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Multidimensional Matrix Inversions and A_r and D_r Basic Hypergeometric Series

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Abstract. We compute the inverse of a specific infinite r-dimensional matrix, thus unifying multidimensional matrix inversions recently found by Milne, Lilly, and Bhatnagar. Our inversion is an r-dimensional extension of a matrix inversion previously found by Krattenthaler. We also compute the inverse of another infinite r-dimensional matrix. As applications of our matrix inversions, we derive new summation formulas for multidimensional basic hypergeometric series.

Keywords: multidimensional matrix inversions, A_r basic hypergeometric series, D_r basic hypergeometric series

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1. Introduction

Matrix inversions are very important in many fields of combinatorics and special functions. When dealing with combinatorial sums, application of matrix inversion may help to simplify problems, or yield new identities. Andrews [1] discovered that the Bailey transform [3], which is a very powerful tool in the theory of (basic) hypergeometric series, corresponds to the inversion of two infinite lower-triangular matrices. Gessel and Stanton [13] used a bibasic extension of that matrix inversion to derive a number of basic hypergeometric summations and transformations, and identities of Rogers-Ramanujan type. Even earlier, Carlitz [9] had found an even more general matrix inversion though without giving any applications.

Gasper and Rahman [10], [25], [11], [12, sec. 3.6] used another bibasic matrix inversion together with an indefinite bibasic sum to derive numerous beautiful hypergeometric summation and transformation formulas.

The most general (1-dimensional) matrix inversion, however, which contained all the inversions aforementioned, was found by Krattenthaler [16] who applied his inversion to derive a number of hypergeometric summation formulas. The inverse matrices he gave are basically $(f_{n,k})_{n,k\in \mathrm{Z}}$ and $(g_{k,l})_{k,l\in \mathrm{Z}}$ (Z denotes the set of integers), where

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j - c_k)}{\prod_{j=k+1}^{n}(c_j - c_k)}, \tag{1.1}$$

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and

$$g_{k,l} = \frac{(a_l - c_l)}{(a_k - c_k)} \frac{\prod_{j=l+1}^{k} (a_j - c_k)}{\prod_{j=l}^{k-1} (c_j - c_k)}. \tag{1.2}$$

In fact, the special case $a_j = aq^{-j}$, $c_k = q^k$ is equivalent to the matrix inversion of Andrews, and the case $a_j = ap^{-j}$, $c_k = q^k$ is equivalent to Gessel and Stanton's. Specializing $c_k = q^k$, we obtain Carlitz's matrix inversion, and $a_j = (bp^{-j}/a) + ap^j$, $c_k = q^{-k} + bq^k$ yields the inversion of Gasper and Rahman.

Multidimensional matrix inversions were found by Milne, Lilly and Bhatnagar. The A_r (or equivalently $U(r+1)$) and C_r inversions (corresponding to the root systems A_r and C_r, respectively) of Milne and Lilly [21, Theorem 3.3], [22], [17], [18], which are higher-dimensional generalizations of Andrews' Bailey transform matrices, were used to derive A_r and C_r extensions [21], [23] of many of the classical hypergeometric summation and transformation formulas. Bhatnagar and Milne [4, Theorem 5.7], [6, Theorem 3.48] were even able to find an A_r extension of Gasper and Rahman's bibasic hypergeometric matrix inversion. They used a special case of their matrix inversion, an A_r extension of Carlitz's inversion, to derive A_r identities of Abel-type. But none of these multidimensional matrix inversions contained Krattenthaler's inversion as a special case.

One of the main results of this paper is a multidimensional extension of Krattenthaler's matrix inverse (see Theorem 3.1). This multidimensional matrix inversion unifies all the matrix inversions mentioned so far as it contains them all as special cases. Besides, we present another interesting multidimensional matrix inversion (see Theorem 4.1) which is of different type.

In order to prove our matrix inversions in Theorems 3.1 and 4.1 we utilize Krattenthaler's operator method [15] which we review in section 2. We adapt a main theorem of [15] and add an appropriate multidimensional corollary (see Corollary 2.14).

The main motivation for finding a multidimensional extension of Krattenthaler's matrix inverse came from prospective applications to basic hypergeometric series. These applications are the contents of section 5. We combine a special case of Theorem 3.1 and a C_r ${}_8\phi_7$ summation theorem of Milne and Lilly [23] to derive a D_r ${}_8\phi_7$ summation theorem, which has been derived independently by Bhatnagar [5] using a different method. We also derive A_r and D_r extensions of a quadratic hypergeometric summation formula of Gessel and Stanton [13]. Finally, we derive a D_r extension of a cubic summation formula of Gasper and Rahman [11].

We are sure that our multidimensional matrix inversions are very useful in the theory of basic hypergeometric series of type A_r, C_r, and D_r, respectively, and will lead to the discovery of many more new identities. This claim is heavily supported by the fact that identities derived in this paper already lead to new C_r and D_r extensions of Bailey's very-well-poised ${}_{10}\phi_9$ transformation [2]. This is ongoing research undertaken jointly with Bhatnagar [7].

Two determinant evaluations, which are elegant generalizations of the classical and "symplectic" Vandermonde determinants, turn out to be crucial for our computations in sections 3 and 4. We decided to give them in a separate appendix.

This work is part of the author's thesis, being written under the supervision of C. Krattenthaler. The author feels especially indebted to his supervisor who patiently has provided a lot of help and ideas.

2. An operator method for proving matrix inversions

Let $F = (f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in \mathrm{Z}^r}$ (as before, Z denotes the set of integers) be an infinite lower-triangular r-dimensional matrix; i.e. $f_{\mathbf{nk}} = 0$ unless $\mathbf{n} \geq \mathbf{k}$, by which we mean $n_i \geq k_i$ for all $i = 1, \dots, r$. The matrix $G = (g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in \mathrm{Z}}$ is said to be the *inverse matrix* of F if and only if

$$\sum_{\mathbf{n}\geq\mathbf{k}\geq\mathbf{l}} f_{\mathbf{nk}} g_{\mathbf{kl}} = \delta_{\mathbf{nl}}$$

for all $\mathbf{n}, \mathbf{l} \in \mathrm{Z}^r$, where $\delta_{\mathbf{n},\mathbf{l}}$ is the usual Kronecker delta.

In [15] Krattenthaler gave a method for solving Lagrange inversion problems, which are closely connected with the problem of inverting lower-triangular matrices. We will use his operator method for proving our new theorems. By a *formal Laurent series* we mean a series of the form $\sum_{\mathbf{n}\geq\mathbf{k}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$, for some $\mathbf{k} \in \mathrm{Z}^r$, where $\mathbf{z}^{\mathbf{n}} = z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}$. Given the formal Laurent series $a(\mathbf{z})$ and $b(\mathbf{z})$ we introduce the bilinear form $\langle\, ,\, \rangle$ by

$$\langle a(\mathbf{z}), b(\mathbf{z})\rangle = \langle \mathbf{z}^{\mathbf{0}}\rangle (a(\mathbf{z}) \cdot b(\mathbf{z})),$$

where $\langle \mathbf{z}^{\mathbf{0}}\rangle c(\mathbf{z})$ denotes the coefficient of $\mathbf{z}^{\mathbf{0}}$ in $c(\mathbf{z})$. Given any linear operator L acting on formal Laurent series, L^* denotes the adjoint of L with respect to $\langle\, ,\, \rangle$; i.e. $\langle La(\mathbf{z}), b(\mathbf{z})\rangle = \langle a(\mathbf{z}), L^* b(\mathbf{z})\rangle$ for all formal Laurent series $a(\mathbf{z})$ and $b(\mathbf{z})$. We need the following special case of [15, Theorem 1].

Lemma 2.1 *Let $F = (f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in \mathrm{Z}^r}$ be an infinite lower-triangular r-dimensional matrix with $f_{\mathbf{kk}} \neq 0$ for all $\mathbf{k} \in \mathrm{Z}^r$. For $\mathbf{k} \in \mathrm{Z}^r$, define the formal Laurent series $f_{\mathbf{k}}(\mathbf{z})$ and $g_{\mathbf{k}}(\mathbf{z})$ by $f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{n}\geq\mathbf{k}} f_{\mathbf{nk}} \mathbf{z}^{\mathbf{n}}$ and $g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\leq\mathbf{k}} g_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}$, where $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in \mathrm{Z}^r}$ is the uniquely determined inverse matrix of F. Suppose that for $\mathbf{k} \in \mathrm{Z}^r$ a system of equations of the form*

$$U_j f_{\mathbf{k}}(\mathbf{z}) = c_j(\mathbf{k}) V f_{\mathbf{k}}(\mathbf{z}), \qquad j = 1, \dots, r,$$

holds, where U_j, V are linear operators acting on formal Laurent series, V being bijective, and $(c_j(\mathbf{k}))_{\mathbf{k}\in \mathrm{Z}^r}$ are arbitrary sequences of constants. Moreover, we suppose that

$$\text{for all } \mathbf{m}, \mathbf{n} \in \mathrm{Z}^r,\ \mathbf{m} \neq \mathbf{n}, \text{ there exists a } j \text{ with } 1 \leq j \leq r \text{ and } c_j(\mathbf{m}) \neq c_j(\mathbf{n}). \tag{2.2}$$

Then, if $h_{\mathbf{k}}(\mathbf{z})$ is a solution of the dual system

$$U_j^* h_{\mathbf{k}}(\mathbf{z}) = c_j(\mathbf{k}) V^* h_{\mathbf{k}}(\mathbf{z}), \qquad j = 1, \dots, r,$$

with $h_{\mathbf{k}}(\mathbf{z}) \not\equiv 0$ *for all* $\mathbf{k} \in \mathrm{Z}^r$*, the series* $g_{\mathbf{k}}(\mathbf{z})$ *are given by*

$$g_{\mathbf{k}}(\mathbf{z}) = \frac{1}{\langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{k}}(\mathbf{z})\rangle} V^* h_{\mathbf{k}}(\mathbf{z}).$$

In our applications we will use a corollary of Lemma 2.1 (see Corollary 2.14). Let $\mathcal{S}_r$ be the symmetric group of order r. For possibly noncommuting operators V_{ij} let us define the *column determinant* by

$$\overrightarrow{\det}_{1\le i,j\le r}\,(V_{ij}) := \sum_{\sigma\in\mathcal{S}_r} \operatorname{sgn}(\sigma) V_{\sigma(r),r} V_{\sigma(r-1),r-1} \cdots V_{\sigma(1),1}. \tag{2.3}$$

An equivalent, recursive definition is by means of the expansion along the first column,

$$\overrightarrow{\det}_{1\le i,j\le r}\,(V_{ij}) = \sum_{k=1}^{r} (-1)^{k+1} \vec{V}^{(k,1)} V_{k1}, \tag{2.4}$$

where $\vec{V}^{(i,j)}$ denotes the column minor with the i-th row and j-th column being omitted.

