COMBINATORIAL FORMULAS FOR RAMANUJAN'S TAU FUNCTION

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In what follows, we shall use the standard q-series notation (cf. [14])

$$(a;q)_0 = 1, \qquad \text{and} \qquad (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

where $a \in \mathbb{C}$, $0 < |q| < 1$, and n is a positive integer or infinity.

Ramanujan's [25] tau function $\tau(n)$, defined as the coefficients in the following q-series,

$$\sum_{n\geq 1} \tau(n) q^n = q(q;q)_\infty^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \cdots, \tag{1}$$

satisfies many remarkable arithmetic properties, see [27]. Ramanujan conjectured that $\tau(n)$ is multiplicative (i.e. $\tau(mn) = \tau(m)\tau(n)$ for coprime m and n) and that for prime p and positive integers k the intriguing relation

$$\tau(p^{k+1}) = \tau(p)\tau(p^k) - p^{11}\tau(p^{k-1})$$

holds which soon afterwards was proved by Mordell [22] using Hecke operator techniques. Ramanujan [25, p. 176] further conjectured that the size of the coefficients is bounded by

$$|\tau(p)| \leq 2p^{\frac{11}{2}}. \tag{2}$$

This conjecture required considerably more efforts to settle; this was first achieved by Deligne [7] as a (non-obvious) consequence of his proof of the Weil conjectures.

The q-series in (1), for $q = e^{2\pi i z}$, can be interpreted as a function in z (whose Fourier coefficients are then $\tau(n)$), more precisely, as the 24-th power of the Dedekind eta function, which in the language of modular forms is a cusp form of weight 12. In 1930, Petersson [24] generalized Ramanujan's conjecture (2) to other modular and automorphic forms. This has become known as the generalized Ramanujan conjecture or the Ramanujan–Petersson conjecture. While the generalized conjecture remains open, it has been proved in various special cases (and analogous settings) and has been the inspiration for a lot of research in number theory and adjacent areas (see e.g. [20, 30]).

The tau function oscillates and (irregularly) assumes positive and negative integer values. Lehmer [19] famously conjectured that $\tau(n) \neq 0$ for any positive integer n, which

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until today remains unresolved. Results going into this direction were recently given by Balakrishnan, Craig and Ono [4] who, as an example of more general theorems, showed that $\tau(n) \notin \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 691\}$ (and in their paper the authors further announced proofs for $\tau(n)$ avoiding additional specific integers).

Dyson [8, Eq. (2), p. 636] found the following elegant formula for $\tau(n)$:

$$\tau(n) = \sum \prod_{1 \le i < j \le 5} \frac{k_j - k_i}{j - i},$$

where the sum runs over all 5-tuples of integers $(k_1, k_2, k_3, k_4, k_5)$ with

$$k_i \equiv i \pmod 5, \qquad \sum_{i=1}^{5} k_i = 0, \qquad \text{and} \quad \sum_{i=1}^{5} k_i^2 = 10n.$$

Dyson was led to this formula after seeing a similar formula by Winquist [31] for the 10th power of the Dedekind eta function. Winquist's formula and Dyson's above are special cases of one of the Macdonald identities [21]. Although these formulas do have a strong combinatorial flavour, as they involve alternating multiple sums of rational numbers it is not clear how they would be related to a direct counting of special combinatorial objects.

Charles [6] and Edixhoven [9] describe algorithms for efficiently computing $\tau(n)$. However, their algorithms do not lead to any insight of how the tau function would interplay with other functions or how it would relate to combinatorial objects.

We now describe a method, explained in more detail and with more examples in [12], to obtain combinatorial interpretations for $\tau(n)$. The general idea is to employ q-series identities of a very specific type, together with the use of generating functions for t-cores and (m, k)-capsids. By the application of this method the generating function of $\tau(n)$ can be split into a non-negative and a non-positive component, where both components admit easy combinatorial interpretations in terms of t-cores and (m, k)-capsids.

Let $\mathbb{N}$ denote the set of positive integers. A *partition* λ of a positive integer n (cf. [2]) is a non-increasing sequence of positive integers (called the *parts*), $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, such that $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. If λ is a partition of n, then we write $\lambda \vdash n$ or $|\lambda| = n$. We can identify a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ with its *Ferrers diagram*, denoted by $[\lambda]$, defined to be the set of cells $\{(i,j) \in \mathbb{N}^2 : 1 \le i \le l, 1 \le j \le \lambda_i\}$. For a cell $(i,j) \in [\lambda]$, the *hook* $H_\lambda(i,j)$ of λ is the following subset of the Ferrers diagram $[\lambda]$: $H_\lambda(i,j) = \{(i,t) \in [\lambda] : t \ge j\} \cup \{(s,j) \in [\lambda] : s > i\}$. The cardinality of the hook $H_\lambda(i,j)$ is the *hooklength* $h_\lambda(i,j)$.

