A NONTERMINATING ${}_8\phi_7$ SUMMATION FOR THE ROOT SYSTEM C_r

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Abstract. Using multiple q-integrals and a determinant evaluation, we establish a nonterminating ${}_8\phi_7$ summation for the root system C_r. We also give some important specializations explicitly.

1. Introduction

Bailey's [6, Eq. (3.3)] nonterminating very-well-poised ${}_8\phi_7$ summation,

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a,\ q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix}; q, q\right] \\
&\quad + \frac{(aq, c, d, e, f, b/a, bq/c, bq/d, bq/e, bq/f; q)_\infty}{(a/b, aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a, b^2q/a; q)_\infty} \\
&\quad \times {}_8\phi_7\left[\begin{matrix} b^2/a,\ qb/\sqrt{a}, -qb/\sqrt{a}, b, bc/a, bd/a, be/a, bf/a \\ b/\sqrt{a}, -b/\sqrt{a}, bq/a, bq/c, bq/d, bq/e, bq/f \end{matrix}; q, q\right] \\
&\qquad = \frac{(aq, b/a, aq/cd, aq/ce, aq/cf, aq/de, aq/df, aq/ef; q)_\infty}{(aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a; q)_\infty},
\end{aligned} \tag{1.1}$$

where $a^2q = bcdef$ (cf. [9, Eq. (2.11.7)]), is one of the deepest results in the classical theory of basic hypergeometric series. It contains many important identities as special cases (such as the nonterminating ${}_3\phi_2$ summation, the terminating ${}_8\phi_7$ summation, and all their specializations including the q-binomial theorem). One way to derive (1.1) is to start with a particular rational function identity, namely Bailey's [5] very-well-poised ${}_{10}\phi_9$ transformation, and apply a nontrivial limit procedure, see the exposition in Gasper and Rahman [9, Secs. 2.10 and 2.11].

Basic hypergeometric series (and, more generally, q-series) have various applications in combinatorics, number theory, representation theory, statistics, and physics, see Andrews [1], [2]. For a general account of the importance of basic hypergeometric series in the theory of special functions see Andrews, Askey, and Roy [4].

There are different types of multivariable series. The one we are concerned with are so-called *multiple basic hypergeometric series associated to root systems* (or, equivalently, to *Lie algebras*). This is mainly just a classification of certain multiple series according to the type of specific factors (such as a Vandermonde determinant)

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appearing in the summand. We omit giving a precise definition here, but instead refer to papers of Bhatnagar [7] or Milne [15, Sec. 5].

The significance of the nonterminating ${}_8\phi_7$ summation (1.1) lies in the fact that it can be used for deriving other nonterminating transformation formulae, see Gasper and Rahman [9, Secs. 2.12 and 3.8], and Schlosser [18]. Thus, it is apparently desirable to find (various) multivariable generalizations of Bailey's nonterminating ${}_8\phi_7$ summation.

In this paper, we give a multivariable nonterminating ${}_8\phi_7$ summation for the root system C_r (or, equivalently, the symplectic group $Sp(r)$), see Corollary 5.1. We deduce this result from an equivalent multiple q-integral evaluation, Theorem 4.1. In our proof of the latter we utilize a simple determinant method, essentially the same which was introduced by Gustafson and Krattenthaler [11] and which we further exploited in [17] to derive a number of identities for multiple basic hypergeometric series. The difference here is that now we apply the method to *integrals* and *q-integrals* whereas in [17] we had only applied it to sums. Our new C_r nonterminating ${}_8\phi_7$ summation is not the first multivariable nonterminating ${}_8\phi_7$ sum that has been found. In fact, Degenhardt and Milne [8] already derived such a result for the root system A_n (or, equivalently, the unitary group $U(n)$), a result we consider to be deeper than ours. While Corollary 5.1 is derived by elementary means, by combining known one-variable results with the argument of interchanging the order of summations, or of summation and (q-)integration (this is what the determinant method in this article really does), Degenhardt and Milne deduce their multivariable summation formula by extending Gasper and Rahman's [9, Sec. 2.10] analysis to higher dimensions which appears to be fairly nontrivial. However, supported by the combinatorial applications (in Krattenthaler [14], and Gessel and Krattenthaler [10]) of identities of type strikingly similar to the one being investigated in this paper, we believe that the identities derived here will have future applications and deserve being written out in detail.

Our paper is organized as follows: In Section 2, we review some basics in the theory of basic hypergeometric series. Further, we also note a determinant lemma which we need as an ingredient in proving our results in Sections 3 and 4. We demonstrate the method of proof in Section 3 by deriving a simple multidimensional beta integral evaluation. In Section 4, we derive an (attractive) multiple q-integral evauation, Theorem 4.1, which in Section 5 is used to explicitly write out a nonterminating ${}_8\phi_7$ summation for the root system C_r, see Corollary 5.1. Finally, in Section 6 we explicitly list several interesting specializations of Theorem 4.1 (and of the equivalent Corollary 5.1).