Proposition 2.5 *Let* $(V_{ij})_{i,j=1}^r$ *be a matrix of linear operators acting on formal Laurent series. Suppose* $V_{ij} = C_{ij} + A_{ij}$ *for* $i, j = 1, \dots, r$*, where the operators* C_{ij}, A_{ij} *obey the following commutation rules*

$$C_{ij} C_{kl} = C_{kl} C_{ij}, \qquad i \neq k; \quad i, j, k, l = 1, \dots, r, \tag{2.6}$$

i.e. C_{ij} *and* C_{kl} *commute when taken from different rows,*

$$C_{ij} A_{kl} = A_{kl} C_{ij}, \qquad i \neq k; \quad i, j, k, l = 1, \dots, r, \tag{2.7}$$

i.e. C_{ij} *and* A_{kl} *commute when taken from different rows, the rules*

$$A_{ij} A_{kl} = A_{kj} A_{il}, \qquad i, j, k, l = 1, \dots, r, \tag{2.8}$$

where the column indices j *and* l *keep their order.*

Then the column determinant $\overrightarrow{\det}_{1\le i,j\le r}(V_{ij})$*, as defined in* (2.3) *or* (2.4)*, can be reduced to a polynomial in the* A_{ij}*'s,* $i, j = 1, \dots, r$*, of degree* ≤ 1*,*

$$\overrightarrow{\det}_{1\le i,j\le r}\,(V_{ij}) = \overrightarrow{\det}_{1\le i,j\le r}\,(C_{ij}) + \sum_{i,j=1}^{r} (-1)^{i+j} \vec{C}^{(i,j)} A_{ij}, \tag{2.9}$$

where $\vec{C}^{(i,j)}$ *again denotes the column minor with the* i*-th row and* j*-th column being omitted.*

Moreover, for any $l = 1, \dots, r$ *the expansion*

$$\overrightarrow{\det}_{1\le i,j\le r}\,(V_{ij}) = \sum_{k=1}^{r} (-1)^{k+l} \vec{V}^{(k,l)} V_{kl} \tag{2.10}$$

holds, which means that the determinant can be expanded along any arbitrary column.

Proof: Our column determinant can be expanded into

$$\overrightarrow{\det}_{1\le i,j\le r}(V_{ij}) = \sum_{\substack{\sigma\in\mathcal{S}_r\\ \mathbf{q}\in\{\gamma,\alpha\}^r}} \operatorname{sgn}(\sigma) X_{\sigma(r),r}(q_r)\cdots X_{\sigma(1),1}(q_1), \tag{2.11}$$

where $\mathbf{q} = (q_1, q_2, \dots, q_r)$ with q_i equal to either γ or α for $i = 1, \dots, r$, and where $X_{ij}(\gamma) = C_{ij}$ and $X_{ij}(\alpha) = A_{ij}$. Due to (2.6) and (2.7), we observe that in every term of (2.11) the $X_{ij}(\gamma)$ commute with all other variables. $X_{ij}(\alpha)$ and $X_{kl}(\alpha)$ do not commute unless $j = l$, but due to (2.8) we have $X_{ij}(\alpha)X_{kl}(\alpha) = X_{kj}(\alpha)X_{il}(\alpha)$. This important fact lets all terms in (2.11) cancel where α occurs more than once, since we can pair the terms

$$\operatorname{sgn}(\sigma)\dots X_{\sigma(i),i}(\alpha)\dots X_{\sigma(j),j}(\alpha)\dots$$

and

$$\operatorname{sgn}(\sigma\cdot(i,j))\dots X_{\sigma(j),i}(\alpha)\dots X_{\sigma(i),j}(\alpha)\dots,$$

having chosen the first two occurrences of α, for instance, and where all other factors are unchanged. After cancellation, we are left with terms with at most one occurrence of α and where all factors commute. This implies the first assertion of the proposition.

For proving the second assertion, we define for a given permutation $\tau \in \mathcal{S}_r$ a modified column determinant by

$$\overset{\to(\tau)}{\underset{1\le i,j\le r}{\det}}(V_{ij}) = \sum_{\sigma\in\mathcal{S}_r} \operatorname{sgn}(\sigma) V_{\sigma(\tau(r)),\tau(r)}\cdots V_{\sigma(\tau(1)),\tau(1)}, \tag{2.12}$$

i.e. the sequence of columns in expanding the determinant is determined by τ. Expanding $\overrightarrow{\det}^{(\tau)}_{1\le i,j\le r}(V_{ij})$ as in (2.11) we see that the same cancellation argument applies. Thus we have $\overrightarrow{\det}^{(\tau)}_{1\le i,j\le r}(V_{ij}) = \overrightarrow{\det}_{1\le i,j\le r}(V_{ij})$, which proves (2.10). ■

Remark 2.13. Note that in the determinant of Proposition 2.5, we may not expand along rows because then the cancellation argument does not apply. (For a counterexample, consider the $r = 2$ case.)

Corollary 2.14 *Let W_i, V_{ij} be linear operators acting on formal Laurent series, $c_j(\mathbf{k})$ arbitrary constants for $\mathbf{k} \in \mathbb{Z}^r$ and $i, j = 1, \dots, r$. Suppose $V_{ij} = C_{ij} + A_{ij}$, with the operators C_{ij}, A_{ij}, $i, j = 1, \dots, r$, satisfying the conditions* (2.6), (2.7)*, and* (2.8) *of Proposition 2.5. Suppose $W_i = W_i^{(c)} + W_i^{(a)}$, with the operators $W_i, W_i^{(c)}, W_i^{(a)}$, $i = 1, \dots, r$, satisfying*

$$C_{kl}W_i = W_iC_{kl}, \qquad i \ne k; \quad i,k,l = 1,\dots,r, \tag{2.15}$$

$$A_{kl}W_i^{(c)} = W_i^{(c)}A_{kl}, \qquad i \ne k; \quad i,k,l = 1,\dots,r, \tag{2.16}$$

$$A_{kl}W_i^{(a)} = A_{il}W_k^{(a)}, \qquad i,k,l = 1,\dots,r. \tag{2.17}$$

Moreover the $c_j(\mathbf{k})$ *are assumed to satisfy* (2.2), *and* $\overrightarrow{\det}_{1\le i,j\le r}(V_{ij})$ *is assumed to be invertible. With the notation of Lemma 2.1, if*

$$\sum_{j=1}^{r} c_j(\mathbf{k}) V_{ij} f_{\mathbf{k}}(\mathbf{z}) = W_i f_{\mathbf{k}}(\mathbf{z}), \qquad i = 1, \dots, r, \tag{2.18}$$

then

$$g_{\mathbf{k}}(\mathbf{z}) = \frac{1}{\langle f_{\mathbf{k}}(\mathbf{z}), \overrightarrow{\det}\,(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z})\rangle}\, \overrightarrow{\det}\,(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}), \tag{2.19}$$

where $h_{\mathbf{k}}(\mathbf{z})$ *is a solution of*

$$\sum_{j=1}^{r} c_j(\mathbf{k}) V_{ij}^* h_{\mathbf{k}}(\mathbf{z}) = W_i^* h_{\mathbf{k}}(\mathbf{z}), \qquad i = 1, \dots, r, \tag{2.20}$$

with $h_{\mathbf{k}}(\mathbf{z}) \not\equiv 0$ *for all* $\mathbf{k} \in \mathbb{Z}^r$.

Proof: Since it follows by Proposition 2.5 that the column determinant $\overrightarrow{\det}_{1\le i,j\le r}(V_{ij})$ may be expanded along any column, we can apply Cramer's rule to (2.18) to obtain

$$c_j(\mathbf{k})\, \overrightarrow{\det}_{1\le i,l\le r}(V_{il}) f_{\mathbf{k}}(\mathbf{z}) = \sum_{i=1}^{r} (-1)^{i+j} \vec{V}^{(i,j)} W_i f_{\mathbf{k}}(\mathbf{z}),$$

for $j = 1, \dots, r$. The dual system reads

$$\begin{aligned} c_j(\mathbf{k})\, \overrightarrow{\det}_{1\le i,l\le r}(V_{il}^*) h_{\mathbf{k}}(\mathbf{z}) &= \sum_{i=1}^{r} (-1)^{i+j} W_i^* \vec{V^*}^{(i,j)} h_{\mathbf{k}}(\mathbf{z}) \\ &= \sum_{i=1}^{r} (-1)^{i+j} \vec{V^*}^{(i,j)} W_i^* h_{\mathbf{k}}(\mathbf{z}), \end{aligned} \tag{2.21}$$

for $j = 1, \dots, r$, and is equivalent to (2.20). Notice that, because of (2.15), (2.16), and (2.17), we may apply Proposition 2.5 in (2.21) and shift W_i^* to the right. Now apply Lemma 2.1 with $V = \overrightarrow{\det}\,(V_{ij})$ and $U_j = \sum_{i=1}^{r} (-1)^{i+j} \vec{V}^{(i,j)} W_i f_{\mathbf{k}}(\mathbf{z})$. ■

3. A multidimensional matrix inversion

For convenience, we introduce the notation $|\mathbf{n}| = n_1 + n_2 + \cdots + n_r$.

Theorem 3.1 *Let* $(a_t)_{t\in\mathbb{Z}}$, $(c_i(t_i))_{t_i\in\mathbb{Z}}$, $i = 1, \dots, r$, *be arbitrary sequences,* b *arbitrary, such that none of the denominators in* (3.2) *or* (3.3) *vanish. Then* $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ *are inverses of each other, where*

$$f_{\mathbf{nk}} = \frac{\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}(a_t - b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r \prod_{t_i=k_i+1}^{n_i}(c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \frac{\prod_{i=1}^r \prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}(a_t - c_i(k_i))}{\prod_{i,j=1}^r \prod_{t_i=k_i+1}^{n_i}(c_i(t_i) - c_j(k_j))} \tag{3.2}$$

and

$$\begin{aligned} g_{\mathbf{kl}} = & \prod_{1\le i<j\le r} \frac{(c_i(l_i) - c_j(l_j))}{(c_i(k_i) - c_j(k_j))} \\ & \times \frac{(b - a_{|\mathbf{l}|}\prod_{j=1}^r c_j(l_j))}{(b - a_{|\mathbf{k}|}\prod_{j=1}^r c_j(k_j))} \prod_{i=1}^r \frac{(a_{|\mathbf{l}|} - c_i(l_i))}{(a_{|\mathbf{k}|} - c_i(k_i))} \\ & \times \frac{\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}(a_t - b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r \prod_{t_i=l_i}^{k_i-1}(c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \frac{\prod_{i=1}^r \prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}(a_t - c_i(k_i))}{\prod_{i,j=1}^r \prod_{t_i=l_i}^{k_i-1}(c_i(t_i) - c_j(k_j))}. \end{aligned} \tag{3.3}$$

Remark 3.4. The special case $a_t = 0$, $c_j(k_j) = x_j^{-1}q^{-k_j}$ is equivalent to the A_r Bailey transform of Milne and Lilly [21], [22], the specialization $a_t = 0$, $c_j(k_j) = x_j^{-1}q^{-k_j} + x_j q^{k_j}$, $b = 0$ is equivalent to their C_r Bailey transform [22], [17], [18]. The limiting case $a_t = b\,aq^{-t}$, $c_j(k_j) = x_j^{-1}q^{-k_j}$, then $b \to 0$, is equivalent to a second A_r Bailey transform of Milne [21, Theorem 8.26]. The specialization $c_j(k_j) = x_j^{-1}q^{-k_j}$ is equivalent to the A_r matrix inverse of Bhatnagar and Milne [4, Theorem 5.7], [6, Theorem 3.48]. Moreover, the $r = 1$ case is a restatement of Krattenthaler's matrix inversion (eqs. (1.1) and (1.2)). Due to the fact that Theorem 3.1 covers all known A_r matrix inversions (to the author's knowledge), we view Theorem 3.1 as an A_r matrix inversion theorem (also see Remark 4.4).