Alternatively, we may denote a partition as an increasing sequence of integers with finitely many non-zero multiplicities, formally written in the form $\lambda = (1^{\mu_1}, 2^{\mu_2}, \dots)$. Here, each positive integer $j \in \mathbb{N}$ has an non-negative multiplicity μ_j, corresponding to the number of times the part j occurs in λ. If $\lambda \vdash n$, we must have $\sum_{j \ge 1} j\mu_j = n$.

t-Cores were introduced (originally for t prime) by Nakayama [23] in 1941 in his study of the modular representation of the symmetric group. See [16, Sec. 2.7] for their use in the recursive evaluation of the irreducible characters of the symmetric group. A partition

λ is a t-*core* if, and only if, λ contains no hooks whose lengths are multiples of t. (Here t is any integer greater than 1, not necessarily prime.) Let $c(t,n)$ denote the number of t-core partitions of n. The generating function for t-cores was first obtained by Klyachko [18] in 1982:

$$T_t(q) := 1 + \sum_{n=1}^{\infty} c(t,n) q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}.$$

t-Cores have attracted broad interest and were studied from various points of view, in particular, with focus on some of their combinatorial [10] or analytic [1, 15] properties.

(m,k)-Capsids were, more recently, introduced in [12, Sec. 2]. Their construction involves congruence conditions. Let $m,k \in \mathbb{N}$, $m \geq 2$. For $0 < k < m$ we say that a partition π is an (m,k)-*capsid* if, and only if, the possible parts of π are $m-k$ or are congruent to 0 or k mod m, and satisfy the following two conditions:

(i) if $\mu_{m-k} = 0$, i.e., $m-k$ is not a part, then all parts are congruent to k mod m

(ii) if $\mu_{m-k} > 0$, then $m-k$ is the smallest part and the largest part congruent to 0 mod m is $\leq m \cdot \mu_{m-k}$ and all parts congruent to k mod m (different from $m-k$, if $k = m/2$) are $> m \cdot \mu_{m-k}$.

Let $\gamma(m,k,n)$ be the number of (m,k)-capsid partitions of n. The generating function for (m,k)-capsids of n is

$$C_{m,k}(q) := 1 + \sum_{n=1}^{\infty} \gamma(m,k,n) q^n = \sum_{n=0}^{\infty} \frac{q^{(m-k)n}}{(q^m;q^m)_n (q^{mn+k};q^m)_\infty} = \frac{(q^m;q^m)_\infty}{(q^k;q^m)_\infty (q^{m-k};q^m)_\infty}.$$

The second equality comes from the combinatorial definition while the third one follows from an application of the q-binomial theorem [14, Eq. (II.3)].

From the above product formula one has, by inspection, $C_{m,k}(q) = C_{m,m-k}(q)$, from which one deduces $\gamma(m,k,n) = \gamma(m,m-k,n)$, i.e., the numbers of (m,k)- and $(m,m-k)$-capsid partitions of n are equinumerous.

Now, for convenience, define, for integers $0 < k < m$, the product

$$P_{m,k}(q) := (q^k;q^m)_\infty (q^{m-k};q^m)_\infty. \tag{3}$$

The generating function for (m,k)-capsid partitions hence can be written as

$$C_{m,k}(q) = \frac{(q^m;q^m)_\infty}{P_{m,k}(q)}.$$

The products in (3) obviously satisfy the symmetry

$$P_{m,k}(q) = P_{m,m-k}(q),$$

the scaling of parameters

$$P_{sm,sk}(q) = P_{m,k}(q^s),$$

and, by using $(a;q)_\infty = (a;q^2)_\infty (aq;q^2)_\infty$, the duplication formula

$$P_{m,k}(q) = P_{2m,k}(q) P_{2m,m-k}(q). \tag{4}$$

We turn to another essential ingredient. Ramanujan, in one of his letters to Hardy [26], stated without proof the beautiful identity

$$1 = H(q)G(q^{11}) - q^2 G(q)H(q^{11}), \tag{5}$$

where

$$G(q) := \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \quad \text{and} \quad H(q) := \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q;q)_k} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

are the two Rogers–Ramanujan functions with corresponding product forms following from the two Rogers–Ramanujan identities [28]. The first published proof of (5) was given by Rogers [29].

Many other identities like (5) exist. They belong to a class of identities called "shifted partition identities" and are related to the counting of partitions with respect to specific congruence classes modulo some fixed integer m. In case of (5), $m = 5$. Other such identities (related to other moduli) have been discussed and established in [3, 5, 11, 13, 17]. The general form of a shifted partition identity is

$$1 = A_m(q) - q^d B_m(q)$$

where m and d are positive integers and $A_m(q)$ and $B_m(q)$ consist of finite products, over different $0 < k < m$, of the reciprocal of factors of the form $P_{m,k}(q)$, defined in (3).