2. Basic hypergeometric series and a determinant lemma

Here we recall some standard notation for q-series, and basic hypergeometric series (cf. [9]).

Let q be a complex number such that $0 < |q| < 1$. We define the *q-shifted factorial* for all integers k by

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j) \qquad \text{and} \qquad (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

For brevity, we employ the condensed notation

$$(a_1,\dots,a_m;q)_k \equiv (a_1;q)_k \dots (a_m;q)_k$$

where k is an integer or infinity. Further, we utilize

$$ {}_s\phi_{s-1}\left[\begin{matrix} a_1, a_2, \dots, a_s \\ b_1, b_2, \dots, b_{s-1} \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_s; q)_k}{(q, b_1, \dots, b_{s-1}; q)_k} z^k, \tag{2.1} $$

to denote the *basic hypergeometric* $_s\phi_{s-1}$ *series.* In (2.1), $a_1, \dots, a_s$ are called the *upper parameters*, $b_1, \dots, b_{s-1}$ the *lower parameters*, z is the *argument*, and q the *base* of the series. The series in (2.1) terminates if one of the upper parameters, say a_s, equals q^{-n} where n is a nonnegeative integer. If the series does not terminate, we need $|z| < 1$ for convergence.

The classical theory of basic hypergeometric series contains numerous summation and transformation formulae involving $_s\phi_{s-1}$ series. Many of these summation theorems require that the parameters satisfy the condition of being either balanced and/or very-well-poised. An $_s\phi_{s-1}$ basic hypergeometric series is called *balanced* if $b_1 \cdots b_{s-1} = a_1 \cdots a_s q$ and $z = q$. An $_s\phi_{s-1}$ series is *well-poised* if $a_1 q = a_2 b_1 = \cdots = a_s b_{s-1}$ and *very-well-poised* if it is well-poised and if $a_2 = -a_3 = q\sqrt{a_1}$. Note that the factor

$$ \frac{1 - a_1 q^{2k}}{1 - a_1} $$

appears in a very-well-poised series. The parameter a_1 is usually referred to as the *special parameter* of such a series.

One of the most important summation theorems in the theory of basic hypergeometric series is Jackson's [13] terminating very-well-poised balanced $_8\phi_7$ summation (cf. [9, Eq. (2.6.2)]):

$$ {}_8\phi_7\left[\begin{matrix} a,\ q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2q^{1+n}/bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcdq^{-n}/a, aq^{1+n} \end{matrix}; q, q\right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}. \tag{2.2} $$

Clearly, (2.2) is the special case $f \to q^{-n}$ of (1.1).

For studying nonterminating basic hypergeometric series it is often convenient to utilize Jackson's [12] q-integral notation, defined by

$$ \int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t, \tag{2.3} $$

where

$$ \int_0^a f(t)\, d_q t = a(1-q) \sum_{k=0}^{\infty} f(aq^k) q^k. \tag{2.4} $$

If f is continuous on $[0, a]$, then it is easily seen that

$$ \lim_{q \to 1^-} \int_0^a f(t)\, d_q t = \int_0^a f(t)\, dt, $$

see [9, Eq. (1.11.6)].

Using the above q-integral notation, the nonterminating $_8\phi_7$ summation (1.1) can be conveniently expressed as

$$ \int_a^b \frac{(qt/a, qt/b, t/\sqrt{a}, -t/\sqrt{a}, qt/c, qt/d, qt/e, qt/f; q)_\infty}{(t, bt/a, qt/\sqrt{a}, -qt/\sqrt{a}, ct/a, dt/a, et/a, ft/a; q)_\infty}\, d_q t $$

$$= \frac{b(1-q)(q, a/b, bq/a, aq/cd, aq/ce, aq/cf, aq/de, aq/df, aq/ef; q)_\infty}{(b, c, d, e, f, bc/a, bd/a, be/a, bf/a; q)_\infty}, \quad (2.5)$$

where $a^2q = bcdef$ (cf. [9, Eq. (2.11.8)]).

A standard reference for basic hypergeometric series is Gasper and Rahman's text [9]. In our computations in the subsequent sections we frequently use some elementary identities of q-shifted factorials, listed in [9, Appendix I].

The following determinant evaluation was given as Lemma A.1 in [17] where it was derived from a determinant lemma of Krattenthaler [14, Lemma 34].

Lemma 2.1. *Let $X_1, \dots, X_r$, A, B, and C be indeterminate. Then there holds*

$$\det_{1\le i,j\le r}\left(\frac{(AX_i, AC/X_i; q)_{r-j}}{(BX_i, BC/X_i; q)_{r-j}}\right) = \prod_{1\le i<j\le r}(X_j - X_i)(1 - C/X_iX_j)$$
$$\times A^{\binom{r}{2}} q^{\binom{r}{3}} \prod_{i=1}^{r} \frac{(B/A, ABCq^{2r-2i}; q)_{i-1}}{(BX_i, BC/X_i; q)_{r-1}}. \quad (2.6)$$

The above determinant evaluation was generalized to the elliptic case (more precisely, to an evaluation involving Jacobi theta functions) by Warnaar [20, Cor. 5.4].

