frac{w+\ell}{d}};q}\left(\frac{w(s-1)+\ell(|x|-1)}{d}\right).$$

Define $\mathrm{ddeg}_{\mathsf{a;q}}(x)$ to be the sum of the weights $\mathrm{wt}(x,y)$ for all $y\in[\emptyset,\lambda]$ satisfying $y\lessdot x$. Further, define the $\mathsf{a};q$-weight generating function of $[\emptyset,\lambda]$ by

$$R(\lambda|\,\mathsf{a};q) = \sum_{x\in[\emptyset,\lambda]} W_{\mathsf{a}q^{\frac{w\ell}{d}};q}\left(\frac{\ell|x|}{d}\right).$$

Now define $\mathbb{E}_{\mathsf{a};q}(X)$ by

$$\mathbb{E}_{\mathsf{a};q}(X) = \frac{\sum_{x\in[\emptyset,\lambda]}\mathrm{ddeg}_{\mathsf{a;q}}(x)}{R(\lambda|\,\mathsf{a};q)}. \tag{A.2}$$

Notice that for $\mathsf{a}\to\infty$, the $\mathsf{a};q$-weight generating function $R(\lambda|\,\mathsf{a};q)$ reduces to $R\big(\lambda,q^{\frac{\ell}{d}}\big)$, not to $R(\lambda,q)$. With the above definitions the following conjecture is easy to state and takes a symmetric form.

Conjecture A.2. If λ is a balanced partition (of any slope), then the product formula

$$\mathbb{E}_{\mathsf{a};q}(X) = \frac{\left[\frac{w\ell}{d}\right]_{\mathsf{a}q^{\frac{w+\ell}{d}};q}}{\left[\frac{w+\ell}{d}\right]_{\mathsf{a}q^{\frac{w\ell}{d}};q}} \tag{A.3}$$

holds, where $w=\lambda_1$, $\ell=\ell(\lambda)$, and $d=\gcd(w,\ell)$.

It is not difficult to verify directly, by using the definition for the $\mathsf{a};q$-numbers, that (A.3) can be alternatively written as

$$\mathbb{E}_{\mathsf{a};q}(X) = \frac{\left[\frac{w\ell}{d}\right]_{\mathsf{a}q;q^{\frac{w+\ell}{d}}}}{\left[\frac{w+\ell}{d}\right]_{\mathsf{a}q;q^{\frac{w\ell}{d}}}}. \tag{A.4}$$

It is interesting to see from (A.3) (or from (A.4)) that for balanced partitions λ, the down-degree expectation $\mathbb{E}_{\mathsf{a};q}(X)$ on $[\emptyset,\lambda]$ is the same as on the poset $[\emptyset,\lambda']$, the lower interval poset related to the conjugate λ' of λ. This symmetry is not obvious from (A.2) and an explanation of this fact itself would be desirable.

Conjecture A.2 is easy to confirm in the cases of λ consisting of one row or of one column, by induction. In the case of λ consisting of one row (resp. one column), the numerator in (A.2) simplifies to the numerator in (A.3) (resp. (A.4)), while the denominator in (A.2) then simplifies to the denominator in (A.3) (resp. (A.4)).

It is also easy to show that if $\lambda = (w^\ell)$ is a rectangular shape, the denominator $R(\lambda|\,\mathsf{a};q)$ simplifies into a product as follows. First of all, it is well-known (and corresponds to a classical result by MacMahon) that

$$\sum_{x\in[\emptyset,(w^\ell)]} q^{|x|} = \begin{bmatrix} w+\ell \\ \ell \end{bmatrix}_q,$$

where

$$\begin{bmatrix} w+\ell \\ \ell \end{bmatrix}_q = \frac{(q;q)_{w+\ell}}{(q;q)_w(q;q)_\ell}$$

is the q-binomial coefficient. By definition one can easily see that

$$\begin{bmatrix} w+\ell \\ \ell \end{bmatrix}_{q^{-1}} = \begin{bmatrix} w+\ell \\ \ell \end{bmatrix}_q q^{-w\ell}.$$

Now

$$\begin{aligned}\sum_{x\in[\emptyset,(w^\ell)]} W_{\mathsf{a};q}(|x|) &= \sum_{x\in[\emptyset,(w^\ell)]} \frac{1-\mathsf{a}q^{1+2|x|}}{1-\mathsf{a}q} q^{-|x|} = (1-\mathsf{a}q)^{-1}\left(\sum_{x\in[\emptyset,(w^\ell)]} q^{-|x|} - \mathsf{a}q \sum_{x\in[\emptyset,(w^\ell)]} q^{|x|}\right) \\ &= (1-\mathsf{a}q)^{-1}\left(\begin{bmatrix} w+\ell \\ \ell \end{bmatrix}_{q^{-1}} - \mathsf{a}q \begin{bmatrix} w+\ell \\ \ell \end{bmatrix}_q\right) = \begin{bmatrix} w+\ell \\ \ell \end{bmatrix}_q \frac{1-\mathsf{a}q^{1+w\ell}}{1-\mathsf{a}q} q^{-w\ell}.\end{aligned}$$