In particular, Ramanujan's identity (5) can be written, using (3) and (4), as

$$1 = \frac{1}{P_{10,2}(q)P_{10,3}(q)P_{110,11}(q)P_{110,44}(q)} - \frac{q^2}{P_{10,1}(q)P_{10,4}(q)P_{110,22}(q)P_{110,33}(q)}.$$

Multiplying both sides of this identity by $q^{110}(q^{110};q^{110})_\infty^{24}$, the resulting identity can be written in terms of $(10,k)$-capsid and 11-core generating functions:

$$\begin{aligned}
\sum_{n=1}^{\infty} \tau(n) q^{110n} &= q^{110} \frac{(q^{10};q^{10})_\infty^2}{P_{10,2}(q)P_{10,3}(q)} \frac{(q^{110};q^{110})_\infty^2}{P_{10,1}(q^{11})P_{10,4}(q^{11})} \frac{(q^{110};q^{110})_\infty^{22}}{(q^{10};q^{10})_\infty^2} \\
&\quad - q^{112} \frac{(q^{10};q^{10})_\infty^2}{P_{10,1}(q)P_{10,4}(q)} \frac{(q^{110};q^{110})_\infty^2}{P_{10,2}(q^{11})P_{10,3}(q^{11})} \frac{(q^{110};q^{110})_\infty^{22}}{(q^{10};q^{10})_\infty^2} \\
&= q^{110}\, C_{10,2}(q)\, C_{10,3}(q)\, C_{10,1}(q^{11})\, C_{10,4}(q^{11})\, T_{11}(q^{10})^2 \\
&\quad - q^{112}\, C_{10,1}(q)\, C_{10,4}(q)\, C_{10,2}(q^{11})\, C_{10,3}(q^{11})\, T_{11}(q^{10})^2.
\end{aligned} \tag{6}$$

Let $\mathcal{C}_{m,k}$ be the set of (m,k)-capsids and $\mathcal{T}_t$ be the set of t-cores. Further, define two sets of vector partitions:

$$\begin{aligned}
\mathcal{A} &:= \mathcal{C}_{10,2} \times \mathcal{C}_{10,3} \times \mathcal{C}_{10,1} \times \mathcal{C}_{10,4} \times \mathcal{T}_{11} \times \mathcal{T}_{11}, \\
\mathcal{B} &:= \mathcal{C}_{10,1} \times \mathcal{C}_{10,4} \times \mathcal{C}_{10,2} \times \mathcal{C}_{10,3} \times \mathcal{T}_{11} \times \mathcal{T}_{11}.
\end{aligned}$$

For a partition π let $|\pi|$ denote the sum of its parts. For $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6)$ in $\mathcal{A}$ or in $\mathcal{B}$ define

$$|\vec{\pi}| := |\pi_1| + |\pi_2| + 11 \cdot |\pi_3| + 11 \cdot |\pi_4| + 10 \cdot |\pi_5| + 10 \cdot |\pi_6|.$$

If $|\vec{\pi}| = n$ call $\vec{\pi}$ a vector partition of n. Let

$$a(n) := \text{the number of vector partitions in } \mathcal{A} \text{ of } n,$$
$$b(n) := \text{the number of vector partitions in } \mathcal{B} \text{ of } n.$$

Then the following two identities are clear:

$$1 + \sum_{n=1}^{\infty} a(n)q^n = C_{10,2}(q)\, C_{10,3}(q)\, C_{10,1}(q^{11})\, C_{10,4}(q^{11})\, T_{11}(q^{10})^2,$$
$$1 + \sum_{n=1}^{\infty} b(n)q^n = C_{10,1}(q)\, C_{10,4}(q)\, C_{10,2}(q^{11})\, C_{10,3}(q^{11})\, T_{11}(q^{10})^2.$$

Finally, define $a(0) = 1$, $b(-2) = b(-1) = 0$, and $b(0) = 1$. From (6) one has the following result (given in [12, Thm. 4]):

Theorem 1 (A combinatorial formula for Ramanujan's tau function)**.** *For $n \geq 1$, we have*

$$(i) \qquad \tau(n) = a(110n - 110) - b(110n - 112),$$

and

$$(ii) \qquad 0 = a(n) - b(n-2), \qquad \text{if } n \not\equiv 0 \pmod{110}.$$

This formula for $\tau(n)$ has the advantage that it is compact and combinatorial, but the numbers obtained when computing $a(m)$ and $b(m)$ get large. For example, application of the theorem gives $\tau(2) = a(110) - b(108) = 174780 - 174804 = -24$.

Other combinatorial formulas for $\tau(n)$ can be obtained by the same method that was used to derive Theorem 1 by using other shifted partition identities in place of (5).

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