3. A multidimensional beta integral evaluation

Here, we present a simple multivariable extension of Euler's beta integral evaluation. The proof serves as an illustration of the determinant method which we use in Section 4 to derive a multivariable extension of (2.5).

Proposition 3.1. *Let a, b, and $x_1, \dots, x_r$ be indeterminate. Then*

$$\int_0^1 \dots \int_0^1 \prod_{1\le i<j\le r}(u_i - u_j) \prod_{i=1}^{r} u_i^{a-1+x_i}(1-u_i)^{b-1}\, du_r \dots du_1$$
$$= \prod_{1\le i<j\le r}(x_i - x_j) \prod_{i=1}^{r} \frac{\Gamma(a+x_i)\,\Gamma(b+i-1)}{\Gamma(a+b+x_i+r-1)}, \quad (3.1)$$

provided $\Re(a+x_i), \Re(b) > 0$, for $i = 1, \dots, r$.

Remark 3.2. We note the differences between Proposition 3.1 and Selberg's [19] integral,

$$\int_0^1 \dots \int_0^1 \prod_{1\le i<j\le r}|u_i - u_j|^{2c} \prod_{i=1}^{r} u_i^{a-1}(1-u_i)^{b-1}\, du_r \dots du_1$$
$$= \prod_{i=1}^{r} \frac{\Gamma(a+(i-1)c)\,\Gamma(b+(i-1)c)\,\Gamma(ic+1)}{\Gamma(a+b+(r+i-2)c)\,\Gamma(c+1)}, \quad (3.2)$$

where $\Re(a), \Re(b) > 0$, and $\Re(c) > \max(-1/r, -\Re(a)/(r-1), -\Re(b)/(r-1))$. In (3.1) we have additional parameters $x_1, \dots, x_r$, while in (3.2) the absolute value of the discriminant $\prod_{1\le i<j\le r}(u_i - u_j)$ in the integrand is taken to an arbitrary power $2c$, which makes the computation considerably more difficult.

Proof of Proposition 3.1. First, we note that if in Lemma 2.1 we let $C \to 0$, replace A, B, and X_i by q^a, q^{a+b}, and q^{x_i}, for $i = 1, \dots, r$, respectively, and then let $q \to 1$, we have

$$\det_{1\le i,j\le r}\left(\frac{(a+x_i)_{r-j}}{(a+b+x_i)_{r-j}}\right) = \prod_{1\le i<j\le r}(x_i - x_j)\prod_{i=1}^r \frac{(b)_{i-1}}{(a+b+x_i)_{r-1}}, \tag{3.3}$$

where

$$(\alpha)_k := \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \tag{3.4}$$

is the *shifted factorial*. Thus,

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 \prod_{1\le i<j\le r}(u_i - u_j)\prod_{i=1}^r u_i^{a-1+x_i}(1-u_i)^{b-1}\, du_r \dots du_1\\
&= \det_{1\le i,j\le r}\left(\int_0^1 u_i^{a-1+x_i+r-j}(1-u_i)^{b-1}\, du_i\right)\\
&= \det_{1\le i,j\le r}\left(\frac{\Gamma(a+x_i+r-j)\,\Gamma(b)}{\Gamma(a+b+x_i+r-j)}\right) = \det_{1\le i,j\le r}\left(\frac{\Gamma(a+x_i)\,\Gamma(b)}{\Gamma(a+b+x_i)}\frac{(a+x_i)_{r-j}}{(a+b+x_i)_{r-j}}\right)\\
&= \prod_{i=1}^r \frac{\Gamma(a+x_i)\,\Gamma(b)}{\Gamma(a+b+x_i)}\cdot \det_{1\le i,j\le r}\left(\frac{(a+x_i)_{r-j}}{(a+b+x_i)_{r-j}}\right)\\
&= \prod_{i=1}^r \frac{\Gamma(a+x_i)\,\Gamma(b)}{\Gamma(a+b+x_i)}\prod_{1\le i<j\le r}(x_i - x_j)\prod_{i=1}^r \frac{(b)_{i-1}}{(a+b+x_i)_{r-1}}\\
&= \prod_{1\le i<j\le r}(x_i - x_j)\prod_{i=1}^r \frac{\Gamma(a+x_i)\,\Gamma(b+i-1)}{\Gamma(a+b+x_i+r-1)},
\end{aligned}$$

where we have used linearity of the determinant with respect to rows, the Vandermonde determinant evaluation $\det_{1\le i,j\le r}(u_i^{r-j}) = \prod_{1\le i<j\le r}(u_i - u_j)$, Euler's beta integral evaluation $\int_0^1 u^{a-1}(1-u)^{b-1}dt = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$, for $\Re(a), \Re(b) > 0$, the definition of the shifted factorial (3.4), and the determinant evaluation (3.3). □