This immediately implies

$$R\big((w^\ell)|\,\mathsf{a};q\big) = \begin{bmatrix} w+\ell \\ \ell \end{bmatrix}_{q^{\frac{\ell}{d}}} \frac{1-\mathsf{a}q^{1+\frac{w\ell(\ell+1)}{d}}}{1-\mathsf{a}q^{1+\frac{w\ell}{d}}} q^{-\frac{w\ell^2}{d}}. \tag{A.5}$$

We finally give three concrete examples which illustrate cancellation of *non-linear* factors in the computation of $\mathbb{E}_{\mathsf{a};q}(X)$. (At the same time, they explain why the parameter a in the numerator and denominator is shifted differently. Without these shifts, the a-dependent factors would not cancel each other.)

Example A.3. Let $\lambda = (4,2)$. Then $(w,\ell,d) = (4,2,2)$ and

$$\begin{aligned}\sum_{x\in[\emptyset,\lambda]} \mathrm{ddeg}_{\mathsf{a};\mathsf{q}}(x) &= 0 + 1 + W_{\mathsf{a}q^3;q}(1) + 2W_{\mathsf{a}q^3;q}(2) + 3W_{\mathsf{a}q^3;q}(3) \\ &\quad + 3W_{\mathsf{a}q^3;q}(4) + 3W_{\mathsf{a}q^3;q}(5) + 2W_{\mathsf{a}q^3;q}(6) + W_{\mathsf{a}q^3;q}(7) \\ &= \frac{(1-q^4)(1+q+q^2+q^4-\mathsf{a}q^{11}-\mathsf{a}q^{13}-\mathsf{a}q^{14}-\mathsf{a}q^{15})}{(1-q)(1-\mathsf{a}q^4)} q^{-7},\end{aligned}$$

and

$$\begin{aligned}R(\lambda|\,\mathsf{a};q) &= 1 + W_{\mathsf{a}q^4;q}(1) + 2W_{\mathsf{a}q^4;q}(2) + 2W_{\mathsf{a}q^4;q}(3) + 3W_{\mathsf{a}q^4;q}(4) + 2W_{\mathsf{a}q^4;q}(5) + W_{\mathsf{a}q^4;q}(6) \\ &= \frac{(1-q^3)(1+q+q^2+q^4-\mathsf{a}q^{11}-\mathsf{a}q^{13}-\mathsf{a}q^{14}-\mathsf{a}q^{15})}{(1-q)(1-\mathsf{a}q^5)} q^{-6},\end{aligned}$$

so according to (A.2),

$$\mathbb{E}_{\mathsf{a};q}(X) = \frac{\sum_{x\in[\emptyset,\lambda]} \mathrm{ddeg}_{\mathsf{a};\mathsf{q}}(x)}{R(\lambda|\,\mathsf{a};q)} = \frac{(1-q^4)(1-\mathsf{a}q^5)}{(1-q^3)(1-\mathsf{a}q^4)} q^{-1} = \frac{[4]_{\mathsf{a}q^3;q}}{[3]_{\mathsf{a}q^4;q}},$$

which agrees with the $(w,\ell,d) = (4,2,2)$ case of (A.3).

Example A.4. Let $\lambda = (2,2,1,1)$. Then $(w,\ell,d) = (2,4,2)$ and

$$\begin{aligned}\sum_{x\in[\emptyset,\lambda]} \mathrm{ddeg}_{\mathsf{a};\mathsf{q}}(x) &= 0 + 1 + W_{\mathsf{a}q^3;q}(2) + W_{\mathsf{a}q^3;q}(3) + W_{\mathsf{a}q^3;q}(4) + W_{\mathsf{a}q^3;q}(5) + 2W_{\mathsf{a}q^3;q}(6) + W_{\mathsf{a}q^3;q}(7)\\ &\quad + 2W_{\mathsf{a}q^3;q}(8) + 2W_{\mathsf{a}q^3;q}(9) + W_{\mathsf{a}q^3;q}(10) + 2W_{\mathsf{a}q^3;q}(11) + W_{\mathsf{a}q^3;q}(13)\\ &= \frac{(1-q^4)(1-q^6)(1+q^2+q^4+q^8-\mathsf{a}q^{17}-\mathsf{a}q^{21}-\mathsf{a}q^{23}-\mathsf{a}q^{25})}{(1-q^2)(1-q^3)(1-\mathsf{a}q^4)}q^{-13},\end{aligned}$$