Another important special case of Theorem 3.1 is a new multidimensional bibasic hypergeometric matrix inversion, stated seperately as Theorem 5.10 in section 5, which we utilize in our applications.

Proof of Theorem 3.1: We will use the operator method of section 2. From (3.2) we deduce for $\mathbf{n} \geq \mathbf{k}$ the recursion

$$\begin{aligned} & (c_i(n_i) - b/\textstyle\prod_{j=1}^r c_j(k_j)) \displaystyle\prod_{s=1}^r (c_i(n_i) - c_s(k_s)) f_{\mathbf{nk}} \\ & \qquad = (a_{|\mathbf{n}|-1} - b/\textstyle\prod_{j=1}^r c_j(k_j)) \displaystyle\prod_{s=1}^r (a_{|\mathbf{n}|-1} - c_s(k_s)) f_{\mathbf{n}-\mathbf{e}_i,\mathbf{k}}, \end{aligned} \tag{3.5}$$

for $i = 1, \dots, r$, where $\mathbf{e}_i$ denotes the vector of Z^r where all components are zero except the i-th, which is 1. We write

$$f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{n}\ge\mathbf{k}} \frac{\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}(a_t - b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r \prod_{t_i=k_i+1}^{n_i}(c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \frac{\prod_{i=1}^r \prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}(a_t - c_i(k_i))}{\prod_{i,j=1}^r \prod_{t_i=k_i+1}^{n_i}(c_i(t_i) - c_j(k_j))} \mathbf{z}^{\mathbf{n}}.$$

Moreover, we define linear operators $\mathcal{A}, \mathcal{C}_i$ by $\mathcal{A}\mathbf{z}^{\mathbf{n}} = a_{|\mathbf{n}|}\mathbf{z}^{\mathbf{n}}$ and $\mathcal{C}_i\mathbf{z}^{\mathbf{n}} = c_i(n_i)\mathbf{z}^{\mathbf{n}}$ for all $i = 1, \dots, r$. Then we may write (3.5) in the form

$$\begin{aligned}(\mathcal{C}_i - b/\textstyle\prod_{j=1}^r c_j(k_j)) \displaystyle\prod_{s=1}^r(\mathcal{C}_i - c_s(k_s)) f_{\mathbf{k}}(\mathbf{z})& \\ = z_i(\mathcal{A} - b/\textstyle\prod_{j=1}^r c_j(k_j)) \displaystyle\prod_{s=1}^r(\mathcal{A} - c_s(k_s)) f_{\mathbf{k}}(\mathbf{z}),& \qquad (3.6)\end{aligned}$$

valid for all $\mathbf{k} \in \mathrm{Z}^r$. We want to write our system of equations in a way such that Corollary 2.14 is applicable. In order to achieve this, we expand the products on both sides of (3.6) in terms of the elementary symmetric functions (see [19, p.19])

$$e_j(c_1(k_1), c_2(k_2), \dots, c_r(k_r), b/\textstyle\prod_{s=1}^r c_s(k_s))$$

of order j, for which we write $e_j(\mathbf{c}(\mathbf{k}))$ for short. Our recurrence system then reads, using $e_{r+1}(\mathbf{c}(\mathbf{k})) = b$,

$$\begin{aligned}\sum_{j=1}^r e_j(\mathbf{c}(\mathbf{k}))[(-\mathcal{C}_i)^{r+1-j} - z_i(-\mathcal{A})^{r+1-j}] f_{\mathbf{k}}(\mathbf{z})& \\ = [z_i(-\mathcal{A})^{r+1} + bz_i - (-\mathcal{C}_i)^{r+1} - b] f_{\mathbf{k}}(\mathbf{z}), \quad i = 1, \dots, r.& \qquad (3.7)\end{aligned}$$

Now (3.7) is a system of type (2.18) with $V_{ij} = [(-\mathcal{C}_i)^{r+1-j} - z_i(-\mathcal{A})^{r+1-j}]$, $W_i = [z_i(-\mathcal{A})^{r+1} + bz_i - (-\mathcal{C}_i)^{r+1} - b]$, and $c_j(\mathbf{k}) = e_j(\mathbf{c}(\mathbf{k}))$. The operators $C_{ij} = (-\mathcal{C}_i)^{r+1-j}$, $A_{ij} = -z_i(-\mathcal{A})^{r+1-j}$, $W_i^{(c)} = [-(-\mathcal{C}_i)^{r+1} - b]$, $W_i^{(a)} = [z_i(-\mathcal{A})^{r+1} + bz_i]$ satisfy (2.6), (2.7), (2.8), (2.15), (2.16), and (2.17), the functions $c_j(\mathbf{k})$ satisfy (2.2). Hence we may apply Corollary 2.14. The dual system (2.20) for the auxiliary formal Laurent series $h_{\mathbf{k}}(\mathbf{z})$ in this case reads

$$\begin{aligned}\sum_{j=1}^r e_j(\mathbf{c}(\mathbf{k}))[(-\mathcal{C}_i^*)^{r+1-j} - (-\mathcal{A}^*)^{r+1-j} z_i] h_{\mathbf{k}}(\mathbf{z})& \\ = [(-\mathcal{A}^*)^{r+1} z_i + bz_i - (-\mathcal{C}_i^*)^{r+1} - b] h_{\mathbf{k}}(\mathbf{z}), \quad i = 1, \dots, r.&\end{aligned}$$

Equivalently, we have

$$\begin{aligned}(\mathcal{C}_i^* - b/\textstyle\prod_{j=1}^r c_j(k_j)) \displaystyle\prod_{s=1}^r(\mathcal{C}_i^* - c_s(k_s)) h_{\mathbf{k}}(\mathbf{z})& \\ = (\mathcal{A}^* - b/\textstyle\prod_{j=1}^r c_j(k_j)) \displaystyle\prod_{s=1}^r(\mathcal{A}^* - c_s(k_s)) z_i h_{\mathbf{k}}(\mathbf{z}),& \qquad (3.8)\end{aligned}$$

for all $i = 1, \dots, r$ and $\mathbf{k} \in \mathbb{Z}^r$. As is easily seen, we have $\mathcal{A}^* \mathbf{z}^{-\mathbf{l}} = a_{|\mathbf{l}|} \mathbf{z}^{-\mathbf{l}}$ and $\mathcal{C}_i^* \mathbf{z}^{-\mathbf{l}} = c_i(l_i) \mathbf{z}^{-\mathbf{l}}$ for $i = 1, \dots, r$. Thus, with $h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}$, by comparing coefficients of $\mathbf{z}^{-\mathbf{l}}$ in (3.8) we obtain

$$\begin{aligned}(c_i(l_i) - b/\textstyle\prod_{j=1}^r c_j(k_j)) \displaystyle\prod_{s=1}^r (c_i(l_i) - c_s(k_s)) h_{\mathbf{kl}} \\ = (a_{|\mathbf{l}|} - b/\textstyle\prod_{j=1}^r c_j(k_j)) \displaystyle\prod_{s=1}^r (a_{|\mathbf{l}|} - c_s(k_s)) h_{\mathbf{k},\mathbf{l}+\mathbf{e}_i}.\end{aligned}$$

If we set $h_{\mathbf{kk}} = 1$, we get

$$h_{\mathbf{kl}} = \frac{\prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1} (a_t - b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r \prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \frac{\prod_{i=1}^r \prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1} (a_t - c_i(k_i))}{\prod_{i,j=1}^r \prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - c_j(k_j))}.$$

Taking into account (2.19), we have to compute the action of

$$\overrightarrow{\det}_{1 \le i,j \le r} (V_{ij}^*) = \overrightarrow{\det}_{1 \le i,j \le r} \left[(-\mathcal{C}_i^*)^{r+1-j} - (-\mathcal{A}^*)^{r+1-j} z_i \right] \tag{3.9}$$

when applied to

$$h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} \frac{\prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1} (a_t - b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^r \prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \frac{\prod_{i=1}^r \prod_{t=|\mathbf{l}|}^{|\mathbf{k}|-1} (a_t - c_i(k_i))}{\prod_{i,j=1}^r \prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - c_j(k_j))} \mathbf{z}^{-\mathbf{l}}.$$

From Proposition 2.5 it follows that the determinant (3.9) can be written as

$$\overrightarrow{\det}_{1 \le i,j \le r} (V_{ij}^*) = \overrightarrow{\det}_{1 \le i,j \le r} (C_{ij}^*) + \sum_{i,j=1}^r (-1)^{i+j} \overrightarrow{\det}_{\substack{1 \le m \le r, m \ne i \\ 1 \le n \le r, n \ne j}} (C_{mn}^*) \, A_{ij}^*,$$

or more explicitly,

$$\begin{aligned}\overrightarrow{\det}_{1 \le i,j \le r} (V_{ij}^*) &= \overrightarrow{\det}_{1 \le i,j \le r} \left[(-\mathcal{C}_i^*)^{r+1-j} - (-\mathcal{A}^*)^{r+1-j} z_i \right] \\ &= \overrightarrow{\det}_{1 \le i,j \le r} \left[(-\mathcal{C}_i^*)^{r+1-j} \right] \\ &\quad + \sum_{i,j=1}^r (-1)^{i+j} \overrightarrow{\det}_{\substack{1 \le m \le r, m \ne i \\ 1 \le n \le r, n \ne j}} \left[(-\mathcal{C}_m^*)^{r+1-n} \right] \left(-(-\mathcal{A}^*)^{r+1-j} z_i \right).\end{aligned} \tag{3.10}$$