4. A multiple q-integral evaluation

By iteration, the extension of (2.4) to *multiple* q-integrals is straightforward:

$$\begin{aligned}
&\int_0^{a_1} \cdots \int_0^{a_r} f(t_1, \dots, t_r)\, d_q t_r \dots d_q t_1\\
&= a_1 \dots a_r (1-q)^r \sum_{k_1,\dots,k_r=0}^{\infty} f(a_1 q^{k_1}, \dots, a_r q^{k_r})\, q^{k_1+\dots+k_r}.
\end{aligned} \tag{4.1}$$

Similarly, the extension of (2.3) is

$$\begin{aligned}
&\int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} f(t_1, \dots, t_r)\, d_q t_r \dots d_q t_1\\
&= \sum_{S\subseteq\{1,2,\dots,r\}} \left(\prod_{i\in S}(-a_i)\right)\left(\prod_{i\notin S} b_i\right)(1-q)^r
\end{aligned}$$

$$\times \sum_{k_1,\dots,k_r=0}^{\infty} f(c_1(S)q^{k_1},\dots,c_r(S)q^{k_r})q^{k_1+\dots+k_r}, \quad (4.2)$$

where the outer sum runs over all 2^r subsets S of $\{1,2,\dots,r\}$, and where $c_i(S)=a_i$ if $i\in S$ and $c_i(S)=b_i$ if $i\notin S$, for $i=1,\dots,r$.

We give our main result, a C_r extension of (2.5):

Theorem 4.1. *Let $a^2q^{2-r}=bcdef$. Then there holds*

$$\int_{ax_1}^{b}\cdots\int_{ax_r}^{b}\prod_{1\le i<j\le r}(t_i-t_j)(1-t_it_j/a)\prod_{i=1}^{r}(1-t_i^2/a)$$
$$\times\prod_{i=1}^{r}\frac{(qt_i/ax_i,qt_i/b,qt_i/c,qt_i/d,qt_i/e,qt_ix_i/f;q)_\infty}{(t_ix_i,bt_i/a,ct_i/a,dt_i/a,et_i/a,ft_i/ax_i;q)_\infty}\,d_qt_r\dots d_qt_1$$
$$=a^{\binom{r}{2}}b^r(1-q)^r\prod_{1\le i<j\le r}(x_i-x_j)(1-ax_ix_j/f)$$
$$\times\prod_{i=1}^{r}\frac{(q,ax_i/b,bq/ax_i,aq^{2-i}/cd,aq^{2-i}/ce;q)_\infty}{(bx_i,cx_i,dx_i,ex_i,f;q)_\infty}$$
$$\times\prod_{i=1}^{r}\frac{(ax_iq/cf,aq^{2-i}/de,ax_iq/df,ax_iq/ef;q)_\infty}{(bcq^{i-1}/a,bdq^{i-1}/a,beq^{i-1}/a,bf/ax_i;q)_\infty}. \quad (4.3)$$

Proof. We have

$$\prod_{1\le i<j\le r}(t_i-t_j)(1-t_it_j/a)=\prod_{i=1}^{r}\frac{(q^{2-r}t_i/d,aq^{2-r}/dt_i;q)_{r-1}}{(aq^{2-r}/cd,cq^{2+r-2i}/d;q)_{i-1}}t_i^{r-1}$$
$$\times c^{-\binom{r}{2}}q^{-\binom{r}{3}}\det_{1\le i,j\le r}\left(\frac{(ct_i/a,c/t_i;q)_{r-j}}{(q^{2-r}t_i/d,aq^{2-r}/dt_i;q)_{r-j}}\right),$$

due to the $X_i\mapsto t_i$, $A\mapsto c/a$, $B\mapsto q^{2-r}/d$, and $C\mapsto a$ case of Lemma 2.1. Hence, using some elementary identities from [9, Appendix I], we may write the left hand side of (4.3) as

$$\left(\frac{a}{d}\right)^{\binom{r}{2}}q^{-\binom{r}{3}}\prod_{i=1}^{r}(aq^{2-r}/cd,cq^{2+r-2i}/d;q)_{i-1}^{-1}$$
$$\times\det_{1\le i,j\le r}\left(\int_{ax_i}^{b}\frac{(1-t_i^2/a)(t_iq/ax_i,t_iq/b;q)_\infty}{(t_ix_i,bt_i/a;q)_\infty}\right.$$
$$\left.\times\frac{(t_iq^{1-r+j}/c,t_iq^{2-j}/d,t_iq/e,t_ix_iq/f;q)_\infty}{(ct_iq^{r-j}/a,dt_iq^{j-1}/a,et_i/a,ft_i/ax_i;q)_\infty}\,d_qt_i\right).$$