and

$$\begin{aligned}R(\lambda|\,\mathsf{a};q) &= 1 + W_{\mathsf{a}q^4;q}(2) + 2W_{\mathsf{a}q^4;q}(4) + 2W_{\mathsf{a}q^4;q}(6) + 3W_{\mathsf{a}q^4;q}(8) + 2W_{\mathsf{a}q^4;q}(10) + W_{\mathsf{a}q^4;q}(12)\\ &= \frac{(1-q^6)(1+q^2+q^4+q^8-\mathsf{a}q^{17}-\mathsf{a}q^{21}-\mathsf{a}q^{23}-\mathsf{a}q^{25})}{(1-q^2)(1-\mathsf{a}q^5)}q^{-12},\end{aligned}$$

so according to (A.2),

$$\mathbb{E}_{\mathsf{a};q}(X) = \frac{\sum_{x\in[\emptyset,\lambda]}\mathrm{ddeg}_{\mathsf{a};\mathsf{q}}(x)}{R(\lambda|\,\mathsf{a};q)} = \frac{(1-q^4)(1-\mathsf{a}q^5)}{(1-q^3)(1-\mathsf{a}q^4)}q^{-1} = \frac{[4]_{\mathsf{a}q^3;q}}{[3]_{\mathsf{a}q^4;q}},$$

which agrees with the $(w,\ell,d) = (2,4,2)$ case of (A.3).

Example A.5. Let $\lambda = (3,2,1)$. Then $(w,\ell,d) = (3,3,3)$ and

$$\begin{aligned}\sum_{x\in[\emptyset,\lambda]} \mathrm{ddeg}_{\mathsf{a};\mathsf{q}}(x) &= 0 + 1 + W_{\mathsf{a}q^2;q}(1) + 3W_{\mathsf{a}q^2;q}(2) + 3W_{\mathsf{a}q^2;q}(3)\\ &\quad + 5W_{\mathsf{a}q^2;q}(4) + 4W_{\mathsf{a}q^2;q}(5) + 3W_{\mathsf{a}q^2;q}(6) + W_{\mathsf{a}q^2;q}(7)\\ &= \frac{(1-q^3)(1+2q+q^2+2q^3+q^5-\mathsf{a}q^{10}-2\mathsf{a}q^{12}-\mathsf{a}q^{13}-2\mathsf{a}q^{14}-\mathsf{a}q^{15})}{(1-q)(1-\mathsf{a}q^3)}q^{-7},\end{aligned}$$

and

$$\begin{aligned}R(\lambda|\,\mathsf{a};q) &= 1 + W_{\mathsf{a}q^3;q}(1) + 2W_{\mathsf{a}q^3;q}(2) + 3W_{\mathsf{a}q^3;q}(3) + 3W_{\mathsf{a}q^3;q}(4) + 3W_{\mathsf{a}q^3;q}(5) + W_{\mathsf{a}q^3;q}(6)\\ &= \frac{(1-q^2)(1+2q+q^2+2q^3+q^5-\mathsf{a}q^{10}-2\mathsf{a}q^{12}-\mathsf{a}q^{13}-2\mathsf{a}q^{14}-\mathsf{a}q^{15})}{(1-q)(1-\mathsf{a}q^4)}q^{-6}.\end{aligned}$$

Then according to (A.2),

$$\mathbb{E}_{\mathsf{a};q}(X) = \frac{\sum_{x\in[\emptyset,\lambda]}\mathrm{ddeg}_{\mathsf{a};\mathsf{q}}(x)}{R(\lambda|\,\mathsf{a};q)} = \frac{(1-q^3)(1-\mathsf{a}q^4)}{(1-q^2)(1-\mathsf{a}q^3)}q^{-1} = \frac{[3]_{\mathsf{a}q^2;q}}{[2]_{\mathsf{a}q^3;q}},$$

which agrees with the $(w,\ell,d) = (3,3,3)$ case of (A.3).

Remark A.6. Recently, Hopkins, Lazar and Linusson [13] established a q-counting formula for barely set-valued plane partitions of height one by introducing a major-like index which they called the "barely set-valued comajor index". It is not clear how their formula is related to our conjectured formula in Conjecture A.2. However, since we are considering a possible q-analogue of $\mathbb{E}(X)$ as opposed to that their construction corresponds to a q-analogue of $\mathbb{E}(Y)$, if we can find any relation between two formulas, it might lead us to define a q-deformation of the CDE property.

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Department of Mathematics, Sungkyunkwan University, Suwon 16420, South Korea
Email address: jangsookim@skku.edu

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
Email address: michael.schlosser@univie.ac.at

Department of Mathematics, Chungbuk National University, Cheongju 28644, South Korea
Email address: meesueyoo@chungbuk.ac.kr (✉)