Note that after expanding the column determinants by (2.12) all summands in (3.10) have pairwise commuting factors (when regarding $(-(-\mathcal{A}^*)^{r+1-j} z_i)$ as *one* factor). Since

$$z_i h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} \frac{(c_i(l_i) - b/\prod_{j=1}^r c_j(k_j))}{(a_{|\mathbf{l}|} - b/\prod_{j=1}^r c_j(k_j))} \prod_{j=1}^r \frac{(c_i(l_i) - c_j(k_j))}{(a_{|\mathbf{l}|} - c_j(k_j))} h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}},$$

we conclude that

$$\overrightarrow{\det}_{1\le i,j\le r}\,(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z})$$
$$=\sum_{\mathbf{l}\le\mathbf{k}}\Bigg(\overrightarrow{\det}_{1\le i,j\le r}\,(C_{ij}^*)+\sum_{i,j=1}^{r}(-1)^{i+j}\,\overrightarrow{\det}_{\substack{1\le m\le r,m\ne i\\1\le n\le r,n\ne j}}\,(C_{mn}^*)\,A_{ij}^*\Bigg)h_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}$$
$$=\sum_{\mathbf{l}\le\mathbf{k}}\Bigg(\overrightarrow{\det}_{1\le i,j\le r}\,\big((-c_i(l_i))^{r+1-j}\big)$$
$$+\sum_{i,j=1}^{r}(-1)^{i+j}\,\overrightarrow{\det}_{\substack{1\le m\le r,m\ne i\\1\le n\le r,n\ne j}}\,\big((-c_m(l_m))^{r+1-n}\big)$$
$$\cdot\big(-(-a_{|\mathbf{l}|})^{r+1-j}\big)\frac{(c_i(l_i)-b/\prod_{j=1}^r c_j(k_j))}{(a_{|\mathbf{l}|}-b/\prod_{j=1}^r c_j(k_j))}\prod_{j=1}^{r}\frac{(c_i(l_i)-c_j(k_j))}{(a_{|\mathbf{l}|}-c_j(k_j))}\Bigg)h_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}. \quad (3.11)$$

We claim that

$$\overrightarrow{\det}_{1\le i,j\le r}\,(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z})=\sum_{\mathbf{l}\le\mathbf{k}}\det_{1\le i,j\le r}(v_{ij})h_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}, \quad (3.12)$$

where

$$v_{ij}=(-c_i(l_i))^{r+1-j}-(-a_{|\mathbf{l}|})^{r+1-j}\frac{(c_i(l_i)-b/\prod_{s=1}^r c_s(k_s))}{(a_{|\mathbf{l}|}-b/\prod_{s=1}^r c_s(k_s))}\prod_{s=1}^{r}\frac{(c_i(l_i)-c_s(k_s))}{(a_{|\mathbf{l}|}-c_s(k_s))}.$$

The claim follows from the observation that $\det_{1\le i,j\le r}(v_{ij})$ is a determinant of commuting entries and so trivially satisfies the assumptions of Proposition 2.5. Thus $\sum_{\mathbf{l}\le\mathbf{k}}\det_{1\le i,j\le r}(v_{ij})h_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}$ can also be transformed into the right side of (3.11).

For the computation of $\det_{1\le i,j\le r}(v_{ij})$ we utilize Lemma A.1 with $x_i=-c_i(l_i)$, $y_s=-c_s(k_s)$, $a=-a_{|\mathbf{l}|}$, and $c=(-1)^{r+1}b$, obtaining

$$\det_{1\le i,j\le r}(v_{ij}) = \frac{(a_{|\mathbf{l}|}-b/\prod_{j=1}^r c_j(l_j))}{(a_{|\mathbf{l}|}-b/\prod_{j=1}^r c_j(k_j))}\prod_{i=1}^{r}\frac{(a_{|\mathbf{l}|}-c_i(l_i))}{(a_{|\mathbf{l}|}-c_i(k_i))}$$
$$\times\prod_{i=1}^{r}(-c_i(l_i))\prod_{1\le i<j\le r}(c_j(l_j)-c_i(l_i)).$$

Plugging this determinant evaluation into (3.12) leads to

$$\overrightarrow{\det}_{1\le i,j\le r}\,(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}}\Bigg(\prod_{1\le i<j\le r}(c_j(l_j)-c_i(l_i))\prod_{i=1}^{r}(-c_i(l_i))$$
$$\times\frac{(a_{|\mathbf{l}|}-b/\prod_{j=1}^r c_j(l_j))}{(a_{|\mathbf{k}|}-b/\prod_{j=1}^r c_j(k_j))}\prod_{i=1}^{r}\frac{(a_{|\mathbf{l}|}-c_i(l_i))}{(a_{|\mathbf{k}|}-c_i(k_i))}$$
$$\times\frac{\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}(a_t-b/\prod_{j=1}^r c_j(k_j))}{\prod_{i=1}^{r}\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-b/\prod_{j=1}^r c_j(k_j))}\,\frac{\prod_{i=1}^{r}\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}(a_t-c_i(k_i))}{\prod_{i,j=1}^{r}\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_j(k_j))}\mathbf{z}^{-\mathbf{l}}\Bigg). \quad (3.13)$$

Note that since $f_{\mathbf{kk}} = 1$, the pairing $\langle f_{\mathbf{k}}(\mathbf{z}), \overrightarrow{\det}\,(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z})\rangle$ is simply the coefficient of $\mathbf{z}^{-\mathbf{k}}$ in (3.13). Thus, equation (2.19) reads

$$g_{\mathbf{k}}(\mathbf{z}) = \prod_{1\le i<j\le r} (c_j(k_j) - c_i(k_i))^{-1} \prod_{i=1}^{r} (-c_i(k_i))^{-1} \overrightarrow{\det}_{1\le i,j\le r}\,(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}), \tag{3.14}$$

where $g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}$. So, extracting the coefficient of $\mathbf{z}^{-\mathbf{l}}$ in (3.14) we obtain exactly (3.3). ■

4. Another multidimensional matrix inversion

Theorem 4.1 *Let $(c_i(t_i))_{t_i\in\mathbb{Z}}$, $i = 1,\dots,r$, be arbitrary sequences, b arbitrary, such that none of the denominators in (4.2) or (4.3) vanish. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = \prod_{i=1}^{r} \frac{\prod_{t_i=k_i}^{n_i-1} (1 - bc_i(t_i)/\prod_{j=1}^{r} c_j(k_j))}{\prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - b/\prod_{j=1}^{r} c_j(k_j))} \prod_{i,j=1}^{r} \frac{\prod_{t_i=k_i}^{n_i-1} (1 - c_i(t_i)c_j(k_j))}{\prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - c_j(k_j))} \tag{4.2}$$

and

$$\begin{aligned} g_{\mathbf{kl}} \;=\; & \prod_{1\le i<j\le r} \left(\frac{(c_i(l_i) - c_j(l_j))}{(c_i(k_i) - c_j(k_j))} \frac{(1 - c_i(l_i)c_j(l_j))}{(1 - c_i(k_i)c_j(k_j))} \right) \\ & \times \prod_{i=1}^{r} \frac{(1 - c_i(l_i)^2)}{(1 - c_i(k_i)^2)} \prod_{i=1}^{r} \frac{c_i(l_i)}{c_i(k_i)} \\ & \times \prod_{i=1}^{r} \frac{\prod_{t_i=l_i+1}^{k_i} (1 - bc_i(t_i)/\prod_{j=1}^{r} c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - b/\prod_{j=1}^{r} c_j(k_j))} \prod_{i,j=1}^{r} \frac{\prod_{t_i=l_i+1}^{k_i} (1 - c_i(t_i)c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - c_j(k_j))}. \end{aligned} \tag{4.3}$$

Remark 4.4. The special case $c_j(k_j) = x_j^{-1} q^{-k_j}$ is a C_r generalization of Bressoud's matrix inversion formula [8], as pointed out in [18, second remark after Theorem 2.11]. Setting, in addition, $b = 0$ yields a C_r Bailey transform which is equivalent to the one derived in [18]. Therefore, we view Theorem 4.1 as a C_r matrix inversion theorem.

Proof of Theorem 4.1: Again, we will use the operator method of section 2. From (4.2) we deduce for $\mathbf{n} \ge \mathbf{k}$ the recursion

$$(c_i(n_i) - b/\prod_{j=1}^{r} c_j(k_j)) \prod_{s=1}^{r} (c_i(n_i) - c_s(k_s)) f_{\mathbf{nk}}$$

$$= (1 - bc_i(n_i - 1)/\prod_{j=1}^r c_j(k_j)) \prod_{s=1}^r (1 - c_i(n_i - 1)c_s(k_s)) f_{\mathbf{n}-\mathbf{e}_i,\mathbf{k}}, \quad (4.5)$$

for $i = 1, \dots, r$. We write

$$f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} \prod_{i=1}^r \frac{\prod_{t_i=k_i}^{n_i-1} (1 - bc_i(t_i)/\prod_{j=1}^r c_j(k_j))}{\prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \prod_{i,j=1}^r \frac{\prod_{t_i=k_i}^{n_i-1} (1 - c_i(t_i)c_j(k_j))}{\prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - c_j(k_j))} \mathbf{z}^{\mathbf{n}}.$$

Moreover, we define linear operators $\mathcal{C}_i$ by $\mathcal{C}_i \mathbf{z}^{\mathbf{n}} = c_i(n_i)\mathbf{z}^{\mathbf{n}}$ for $i = 1, \dots, r$. Then we may write (4.5) in the form

$$\begin{aligned} &(\mathcal{C}_i - b/\prod_{j=1}^r c_j(k_j)) \prod_{s=1}^r (\mathcal{C}_i - c_s(k_s)) f_{\mathbf{k}}(\mathbf{z}) \\ &\qquad = z_i(I - \mathcal{C}_i b/\prod_{j=1}^r c_j(k_j)) \prod_{s=1}^r (I - \mathcal{C}_i c_s(k_s)) f_{\mathbf{k}}(\mathbf{z}), \end{aligned} \quad (4.6)$$

I being the identity operator, valid for all $\mathbf{k} \in \mathbb{Z}^r$. We will write our system of equations in a way such that Corollary 2.14 is applicable. Again, we expand the products on both sides of (4.6) in terms of the elementary symmetric functions

$$e_j(c_1(k_1), c_2(k_2), \dots, c_r(k_r), b/\prod_{s=1}^r c_s(k_s))$$

of order j, for which we write $e_j(\mathbf{c}(\mathbf{k}))$ for short. Our recurrence system then reads, again using $e_{r+1}(\mathbf{c}(\mathbf{k})) = b$,

$$\begin{aligned} &\sum_{j=1}^r e_j(\mathbf{c}(\mathbf{k}))[(-\mathcal{C}_i)^{r+1-j} - (-1)^{r+1} z_i (-\mathcal{C}_i)^j] f_{\mathbf{k}}(\mathbf{z}) \\ &\qquad = [(-1)^{r+1} z_i + b z_i \mathcal{C}_i^{r+1} - (-\mathcal{C}_i)^{r+1} - b] f_{\mathbf{k}}(\mathbf{z}), \quad i = 1, \dots, r. \end{aligned} \quad (4.7)$$