Now, to the integral inside the determinant we apply the q-integral evaluation (2.5), with the substitution $t\mapsto t_ix_i$, and the replacements $a\mapsto ax_i^2$, $b\mapsto bx_i$, $c\mapsto cq^{r-j}x_i$, $d\mapsto dq^{j-1}x_i$, and $e\mapsto ex_i$. Thus we obtain

$$\left(\frac{a}{d}\right)^{\binom{r}{2}}q^{-\binom{r}{3}}\prod_{i=1}^{r}(aq^{2-r}/cd,cq^{2+r-2i}/d;q)_{i-1}^{-1}$$

$$\times \det_{1\le i,j\le r}\left(\frac{b(1-q)(q,ax_i/b,bq/ax_i,aq^{2-r}/cd,aq^{1-r+j}/ce;q)_\infty}{(bx_i,cx_iq^{r-j},dx_iq^{j-1},ex_i,f;q)_\infty}\right.$$
$$\left.\times\frac{(ax_iq^{1-r+j}/cf,aq^{2-j}/de,ax_iq^{2-j}/df,ax_iq/ef;q)_\infty}{(bcq^{r-j}/a,bdq^{j-1}/a,be/a,bf/ax_i;q)_\infty}\right).$$

Now, by using linearity of the determinant with respect to rows and columns, we take some factors out of the determinant and obtain

$$\left(\frac{a}{d}\right)^{\binom{r}{2}} q^{-\binom{r}{3}} b^r(1-q)^r \prod_{i=1}^r \frac{(q,ax_i/b,bq/ax_i;q)_\infty}{(aq^{2-r}/cd,cq^{2+r-2i}/d;q)_{i-1}}$$
$$\times\prod_{i=1}^r \frac{(aq^{2-r}/cd,aq^{1-r+i}/ce,ax_iq/cf,aq^{2-i}/de,ax_iq^{2-r}/df,ax_iq/ef;q)_\infty}{(bx_i,cx_i,dx_iq^{r-1},ex_i,f,bcq^{r-i}/a,bdq^{i-1}/a,be/a,bf/ax_i;q)_\infty}$$
$$\times\left(\frac{a}{cdf}\right)^{\binom{r}{2}} q^{-3\binom{r}{3}} \det_{1\le i,j\le r}\left(\frac{(cx_i,cf/ax_i;q)_{r-j}}{(ax_iq^{2-r}/df,q^{2-r}/dx_i;q)_{r-j}}\right).$$

The determinant can be evaluated by means of Lemma 2.1 with $X_i\mapsto x_i$, $A\mapsto c$, $B\mapsto aq^{2-r}/df$, and $C\mapsto f/a$; specifically

$$\det_{1\le i,j\le r}\left(\frac{(cx_i,cf/ax_i;q)_{r-j}}{(ax_iq^{2-r}/df,q^{2-r}/dx_i;q)_{r-j}}\right)$$
$$=c^{\binom{r}{2}}q^{\binom{r}{3}}\prod_{1\le i<j\le r}(x_j-x_i)(1-f/ax_ix_j)\prod_{i=1}^r\frac{(aq^{2-r}/cdf,cq^{2+r-2i}/d;q)_{i-1}}{(ax_iq^{2-r}/df,q^{2-r}/dx_i;q)_{r-1}}.$$

Substituting our calculations and performing further elementary manipulations we arrive at the right hand side of (4.3). □

5. A multivariable nonterminating ${}_8\phi_7$ summation

Note that if the integrand $f(t_1,\dots,t_r)$ of the multiple integral in (4.3) were an antisymmetric function in $t_1,\dots,t_r$, the multiple sum in (4.2) would simplify considerably. In fact, if $t_i=bq^{k_i}$ and $t_j=bq^{k_j}$, for a pair $i<j$, we would then have

$$\sum_{k_i,k_j=0}^\infty f(\dots,bq^{k_i},\dots,bq^{k_j},\dots)=0.$$

(A double sum of any function antisymmetric in its two summation indices vanishes.) Thus, the multiple q-integral $\int_{ax_1}^b\dots\int_{ax_r}^b f(t_1,\dots,t_r)\,d_qt_r\dots d_qt_1$, being a sum of 2^r sums according to (4.2), would reduce to a sum of $r+1$ nonzero sums. In particular, we would have

$$\int_{ax_1}^b\dots\int_{ax_r}^b f(t_1,\dots,t_r)\,d_qt_r\dots d_qt_1$$
$$=(-1)^ra^rx_1\dots x_r(1-q)^r\sum_{k_1,\dots,k_r=0}^\infty f(ax_1q^{k_1},\dots,ax_rq^{k_r})\,q^{\sum_{i=1}^r k_i}$$
$$+(-1)^{r-1}a^{r-1}bx_1\dots x_r(1-q)^r\sum_{l=1}^r x_l^{-1}$$

$$\times \sum_{k_1,\dots,k_r=0}^{\infty} f(ax_1q^{k_1},\dots,ax_{l-1}q^{k_{l-1}},bq^{k_l},ax_{l+1}q^{k_{l+1}},\dots,ax_rq^{k_r})\, q^{\sum_{i=1}^r k_i}. \tag{5.1}$$

A very similar situation occurs in the $U(n)$ (or A_r) nonterminating ${}_8\phi_7$ summation by Degenhardt and Milne [8] (however, their argument is reversed, i.e., they first derive a nonterminating summation and then deduce the multiple q-integral evaluation). Unfortunately, in our case the multiple integrand in (4.3) is *not* antisymmetric in $t_1,\dots,t_r$, whence we have all 2^r sums on the right hand side of (4.2).

We write out (4.2) explicitly for our integral in (4.3):

$$\int_{ax_1}^{b}\cdots\int_{ax_r}^{b}\prod_{1\le i<j\le r}(t_i-t_j)(1-t_it_j/a)\prod_{i=1}^r(1-t_i^2/a)$$
$$\times\prod_{i=1}^r\frac{(qt_i/ax_i,qt_i/b,qt_i/c,qt_i/d,qt_i/e,qt_ix_i/f;q)_\infty}{(t_ix_i,bt_i/a,ct_i/a,dt_i/a,et_i/a,ft_i/ax_i;q)_\infty}\,d_qt_r\dots d_qt_1$$
$$=\sum_{S\subseteq\{1,2,\dots,r\}}(-1)^{|S|}a^{|S|}b^{r-|S|}(1-q)^ra^{\binom{|S|}{2}}b^{\binom{r-|S|}{2}}\prod_{i\in S}x_i$$
$$\times\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{\substack{1\le i<j\le r\\ i,j\in S}}(x_iq^{k_i}-x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})\prod_{i\in S}(1-ax_i^2q^{2k_i})$$
$$\times\prod_{\substack{1\le i<j\le r\\ i,j\notin S}}(q^{k_i}-q^{k_j})(1-b^2q^{k_i+k_j}/a)\prod_{i\notin S}(1-b^2q^{2k_i}/a)$$
$$\times\prod_{i\in S,j\notin S}(ax_iq^{k_i}-bq^{k_j})(1-bx_iq^{k_i+k_j})(-1)^{\chi(i>j)}$$
$$\times\prod_{i\in S}\frac{(q^{1+k_i},ax_iq^{1+k_i}/b,ax_iq^{1+k_i}/c,ax_iq^{1+k_i}/d,ax_iq^{1+k_i}/e,ax_i^2q^{1+k_i}/f;q)_\infty}{(ax_i^2q^{k_i},bx_iq^{k_i},cx_iq^{k_i},dx_iq^{k_i},ex_iq^{k_i},fq^{k_i};q)_\infty}$$
$$\times\prod_{i\notin S}\frac{(bq^{1+k_i}/ax_i,q^{1+k_i},bq^{1+k_i}/c,bq^{1+k_i}/d,bq^{1+k_i}/e,bx_iq^{1+k_i}/f;q)_\infty}{(bx_iq^{k_i},b^2q^{k_i}/a,bcq^{k_i}/a,bdq^{k_i}/a,beq^{k_i}/a,bfq^{k_i}/ax_i;q)_\infty}\cdot q^{\sum_{i=1}^r k_i},$$

where $|S|$ denotes the number of elements of S, and χ is the truth function (which evaluates to one if the argument is true and evaluates to zero otherwise). Now, if we set the obtained sum of 2^r sums equal to the right hand side of (4.3) and divide both sides by

$$(-1)^ra^{\binom{r+1}{2}}x_1\dots x_r(1-q)^r\prod_{1\le i<j\le r}(x_i-x_j)(1-ax_ix_j)$$
$$\times\prod_{i=1}^r\frac{(q,ax_iq/b,ax_iq/c,ax_iq/d,ax_iq/e,ax_i^2q/f;q)_\infty}{(ax_i^2q,bx_i,cx_i,dx_i,ex_i,f;q)_\infty},$$

and simplify, we obtain the following result which reduces to (1.1) when $r=1$.

Corollary 5.1 (A C_r nonterminating ${}_8\phi_7$ summation)**.** *Let $a^2q^{2-r}=bcdef$. Then there holds*

$$\sum_{S\subseteq\{1,2,\dots,r\}} \left(\frac{b}{a}\right)^{\binom{r-|S|}{2}} \prod_{i\notin S} \frac{(ax_i^2q, cx_i, dx_i, ex_i, f;q)_\infty}{(ax_i/b, ax_iq/c, ax_iq/d, ax_iq/e, ax_i^2q/f;q)_\infty}$$
$$\times \prod_{i\notin S} \frac{(b/ax_i, bq/c, bq/d, bq/e, bx_iq/f;q)_\infty}{(b^2q/a, bc/a, bd/a, be/a, bf/ax_i;q)_\infty}$$
$$\times \sum_{k_1,\dots,k_r=0}^{\infty} \prod_{\substack{1\le i<j\le r\\ i,j\in S}} \frac{(x_iq^{k_i}-x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)} \prod_{i\in S} \frac{(1-ax_i^2q^{2k_i})}{(1-ax_i^2)}$$
$$\times \prod_{\substack{1\le i<j\le r\\ i,j\notin S}} \frac{(q^{k_i}-q^{k_j})(1-b^2q^{k_i+k_j}/a)}{(x_i-x_j)(1-ax_ix_j)} \prod_{i\notin S} \frac{(1-b^2q^{2k_i}/a)}{(1-b^2/a)}$$
$$\times \prod_{i\in S, j\notin S} \frac{(x_iq^{k_i}-bq^{k_j}/a)(1-bx_iq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)}$$
$$\times \prod_{i\in S} \frac{(ax_i^2, bx_i, cx_i, dx_i, ex_i, f;q)_{k_i}}{(q, ax_iq/b, ax_iq/c, ax_iq/d, ax_iq/e, ax_i^2q/f;q)_{k_i}}$$
$$\times \prod_{i\notin S} \frac{(b^2/a, bx_i, bc/a, bd/a, be/a, bf/ax_i;q)_{k_i}}{(q, bq/ax_i, bq/c, bq/d, bq/e, bx_iq/f;q)_{k_i}} \cdot q^{\sum_{i=1}^r k_i}$$
$$= \prod_{1\le i<j\le r} \frac{(1-ax_ix_j/f)}{(1-ax_ix_j)} \prod_{i=1}^{r} \frac{(ax_i^2q, b/ax_i, aq^{2-i}/cd, aq^{2-i}/ce;q)_\infty}{(ax_iq/c, ax_iq/d, ax_iq/e, ax_i^2q/f;q)_\infty}$$
$$\times \prod_{i=1}^{r} \frac{(ax_iq/cf, aq^{2-i}/de, ax_iq/df, ax_iq/ef;q)_\infty}{(bcq^{i-1}/a, bdq^{i-1}/a, beq^{i-1}/a, bf/ax_i;q)_\infty}. \quad (5.2)$$