Now, (4.7) is a system of type (2.18) with $V_{ij} = [(-\mathcal{C}_i)^{r+1-j} - (-1)^{r+1} z_i (-\mathcal{C}_i)^j]$, $W_i = [(-1)^{r+1} z_i + b z_i \mathcal{C}_i^{r+1} - (-\mathcal{C}_i)^{r+1} - b]$ and $c_j(\mathbf{k}) = e_j(\mathbf{c}(\mathbf{k}))$. The operators $C_{ij} = V_{ij}$, $A_{ij} = 0$, $W_i^{(c)} = W_i$, $W_i^{(a)} = 0$ satisfy (2.6), (2.7), (2.8), (2.15), (2.16), and (2.17), the functions $c_j(\mathbf{k})$ satisfy (2.2). Hence we may apply Corollary 2.14. The dual system (2.20) for the auxiliary formal Laurent series $h_{\mathbf{k}}(\mathbf{z})$ in this case reads

$$\begin{aligned} &\sum_{j=1}^r e_j(\mathbf{c}(\mathbf{k}))[(-\mathcal{C}_i^*)^{r+1-j} - (-1)^{r+1} (-\mathcal{C}_i^*)^j z_i] h_{\mathbf{k}}(\mathbf{z}) \\ &\qquad = [(-1)^{r+1} z_i + b(\mathcal{C}_i^*)^{r+1} z_i - (-\mathcal{C}_i^*)^{r+1} - b] h_{\mathbf{k}}(\mathbf{z}), \quad i = 1, \dots, r. \end{aligned}$$

Equivalently, we have

$$(\mathcal{C}_i^* - b/\prod_{j=1}^r c_j(k_j)) \prod_{s=1}^r (\mathcal{C}_i^* - c_s(k_s)) h_{\mathbf{k}}(\mathbf{z})$$

$$= (I - \mathcal{C}_i^* b / \prod_{j=1}^r c_j(k_j)) \prod_{s=1}^r (I - \mathcal{C}_i^* c_s(k_s)) z_i h_{\mathbf{k}}(\mathbf{z}), \tag{4.8}$$

for all $i = 1, \ldots, r$ and $\mathbf{k} \in \mathrm{Z}^r$. Thus, with $h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}$, by comparing coefficients of $\mathbf{z}^{-\mathbf{l}}$ in (4.8) we obtain

$$\begin{aligned}(c_i(l_i) - b/\prod_{j=1}^r c_j(k_j)) &\prod_{s=1}^r (c_i(l_i) - c_s(k_s)) h_{\mathbf{kl}} \\ &= (1 - bc_i(l_i)/\prod_{j=1}^r c_j(k_j)) \prod_{s=1}^r (1 - c_i(l_i) c_s(k_s)) h_{\mathbf{k},\mathbf{l}+\mathbf{e}_i}.\end{aligned}$$

If we set $h_{\mathbf{kk}} = 1$, we get

$$h_{\mathbf{kl}} = \prod_{i=1}^r \frac{\prod_{t_i=l_i}^{k_i-1} (1 - bc_i(t_i)/\prod_{j=1}^r c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \prod_{i,j=1}^r \frac{\prod_{t_i=l_i}^{k_i-1} (1 - c_i(t_i) c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - c_j(k_j))}.$$

Taking into account (2.19), we have to compute the action of

$$\overrightarrow{\det}_{1 \le i,j \le r} (V_{ij}^*) = \overrightarrow{\det}_{1 \le i,j \le r} [(-\mathcal{C}_i^*)^{r+1-j} - (-1)^{r+1} (-\mathcal{C}_i^*)^j z_i]$$

when applied to

$$h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} \prod_{i=1}^r \frac{\prod_{t_i=l_i}^{k_i-1} (1 - bc_i(t_i)/\prod_{j=1}^r c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - b/\prod_{j=1}^r c_j(k_j))} \prod_{i,j=1}^r \frac{\prod_{t_i=l_i}^{k_i-1} (1 - c_i(t_i) c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - c_j(k_j))} \mathbf{z}^{-\mathbf{l}}.$$

Because $V_{i_1 j_1}^*$ and $V_{i_2 j_2}^*$ commute for $i_1 \neq i_2$ and all j_1, j_2, all the summands in $\overrightarrow{\det}_{1 \le i,j \le r} (V_{ij}^*)$ have pairwise commuting factors. Since

$$z_i h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} \frac{(c_i(l_i) - b/\prod_{j=1}^r c_j(k_j))}{(1 - bc_i(l_i)/\prod_{j=1}^r c_j(k_j))} \prod_{j=1}^r \frac{(c_i(l_i) - c_j(k_j))}{(1 - c_i(l_i) c_j(k_j))} h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}},$$

we conclude that

$$\overrightarrow{\det}_{1 \le i,j \le r} (V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \le \mathbf{k}} \det_{1 \le i,j \le r} (v_{ij}) h_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}, \tag{4.9}$$

where

$$\begin{aligned} v_{ij} \;=\; & (-c_i(l_i))^{r+1-j} \\ & -(-1)^{r+1} (-c_i(l_i))^j \frac{(c_i(l_i) - b/\prod_{s=1}^r c_s(k_s))}{(1 - bc_i(l_i)/\prod_{s=1}^r c_s(k_s))} \prod_{s=1}^r \frac{(c_i(l_i) - c_s(k_s))}{(1 - c_i(l_i) c_s(k_s))}. \end{aligned}$$

For the computation of $\det_{1\le i,j\le r}(v_{ij})$ we utilize Lemma A.11 with $x_i = -c_i(l_i)$, $y_s = -c_s(k_s)$, and $c = (-1)^{r-1}b$, obtaining

$$\begin{aligned}\det_{1\le i,j\le r}(v_{ij}) \;&=\; \prod_{i=1}^{r} \frac{(1-c_i(k_i)b/\prod_{j=1}^{r} c_j(k_j))}{(1-c_i(l_i)b/\prod_{j=1}^{r} c_j(k_j))}\\ &\quad\times \prod_{i=1}^{r}(1-c_i(l_i)^2)\prod_{i=1}^{r}(-c_i(l_i)) \prod_{i,j=1}^{r}(1-c_i(l_i)c_j(k_j))^{-1}\\ &\quad\times \prod_{1\le i<j\le r} [(c_j(l_j)-c_i(l_i))(1-c_i(l_i)c_j(l_j))(1-c_i(k_i)c_j(k_j))].\end{aligned}$$

Plugging this determinant evaluation into (4.9) leads to

$$\begin{aligned}\overrightarrow{\det}_{1\le i,j\le r}\,(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z}) \;&=\; \sum_{\mathbf{l}\le\mathbf{k}} \Bigg(\prod_{1\le i<j\le r} [(c_j(l_j)-c_i(l_i))(1-c_i(l_i)c_j(l_j))]\\ &\quad\times \prod_{i=1}^{r}(1-c_i(l_i)^2)\prod_{i=1}^{r}(-c_i(l_i)) \prod_{i\le j}(1-c_i(k_i)c_j(k_j))^{-1}\\ &\quad\times \prod_{i=1}^{r} \frac{\prod_{t_i=l_i+1}^{k_i}(1-bc_i(t_i)/\prod_{j=1}^{r} c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-b/\prod_{j=1}^{r} c_j(k_j))} \prod_{i,j=1}^{r} \frac{\prod_{t_i=l_i+1}^{k_i}(1-c_i(t_i)c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_j(k_j))}\mathbf{z}^{-\mathbf{l}}\Bigg).\end{aligned} \tag{4.10}$$

Again, the pairing $\langle f_{\mathbf{k}}(\mathbf{z}), \overrightarrow{\det}\,(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z})\rangle$ is simply the coefficient of $\mathbf{z}^{-\mathbf{k}}$ in (4.10). Thus, equation (2.19) reads

$$g_{\mathbf{k}}(\mathbf{z}) = \prod_{1\le i<j\le r}(c_j(k_j)-c_i(k_i))^{-1} \prod_{i=1}^{r}(-c_i(k_i))^{-1}\, \overrightarrow{\det}_{1\le i,j\le r}\,(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z}), \tag{4.11}$$

where $g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}$. So, extracting the coefficient of $\mathbf{z}^{-\mathbf{l}}$ in (4.11) we obtain exactly (4.3). ■

5. Applications to A_r and D_r basic hypergeometric series

Probably, the most important application of matrix inversion is the derivation of hypergeometric series identities. There is a standard technique for deriving new summation formulas from known ones by using inverse matrices (cf. [1], [13], [26]). If $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in Z^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in Z^r}$ are lower triangular matrices being inverses of each other, then of course the following is true:

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} f_{\mathbf{nk}}a_{\mathbf{k}} = b_{\mathbf{n}} \tag{5.1}$$

if and only if

$$\sum_{\mathbf{0}\le\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}} b_{\mathbf{l}} = a_{\mathbf{k}}. \tag{5.2}$$

We expect that applications of our matrix inversions in Theorems 3.1 and 4.1 will lead to many new identities for multidimensional (basic) hypergeometric series. As an illustration, we use special cases of our Theorem 3.1 to derive A_r and D_r extensions of a terminating quadratic summation of Gessel and Stanton [13], D_r extensions of Jackson's ${}_8\phi_7$ summation [14], and a D_r extension of a cubic summation of Gasper and Rahman [11].

We recall the standard definition of the rising q-factorial (cf. [12]). Define

$$(a;q)_\infty := \prod_{j\ge 0}(1-aq^j),$$

and for any integer k,

$$(a;q)_k := \prod_{j=0}^{k-1}(1-aq^j) = \frac{(a;q)_\infty}{(aq^k;q)_\infty}. \tag{5.3}$$

Theorem 5.4 (An A_r quadratic sum) *Let $x_1,\dots,x_r$, a, b, and d be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers, let $r\ge 1$, and suppose that none of the denominators in* (5.5) *vanish. Then*

$$\begin{aligned}
\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} &\left(\prod_{i=1}^r\left(\frac{1-ax_iq^{2k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right)\right.\\
&\times\prod_{i,j=1}^r\frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}}\prod_{i=1}^r\frac{(dx_i;q^2)_{k_i}\,(a^2x_iq^{1+2|\mathbf{n}|}/d;q^2)_{k_i}}{(ax_iq^2/b;q^2)_{k_i}\,(abx_iq;q^2)_{k_i}}\\
&\left.\times\prod_{i=1}^r\frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+2n_i};q)_{|\mathbf{k}|}}\cdot\frac{(b;q)_{|\mathbf{k}|}\,(q/b;q)_{|\mathbf{k}|}}{(aq/d;q)_{|\mathbf{k}|}\,(dq^{-2|\mathbf{n}|}/a;q)_{|\mathbf{k}|}}\,q^{-|\mathbf{k}|+2\sum_{i=1}^r ik_i}\right)\\
&=\frac{(aq^2/bd;q^2)_{|\mathbf{n}|}\,(abq/d;q^2)_{|\mathbf{n}|}}{(aq/d;q)_{2|\mathbf{n}|}}\prod_{i=1}^r\frac{(ax_iq;q)_{2n_i}}{(ax_iq^2/b;q^2)_{n_i}\,(abx_iq;q^2)_{n_i}}.
\end{aligned} \tag{5.5}$$

Remark 5.6. This quadratic summation formula is an A_r extension of

$$\begin{aligned}
\sum_{k=0}^n\frac{1-aq^{3k}}{1-a}&\frac{(a;q)_k(b;q)_k(q/b;q)_k\,(d;q^2)_k(a^2q^{1+2n}/d;q^2)_k(q^{-2n};q^2)_k}{(q^2;q^2)_k(aq^2/b;q^2)_k(abq;q^2)_k\,(aq/d;q)_k(dq^{-2n}/a;q)_k(aq^{2n+1};q)_k}\,q^k\\
&=\frac{(aq;q)_{2n}}{(aq/d;q)_{2n}}\frac{(abq/d;q^2)_n(aq^2/bd;q^2)_n}{(aq^2/b;q^2)_n(abq;q^2)_n},
\end{aligned} \tag{5.7}$$

due to Gessel and Stanton [13, eq. (1.4), $q\to q^2$], to which it reduces for $r=1$. Many identities like (5.7), involving bases of different powers of q, are known. Hypergeometric series with several bases were extensively studied by Gasper and Rahman [10], [25], [11], [12, sec. 3.8].