6. Specializations

It is clear that in Corollary 5.1, if we replace e by $a^2q^{2-r}/bcdf$ and then let $f\to q^{-N}$, we obtain a C_r extension of Jackson's $_8\phi_7$ summation (2.2). This result,

$$\sum_{k_1,\dots,k_r=0}^{N} \prod_{1\le i<j\le r} \frac{(x_iq^{k_i}-x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)} \prod_{i=1}^{r} \frac{(1-ax_i^2q^{2k_i})}{(1-ax_i^2)}$$
$$\times \prod_{i=1}^{r} \frac{(ax_i^2, bx_i, cx_i, dx_i, a^2x_iq^{2-r+N}/bcd, q^{-N};q)_{k_i}}{(q, ax_iq/b, ax_iq/c, ax_iq/d, bcdx_iq^{r-1-N}/a, ax_i^2q^{1+N};q)_{k_i}} \cdot q^{\sum_{i=1}^r k_i}$$
$$= \prod_{1\le i<j\le r} \frac{(1-ax_ix_jq^N)}{(1-ax_ix_j)} \prod_{i=1}^{r} \frac{(ax_i^2q, aq^{2-i}/bc, aq^{2-i}/bd, aq^{2-i}/cd;q)_N}{(aq^{2-r}/bcdx_i, ax_iq/d, ax_iq/c, ax_iq/b;q)_N}, \quad (6.1)$$

was given as Theorem 4.3 in [17]. An extension of (6.1) to *elliptic* hypergeometric series was found by Warnaar [20, Theorem 5.1].

Next, if in (4.3), we first replace e by $a^2q^{2-r}/bcdf$, then b by aq/b, do the substitution $t_i\mapsto at_i$, divide both sides by $a^{\binom{r+1}{2}}$, let $a\to 0$, and afterwards do the simultaneous substitutions $x_i\mapsto x_i\sqrt{a}$, $b\mapsto q\sqrt{a}/b$, $c\mapsto cq^{1-r}/bdf\sqrt{a}$, $d\mapsto d\sqrt{a}$, $f\mapsto fa$, then multiply both sides by $\sqrt{a}^{\binom{r+1}{2}}$, and perform the substitution $t_i\mapsto t_i/\sqrt{a}$, we obtain the following multiple q-integral evaluation which is an r-dimensional extension of Eq. (2.10.18) in [9].

Theorem 6.1. *Let $c = abdefq^{r-1}$. Then there holds*

$$\int_{ax_1}^{b} \dots \int_{ax_r}^{b} \prod_{1\le i<j\le r} (t_i - t_j) \prod_{i=1}^{r} \frac{(qt_i/ax_i, qt_i/b, ct_i; q)_\infty}{(dt_i, et_i, ft_i/x_i; q)_\infty}\, d_qt_r \dots d_qt_1$$
$$= a^{\binom{r}{2}} b^r (1-q)^r \prod_{1\le i<j\le r} (x_i - x_j)$$
$$\times \prod_{i=1}^{r} \frac{(q, ax_i/b, bq/ax_i, cq^{1-i}/d, cq^{1-i}/e, cx_i/f; q)_\infty}{(adx_i, aex_i, af, bdq^{i-1}, beq^{i-1}, bf/x_i; q)_\infty}. \quad (6.2)$$

Similarly, we can specialize Corollary 5.1 to a multivariable nonterminating q-Pfaff–Saalschütz summation by first replacing b and e by aq/b and abq^{1-r}/cdf, respectively, then letting $a \to 0$, and finally performing the simultaneous substitutions $b \mapsto e$, $c \mapsto a$, $d \mapsto b$ and $f \mapsto c$. We obtain the following multivariable extension of Eq. (II.24) in [9].

Corollary 6.2 (An A_r nonterminating ${}_3\phi_2$ summation)**.** *Let $ef = abcq^r$. Then there holds*

$$\sum_{S\subseteq\{1,2,\dots,r\}} \left(\frac{q}{e}\right)^{\binom{r}{2}-\binom{|S|}{2}} \prod_{i\notin S} \frac{(ax_i, bx_i, c, q/ex_i, fq/e; q)_\infty}{(aq/e, bq/e, cq/ex_i, ex_i/q, fx_i; q)_\infty}$$
$$\times \sum_{k_1,\dots,k_r=0}^{\infty} \prod_{\substack{1\le i<j\le r\\ i,j\in S}} \frac{(x_iq^{k_i} - x_jq^{k_j})}{(x_i - x_j)} \prod_{\substack{1\le i<j\le r\\ i,j\notin S}} \frac{(q^{k_i} - q^{k_j})}{(x_i - x_j)} \prod_{i\in S, j\notin S} \frac{(ex_iq^{1+k_i} - q^{k_j})}{(x_i - x_j)}$$
$$\times \prod_{i\in S} \frac{(ax_i, bx_i, c; q)_{k_i}}{(q, ex_i, fx_i; q)_{k_i}} \prod_{i\notin S} \frac{(aq/e, bq/e, cq/ex_i; q)_{k_i}}{(q, q^2/ex_i, fq/e; q)_{k_i}} \cdot q^{\sum_{i=1}^r k_i}$$
$$= \prod_{i=1}^{r} \frac{(q/ex_i, fq^{1-i}/a, fq^{1-i}/b, fx_i/c; q)_\infty}{(aq^i/e, bq^i/e, cq/ex_i, fx_i; q)_\infty}. \quad (6.3)$$

It is again clear that Corollary 6.2 above can be specialized to an A_r terminating q-Pfaff–Saalschütz summation, Namely, by first replacing f by $abcq^r/e$ and then letting $c \to q^{-N}$, we obtain

$$\sum_{k_1,\dots,k_r=0}^{N} \prod_{1\le i<j\le r} \frac{(x_iq^{k_i} - x_jq^{k_j})}{(x_i - x_j)} \prod_{i=1}^{r} \frac{(ax_i, bx_i, cx_i, q^{-N}; q)_{k_i}}{(q, cx_i, abx_iq^{r-N}/c; q)_{k_i}} \cdot q^{\sum_{i=1}^r k_i}$$
$$= \prod_{i=1}^{r} \frac{(cq^{1-i}/a, cq^{1-i}/b; q)_N}{(cx_i, cq^{1-r}/abx_i; q)_N}, \quad (6.4)$$

which is Theorem 5.1 in [17].

Finally, we specialize Theorem 6.1 further, for possible future reference. We first replace f by $cq^{1-r}/abde$ and then let $c \to 0$ and replace a, d, and e by $-a$, $-c/a$, and d/b, respectively. The result is the following.

Corollary 6.3. *There holds*

$$\int_{-ax_1}^{b} \dots \int_{-ax_r}^{b} \prod_{1\le i<j\le r} (t_i - t_j) \prod_{i=1}^{r} \frac{(-qt_i/ax_i, qt_i/b; q)_\infty}{(-ct_i/a, dt_i/b; q)_\infty}\, d_qt_r \dots d_qt_1$$

$$= (-a)^{\binom{r}{2}} b^r (1-q)^r \prod_{1\le i<j\le r} (x_i - x_j) \prod_{i=1}^{r} \frac{(q, -ax_i/b, -bq/ax_i, cdq^{r-1}x_i; q)_\infty}{(cx_i, -adx_i/b, -bcq^{i-1}/a, dq^{i-1}; q)_\infty}. \tag{6.5}$$

Corollary 6.3 is a multivariable extension of a q-integral derived by Andrews and Askey [3]. Replacing a, b, c, d, and x_i by c, d, q^a, q^b, and q^{x_i}, for $i = 1, \dots, r$, and letting $q \to 1$, we obtain

$$\int_{-c}^{d} \cdots \int_{-c}^{d} \prod_{1\le i<j\le r} (t_i - t_j) \prod_{i=1}^{r} (1 + t_i/c)^{a-1+x_i} (1 - t_i/d)^{b-1} \, dt_r \dots dt_1$$
$$= \prod_{1\le i<j\le r} (x_i - x_j) \prod_{i=1}^{r} \frac{\Gamma(a + x_i)\,\Gamma(b + i - 1)}{\Gamma(a + b + r - a + x_i)}$$
$$\times c^{r(1-a)-\sum x_i} d^{r(1-b)} (c+d)^{r(a+b-1)+\binom{r}{2}+\sum x_i}, \tag{6.6}$$

which follows from the multiple beta integral evaluation in Proposition 3.1 by the substitutions

$$u_i \mapsto \frac{c + t_i}{c + d}, \qquad i = 1, \dots, r. \tag{6.7}$$

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