Proof of Theorem 5.4: If we substitute $c_i(t_i) \mapsto q^{-2t_i}/x_i$, $i = 1, \dots, r$, $a_t \mapsto aq^t$, and $b \mapsto a^2/dx_1 \cdots x_n$ in Theorem 3.1 (this special case can be also obtained from the inversion [6, Theorem 3.48] of Bhatnagar and Milne) we see that the following pair of matrices are inverses of each other:

$$\begin{aligned} f_{\mathbf{nk}} &= \prod_{i=1}^{r} \left(\frac{1 - q^{1+2k_i+2|\mathbf{k}|} a^2 x_i/d}{1 - qa^2 x_i/d} \right) \prod_{1 \le i < j \le r} \left(\frac{1 - q^{2k_i - 2k_j} x_i/x_j}{1 - x_i/x_j} \right) \\ &\quad \times \prod_{i,j=1}^{r} \frac{(q^{-2n_j} x_i/x_j; q^2)_{k_i}}{(q^2 x_i/x_j; q^2)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i q^{|\mathbf{n}|}; q)_{2k_i}}{(a^2 x_i q^{3+2n_i}/d; q^2)_{|\mathbf{k}|}} \cdot \frac{q^{2\sum_{i=1}^{r} i\, k_i}}{(aq^{2-|\mathbf{n}|}/d; q)_{2|\mathbf{k}|}} \end{aligned}$$

and

$$\begin{aligned} g_{\mathbf{kl}} &= \prod_{i=1}^{r} \left(\frac{1 - ax_i q^{2l_i + |\mathbf{l}|}}{1 - ax_i} \right) \prod_{1 \le i < j \le r} \left(\frac{1 - q^{2l_i - 2l_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{i,j=1}^{r} \frac{(q^{-2k_j} x_i/x_j; q^2)_{l_i}}{(q^2 x_i/x_j; q^2)_{l_i}} \\ &\quad \times \prod_{i=1}^{r} \frac{(ax_i; q)_{|\mathbf{l}|}}{(ax_i q^{1+2k_i}; q)_{|\mathbf{l}|}} \prod_{i=1}^{r} \frac{(a^2 x_i q^{1+2|\mathbf{k}|}/d; q^2)_{l_i}}{(a^2 x_i q^3/d; q^2)_{l_i}} \prod_{i=1}^{r} \frac{(a^2 x_i q/d; q^2)_{|\mathbf{k}|}}{(ax_i q; q)_{2k_i}} \\ &\qquad \times \frac{(1 - q^{1+|\mathbf{l}|} a/d)}{(1 - qa/d)} \frac{(d/aq; q)_{|\mathbf{l}|} (qa/d; q)_{2|\mathbf{k}|}}{(q^{2-2|\mathbf{k}|} a/d; q)_{|\mathbf{l}|}} q^{-|\mathbf{l}| + 2\sum_{i=1}^{r} i\, l_i}. \end{aligned}$$

Now (5.1) holds for

$$a_{\mathbf{k}} = (baq/d; q^2)_{|\mathbf{k}|} (aq^2/bd; q^2)_{|\mathbf{k}|} \prod_{i=1}^{r} \frac{(a^2 x_i q/d; q^2)_{|\mathbf{k}|}}{(ax_i q^2/b; q^2)_{k_i} (abx_i q; q)_{k_i}}$$

and

$$b_{\mathbf{n}} = \frac{(q^{2-|\mathbf{n}|}/b; q^2)_{|\mathbf{n}|} (bq^{1-|\mathbf{n}|}; q^2)_{|\mathbf{n}|}}{(aq^{3-|\mathbf{n}|}/d; q^2)_{|\mathbf{n}|} (dq^{-|\mathbf{n}|}/a; q^2)_{|\mathbf{n}|}} \prod_{i=1}^{r} \frac{(a^2 x_i q^3/d; q^2)_{n_i} (dx_i/a; q^2)_{n_i}}{(ax_i q^2/b; q^2)_{n_i} (abx_i q; q^2)_{n_i}}$$

by means of an A_r extension of Jackson's ${}_8\phi_7$-sum, taken from [20, Theorem 6.14] (or in more convenient notation [24, Theorem A12]). This implies the inverse relation (5.2) which is easily transformed into (5.5). ■

It is not hard to see from a polynomial identity argument that Theorem 5.4 implies the following summation theorem.

Theorem 5.8 (An A_r quadratic sum) *Let $x_1, \dots, x_r$, $c_1, \dots, c_r$, a, and d be indeterminate, let N be a nonnegative integer, let $r \ge 1$, and suppose that none of the denominators* *in* (5.9) *vanish. Then*

$$\sum_{\substack{k_1,\dots,k_r\geq 0\\ \mathbf{0}\leq|\mathbf{k}|\leq N}} \Bigg(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{2k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\leq i<j\leq r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times\prod_{i,j=1}^{r}\frac{(c_jx_i/x_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}}\prod_{i=1}^{r}\frac{(dx_i;q^2)_{k_i}\,(a^2x_iq/d\prod_{j=1}^{r}c_j;q^2)_{k_i}}{(ax_iq^{2+N};q^2)_{k_i}\,(ax_iq^{1-N};q^2)_{k_i}}$$
$$\times\prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq/c_i;q)_{|\mathbf{k}|}}\cdot\frac{(q^{-N};q)_{|\mathbf{k}|}\,(q^{1+N};q)_{|\mathbf{k}|}}{(aq/d;q)_{|\mathbf{k}|}\,(d\prod_{j=1}^{r}c_j/a;q)_{|\mathbf{k}|}}\,q^{-|\mathbf{k}|+2\sum_{i=1}^{r}i\,k_i}\Bigg)$$
$$=\begin{cases}\dfrac{(dq/a;q^2)_M(aq^2/d\prod_{j=1}^{r}c_j;q^2)_M}{(aq^2/d;q^2)_M(dq\prod_{j=1}^{r}c_j/a;q^2)_M}\displaystyle\prod_{i=1}^{r}\frac{(ax_iq^2;q^2)_M(c_iq/ax_i;q^2)_M}{(q/ax_i;q^2)_M(ax_iq^2/c_i;q^2)_M} & (N=2M),\\[2ex] \dfrac{(d/a;q^2)_M(aq/d\prod_{j=1}^{r}c_j;q^2)_M}{(aq/d;q^2)_M(d\prod_{j=1}^{r}c_j/a;q^2)_M}\displaystyle\prod_{i=1}^{r}\frac{(ax_iq;q^2)_M(c_i/ax_i;q^2)_M}{(1/ax_i;q^2)_M(ax_iq/c_i;q^2)_M} & (N=2M-1).\end{cases}\tag{5.9}$$

Proof: First we write the right sides of (5.9) as quotients of infinite products using (5.3). Then by the $b=q^{-N}$ case of Theorem 5.4 it follows that the identity (5.9) holds for $c_j=q^{-2n_j}$, $j=1,\dots,r$. By clearing out denominators in (5.9), we get a polynomial equation in c_1, which is true for q^{-2n_1}, $n_1=0,1,\dots$. Thus we obtain an identity in c_1. By carrying out this process for $c_2,c_3,\dots,c_r$ also, we obtain Theorem 5.8. ■

By another specialization of Theorem 3.1 we obtain an interesting bibasic hypergeometric matrix inversion. We use this inversion to derive D_r basic hypergeometric summation formulas. For explanations why we associate D_r with these formulas the reader is referred to [5].

Theorem 5.10 *Let*

$$f_{\mathbf{nk}}=\frac{\prod_{i=1}^{r}\left[(ap^{|\mathbf{k}|}q^{k_i}x_i;p)_{|\mathbf{n}|-|\mathbf{k}|}\,(ap^{|\mathbf{k}|}q^{-k_i}/x_i;p)_{|\mathbf{n}|-|\mathbf{k}|}\right]}{\prod_{i,j=1}^{r}\left[(q^{1+k_i-k_j}x_i/x_j;q)_{n_i-k_i}\,(q^{1+k_i+k_j}x_ix_j;q)_{n_i-k_i}\right]}\tag{5.11}$$

and

$$g_{\mathbf{kl}}=(-1)^{|\mathbf{k}|-|\mathbf{l}|}\,q^{\binom{|\mathbf{k}|-|\mathbf{l}|}{2}}\prod_{1\leq i<j\leq r}\frac{(1-x_ix_jq^{l_i+l_j})}{(1-x_ix_jq^{k_i+k_j})}$$
$$\times\prod_{i=1}^{r}\frac{(1-ap^{|\mathbf{l}|}q^{l_i}x_i)(1-ap^{|\mathbf{l}|}q^{-l_i}/x_i)}{(1-ap^{|\mathbf{k}|}q^{k_i}x_i)(1-ap^{|\mathbf{k}|}q^{-k_i}/x_i)}$$

$$\times \frac{\prod_{i=1}^{r} \left[(ap^{1+|\mathbf{l}|}q^{k_i}x_i;p)_{|\mathbf{k}|-|\mathbf{l}|}\,(ap^{1+|\mathbf{l}|}q^{-k_i}/x_i;p)_{|\mathbf{k}|-|\mathbf{l}|}\right]}{\prod_{i,j=1}^{r} \left[(q^{1+l_i-l_j}x_i/x_j;q)_{k_i-l_i}\,(q^{l_i+k_j}x_ix_j;q)_{k_i-l_i}\right]}. \tag{5.12}$$

Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in Z^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in Z^r}$ are infinite lower-triangular r-dimensional matrices being inverses of each other.

Proof: In Theorem 3.1 we set $b=0$, $a_t = ap^t + p^{-t}/a$, and $c_i(t_i) = x_iq^{t_i} + q^{-t_i}/x_i$ for $i = 1,\ldots,r$. After some elementary manipulations we obtain the inverse pair (5.11) and (5.12). ∎

Remark 5.13. The inversion in Theorem 5.10 is a D_r extension of Gasper and Rahman's bibasic matrix inversion [12, (3.6.19) and (3.6.20)], to which it reduces for $r=1$.

Theorem 5.14 (A D_r Jackson's sum) *Let $x_1,\ldots,x_r$, a, b, and c be indeterminate, let $n_1,\ldots,n_r$ be nonnegative integers, let $r\geq 1$, and suppose that none of the denominators in (5.15) vanish. Then*

$$\sum_{\substack{0\leq k_i\leq n_i\\ i=1,\ldots,r}} \Bigg(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\leq i<j\leq r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times \prod_{1\leq i<j\leq r}(x_ix_j;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^{r}\frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(x_ix_jq^{n_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$
$$\times \prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-k_i}}{(aq^{1+n_i}x_i;q)_{|\mathbf{k}|}\,(aq^{1-n_i}/x_i;q)_{|\mathbf{k}|}}$$
$$\times \frac{(b;q)_{|\mathbf{k}|}\,(c;q)_{|\mathbf{k}|}\,(a^2q/bc;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}\left[(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(bcx_i/a;q)_{k_i}\right]}\, q^{\sum_{i=1}^{r} i\,k_i} \Bigg)$$
$$= \prod_{i=1}^{r}\frac{(ax_iq;q)_{n_i}\,(ax_iq/bc;q)_{n_i}\,(bx_i/a;q)_{n_i}\,(cx_i/a;q)_{n_i}}{(x_i/a;q)_{n_i}\,(bcx_i/a;q)_{n_i}\,(ax_iq/b;q)_{n_i}\,(ax_iq/c;q)_{n_i}}. \tag{5.15}$$

Remark 5.16. For $r=1$ Theorem 5.14 reduces to Jackson's very-well-poised ${}_8\phi_7$ summation formula [14], [12, (II.22)].

Proof of Theorem 5.14: Setting $p=q$ in Theorem 5.10 (i.e. here we consider a D_r extension of Bressoud's matrix inverse [8]) we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}} = \prod_{1\leq i<j\leq r}\frac{(1-q^{k_i-k_j}x_i/x_j)(1-x_ix_jq^{k_i+k_j})}{(1-x_i/x_j)(1-x_ix_j)} \prod_{i=1}^{r}\frac{(1-x_i^2q^{2k_i})}{(1-x_i^2)}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(x_ix_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(x_ix_jq^{1+n_j};q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_iq^{|\mathbf{n}|};q)_{k_i}}{(x_iq^{1-|\mathbf{n}|}/a;q)_{k_i}} \;\cdot\; q^{\sum_{i=1}^{r} i\,k_i}$$

and

$$\begin{aligned} g_{\mathbf{kl}} \;=\; & \prod_{1\le i<j\le r} \frac{(1-q^{l_i-l_j}x_i/x_j)(1-x_ix_jq^{l_i+l_j})}{(1-x_i/x_j)(1-x_ix_j)} \\ & \times \prod_{i,j=1}^{r} \frac{(q^{-k_j}x_i/x_j;q)_{l_i}\,(x_ix_jq^{k_j};q)_{l_i}}{(qx_i/x_j;q)_{l_i}\,(x_ix_jq;q)_{l_i}} \prod_{i=1}^{r} \frac{(1-aq^{|\mathbf{l}|+l_i}x_i)(1-aq^{|\mathbf{l}|-l_i}/x_i)}{(1-ax_i)(1-a/x_i)} \\ & \times \prod_{i=1}^{r} \frac{(x_i/a;q)_{k_i}}{(ax_iq;q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{l}|}\,(a/x_i;q)_{|\mathbf{l}|}}{(aq^{1+k_i}x_i;q)_{|\mathbf{l}|}\,(aq^{1-k_i}/x_i;q)_{|\mathbf{l}|}} \;\cdot\; q^{\sum_{i=1}^{r} i\,l_i}. \end{aligned}$$

Now (5.1) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^{r} \frac{(bx_i/a;q)_{k_i}\,(cx_i/a;q)_{k_i}\,(ax_iq/bc;q)_{k_i}}{(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(bcx_i/a;q)_{k_i}}$$

and

$$\begin{aligned} b_{\mathbf{n}} = & \prod_{i=1}^{r} \frac{(x_i^2q;q)_{n_i}}{(bcx_i/a;q)_{n_i}\,(x_iq^{1-|\mathbf{n}|}/a;q)_{n_i}} \prod_{1\le i<j\le r} \frac{(x_ix_jq;q)_{n_i}}{(x_ix_jq^{1+n_j};q)_{n_i}} \\ & \times \frac{(bcq^{-|\mathbf{n}|}/a^2;q)_{|\mathbf{n}|}\,(q^{1-|\mathbf{n}|}/b;q)_{|\mathbf{n}|}\,(c;q)_{|\mathbf{n}|}}{\prod_{i=1}^{r}\left[(ax_iq/b;q)_{n_i}\,(cq^{-n_i}/ax_i;q)_{n_i}\right]} \end{aligned}$$

by Milne and Lilly's $C_r\ {}_8\phi_7$ summation [23, Theorem 6.13]. This implies the inverse relation (5.2) which is easily transformed into (5.15). ∎

By using a polynomial argument we get

Theorem 5.17 (A D_r Jackson's sum) *Let $x_1,\ldots,x_r$, $c_1,\ldots,c_r$, a, and b be indeterminate, let N be a nonnegative integer, let $r\ge 1$, and suppose that none of the denominators in (5.18) vanish. Then*

$$\begin{aligned} \sum_{\substack{k_1,\ldots,k_r\ge 0\\ 0\le|\mathbf{k}|\le N}} & \left(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\right. \\ & \times \prod_{1\le i<j\le r}(x_ix_j;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^{r} \frac{(c_jx_i/x_j;q)_{k_i}\,(x_ix_j/c_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \\ & \times \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-k_i}}{(ax_iq/c_i;q)_{|\mathbf{k}|}\,(ac_iq/x_i;q)_{|\mathbf{k}|}} \end{aligned}$$

$$\times \frac{(b;q)_{|\mathbf{k}|}\,(a^2q^{1+N}/b;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}\left[(ax_iq/b;q)_{k_i}\,(bx_iq^{-N}/a;q)_{k_i}\,(ax_iq^{1+N};q)_{k_i}\right]}\; q^{\sum_{i=1}^{r} i\,k_i}\Bigg)$$

$$= \prod_{i=1}^{r} \frac{(ax_iq;q)_N\,(aq/x_i;q)_N\,(ax_iq/bc_i;q)_N\,(ac_iq/bx_i;q)_N}{(aq/bx_i;q)_N\,(ax_iq/b;q)_N\,(ac_iq/x_i;q)_N\,(ax_iq/c_i;q)_N}. \tag{5.18}$$

Proof: First we write the right side of (5.18) as quotient of infinite products using (5.3). Then by the $c = q^{-N}$ case of Theorem 5.14 it follows that the identity (5.18) holds for $c_j = q^{-n_j}$, $j = 1, \ldots, r$. By clearing out denominators in (5.18), we get a polynomial equation in c_1, which is true for q^{-n_1}, $n_1 = 0, 1, \ldots$. Thus we obtain an identity in c_1. By carrying out this process for $c_2, c_3, \ldots, c_r$ also, we obtain Theorem 5.17. ∎

Limiting cases of Theorem 5.14 or Theorem 5.17 include various D_r summations. By reversing the multisum in Theorem 5.14 we obtain another D_r Jackson's sum which was independently derived by G. Bhatnagar [5] using a different method. D_r extensions of many of the classical basic hypergeometric summation theorems are given in [5]. Further consequences of the new D_r ${}_8\phi_7$ summations, such as C_r and D_r extensions of Bailey's very-well-poised ${}_{10}\phi_9$ transformation formula [2], [12, (III.28)] will be given in [7].

Remark 5.19. We note that (5.15) and (5.18) could be written (with $a = 1/x_{r+1}$ and $k_{r+1} := -|\mathbf{k}|$) more compactly as

$$\sum_{k_1+\cdots+k_{r+1}=0} \Bigg(\prod_{1\le i<j\le r+1} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r+1} (x_ix_j;q)^{-1}_{k_i+k_j}$$

$$\times \prod_{i=1}^{r+1}\prod_{j=1}^{r} \frac{(c_jx_i/x_j;q)_{k_i}\,(x_ix_j/c_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$

$$\times \prod_{i=1}^{r+1} \left[(x_iq/bx_{r+1};q)_{k_i}\,(x_iq/cx_{r+1};q)_{k_i}\,(bcx_ix_{r+1};q)_{k_i}\right]^{-1}$$

$$\times \prod_{i=1}^{r+1} x_i^{k_i} \;\cdot\; q^{\sum_{i=1}^{r+1}\left[\binom{k_i}{2}+i\,k_i\right]}\Bigg)$$

$$= \prod_{i=1}^{r} \frac{(x_iq/x_{r+1};q)_\infty(q/x_ix_{r+1};q)_\infty(x_iq/bc_ix_{r+1};q)_\infty(c_iq/bx_ix_{r+1};q)_\infty}{(q/bx_ix_{r+1};q)_\infty(x_iq/bx_{r+1};q)_\infty(c_iq/x_ix_{r+1};q)_\infty(x_iq/c_ix_{r+1};q)_\infty}$$

$$\times \prod_{i=1}^{r} \frac{(q/bcx_ix_{r+1};q)_\infty(x_iq/bcx_{r+1};q)_\infty(c_iq/cx_ix_{r+1};q)_\infty(x_iq/cc_ix_{r+1};q)_\infty}{(x_iq/cx_{r+1};q)_\infty(q/cx_ix_{r+1};q)_\infty(x_iq/bcc_ix_{r+1};q)_\infty(c_iq/bcx_ix_{r+1};q)_\infty}, \tag{5.20}$$

provided the series terminates. However, we feel that the forms (5.15) and (5.18) are preferable since the dependence of summation indices in (5.20) (which just hides what is really going on in the sum) is removed.

The remainder of this section is devoted to D_r quadratic and cubic summation formulas.

Theorem 5.21 (A D_r quadratic sum) *Let $x_1,\dots,x_r$, a, and b be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers, let $r\ge 1$, and suppose that none of the denominators in (5.22) vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\Bigg(\prod_{i=1}^r\left(\frac{1-ax_iq^{2k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times\prod_{1\le i<j\le r}(x_ix_j;q^2)^{-1}_{k_i+k_j}\prod_{i,j=1}^r\frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}\,(x_ix_jq^{2n_j};q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}}$$
$$\times\prod_{i=1}^r\frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-2k_i}}{(ax_iq^{1+2n_i};q)_{|\mathbf{k}|}\,(aq^{1-2n_i}/x_i;q)_{|\mathbf{k}|}}\cdot\frac{(a^2q;q^2)_{|\mathbf{k}|}\,(b;q)_{|\mathbf{k}|}\,(q/b;q)_{|\mathbf{k}|}}{\prod_{i=1}^r\left[(abx_iq;q^2)_{k_i}\,(ax_iq^2/b;q^2)_{k_i}\right]}$$
$$\times(-a)^{|\mathbf{k}|}\prod_{i=1}^r x_i^{-k_i}\cdot q^{2e_2(\mathbf{k})-\binom{|\mathbf{k}|+1}{2}+2\sum_{i=1}^r i\,k_i}\Bigg)$$
$$=\prod_{i=1}^r\frac{(ax_iq;q)_{2n_i}\,(x_iq/ab;q^2)_{n_i}\,(bx_i/a;q^2)_{n_i}}{(x_i/a;q)_{2n_i}\,(abx_iq;q^2)_{n_i}\,(ax_iq^2/b;q^2)_{n_i}},\tag{5.22}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\dots,k_r\}$.

Remark 5.23. This quadratic summation formula is a D_r extension of Gessel and Stanton's summation [13, eq. (1.4)], displayed in (5.7), to which it reduces for $r=1$.

Proof of Theorem 5.21: Doing the replacements $q\to q^2$, $p\to q$ in Theorem 5.10 we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}}=\prod_{1\le i<j\le r}\frac{(1-q^{2k_i-2k_j}x_i/x_j)(1-x_ix_jq^{2k_i+2k_j})}{(1-x_i/x_j)(1-x_ix_j)}\prod_{i=1}^r\frac{(1-x_i^2q^{4k_i})}{(1-x_i^2)}$$
$$\times\prod_{i,j=1}^r\frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}\,(x_ix_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}\,(x_ix_jq^{2+2n_j};q^2)_{k_i}}\prod_{i=1}^r\frac{(ax_iq^{|\mathbf{n}|};q)_{2k_i}}{(x_iq^{1-|\mathbf{n}|}/a;q)_{2k_i}}\cdot q^{2\sum_{i=1}^r i\,k_i}$$

and

$$g_{\mathbf{kl}}=\prod_{1\le i<j\le r}\frac{(1-q^{2l_i-2l_j}x_i/x_j)(1-x_ix_jq^{2l_i+2l_j})}{(1-x_i/x_j)(1-x_ix_j)}$$
$$\times\prod_{i,j=1}^r\frac{(q^{-2k_j}x_i/x_j;q^2)_{l_i}\,(x_ix_jq^{2k_j};q^2)_{l_i}}{(q^2x_i/x_j;q^2)_{l_i}\,(x_ix_jq^2;q^2)_{l_i}}\prod_{i=1}^r\frac{(1-aq^{|\mathbf{l}|+2l_i}x_i)(1-aq^{|\mathbf{l}|-2l_i}/x_i)}{(1-ax_i)(1-a/x_i)}$$
$$\times\prod_{i=1}^r\frac{(x_i/a;q)_{2k_i}}{(ax_iq;q)_{2k_i}}\prod_{i=1}^r\frac{(ax_i;q)_{|\mathbf{l}|}\,(a/x_i;q)_{|\mathbf{l}|}}{(aq^{1+2k_i}x_i;q)_{|\mathbf{l}|}\,(aq^{1-2k_i}/x_i;q)_{|\mathbf{l}|}}\cdot q^{2\sum_{i=1}^r i\,l_i}.$$

Now (5.1) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^{r} \frac{(x_i q/ab;q^2)_{k_i}\,(bx_i/a;q^2)_{k_i}}{(abx_i q;q^2)_{k_i}\,(ax_i q^2/b;q^2)_{k_i}}$$

and

$$b_{\mathbf{n}} = \prod_{i=1}^{r} \frac{(x_i^2 q^2;q^2)_{n_i}}{(abx_i q;q^2)_{n_i}\,(x_i q^{2-|\mathbf{n}|}/a;q^2)_{n_i}} \prod_{1\le i<j\le r} \frac{(x_i x_j q^2;q^2)_{n_i}}{(x_i x_j q^{2+2n_j};q^2)_{n_i}} \\ \times \frac{(bq^{1-|\mathbf{n}|};q^2)_{|\mathbf{n}|}\,(q^{2-|\mathbf{n}|}/b;q^2)_{|\mathbf{n}|}\,(a^2q;q^2)_{|\mathbf{n}|}}{\prod_{i=1}^{r}\left[(ax_i q^2/b;q^2)_{n_i}\,(aq^{1+|\mathbf{n}|-2n_i}/x_i;q^2)_{n_i}\right]}$$

by Milne and Lilly's C_r ${}_8\phi_7$ summation [23, Theorem 6.13]. This implies the inverse relation (5.2) which is easily transformed into (5.22). ∎

Remark 5.24. By reversing the multisum in (5.22) we may obtain another, differently looking, extension of Gessel and Stanton's quadratic summation formula (5.7).

Using the same technique as in the proofs of Theorems 5.8 and 5.17 we obtain from Theorem 5.21 the next two quadratic summation theorems (with minor substitutions of variables in Theorem 5.27).

Theorem 5.25 (A D_r quadratic sum) *Let $x_1,\dots,x_r$, $c_1,\dots,c_r$, and a be indeterminate, let N be a nonnegative integer, let $r\ge 1$, and suppose that none of the denominators in* (5.26) *vanish. Then*

$$\sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le N}} \left(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{2k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right) \right. \\ \times \prod_{1\le i<j\le r}(x_ix_j;q^2)^{-1}_{k_i+k_j} \prod_{i,j=1}^{r} \frac{(c_jx_i/x_j;q^2)_{k_i}\,(x_ix_j/c_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}} \\ \times \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-2k_i}}{(ax_iq/c_i;q)_{|\mathbf{k}|}\,(ac_iq/x_i;q)_{|\mathbf{k}|}} \cdot \frac{(a^2q;q^2)_{|\mathbf{k}|}\,(q^{1+N};q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}\left[(ax_iq^{2+N};q^2)_{k_i}\,(ax_iq^{1-N};q^2)_{k_i}\right]} \\ \left. \times(-a)^{|\mathbf{k}|}\prod_{i=1}^{r}x_i^{-k_i} \cdot q^{2e_2(\mathbf{k})-\binom{|\mathbf{k}|+1}{2}+2\sum_{i=1}^{r} i\,k_i} \right) \\ = \begin{cases} \displaystyle\prod_{i=1}^{r} \frac{(ax_iq^2;q^2)_M\,(aq^2/x_i;q^2)_M\,(c_iq/ax_i;q^2)_M\,(x_iq/ac_i;q^2)_M}{(q/ax_i;q^2)_M\,(x_iq/a;q^2)_M\,(ax_iq^2/c_i;q^2)_M\,(ac_iq^2/x_i;q^2)_M} & (N=2M), \\ \displaystyle\prod_{i=1}^{r} \frac{(ax_iq;q^2)_M\,(aq/x_i;q^2)_M\,(c_i/ax_i;q^2)_M\,(x_i/ac_i;q^2)_M}{(1/ax_i;q^2)_M\,(x_i/a;q^2)_M\,(ax_iq/c_i;q^2)_M\,(ac_iq/x_i;q^2)_M} & (N=2M-1), \end{cases} \tag{5.26}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\dots,k_r\}$.

Theorem 5.27 (A D_r quadratic sum) *Let $x_1,\dots,x_r$, $c_1,\dots,c_r$, and b be indeterminate, let N be a nonnegative integer, let $r\ge 1$, and suppose that none of the denominators in* (5.28) *vanish. Then*

$$\sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le N}}\Bigg(\prod_{i=1}^r\left(\frac{1-x_iq^{2k_i+|\mathbf{k}|}}{1-x_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times\prod_{1\le i<j\le r}(x_ix_jq^{1+2N};q^2)^{-1}_{k_i+k_j}\prod_{i,j=1}^r\frac{(c_jx_i/x_j;q^2)_{k_i}\,(x_ix_jq^{1+2N}/c_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}}$$
$$\times\prod_{i=1}^r\frac{(x_i;q)_{|\mathbf{k}|}\,(q^{-2N}/x_i;q)_{|\mathbf{k}|-2k_i}}{(x_iq/c_i;q)_{|\mathbf{k}|}\,(c_iq^{-2N}/x_i;q)_{|\mathbf{k}|}}\cdot\frac{(b;q)_{|\mathbf{k}|}\,(q/b;q)_{|\mathbf{k}|}\,(q^{-2N};q^2)_{|\mathbf{k}|}}{\prod_{i=1}^r[(bx_iq;q^2)_{k_i}\,(x_iq^2/b;q^2)_{k_i}]}$$
$$\times\prod_{i=1}^r(-x_i)^{-k_i}\cdot q^{2e_2(\mathbf{k})-\binom{|\mathbf{k}|}{2}-2N|\mathbf{k}|+2\sum_{i=1}^r(i-1)k_i}\Bigg)$$
$$=\prod_{i=1}^r\frac{(x_iq;q)_{2N}\,(bx_iq/c_i;q^2)_N\,(x_iq^2/bc_i;q^2)_N}{(x_iq/c_i;q)_{2N}\,(x_iq^2/b;q^2)_N\,(bx_iq;q^2)_N},\tag{5.28}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\dots,k_r\}$.

Finally, we derive some cubic summations.

Theorem 5.29 (A D_r cubic sum) *Let $x_1,\dots,x_r$, and a be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers, let $r\ge 1$, and suppose that none of the denominators in* (5.30) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\Bigg(\prod_{i=1}^r\left(\frac{1-ax_iq^{3k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{3k_i-3k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times\prod_{1\le i<j\le r}(x_ix_j;q^3)^{-1}_{k_i+k_j}\prod_{i,j=1}^r\frac{(q^{-3n_j}x_i/x_j;q^3)_{k_i}\,(x_ix_jq^{3n_j};q^3)_{k_i}}{(q^3x_i/x_j;q^3)_{k_i}}$$
$$\times\prod_{i=1}^r\frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-3k_i}}{(ax_iq^{1+3n_i};q)_{|\mathbf{k}|}\,(aq^{1-3n_i}/x_i;q)_{|\mathbf{k}|}}\cdot\frac{(1/a^2;q)_{|\mathbf{k}|}\,(a^2q;q)_{2|\mathbf{k}|}}{\prod_{i=1}^r(a^3x_iq^3;q^3)_{k_i}}$$
$$\times a^{2|\mathbf{k}|}\prod_{i=1}^r x_i^{-2k_i}\cdot q^{6e_2(\mathbf{k})-\binom{2|\mathbf{k}|+1}{2}+3\sum_{i=1}^r i\,k_i}\Bigg)$$
$$=\prod_{i=1}^r\frac{(ax_iq;q)_{3n_i}\,(x_i/a^3;q^3)_{n_i}}{(x_i/a;q)_{3n_i}\,(a^3x_iq^3;q^3)_{n_i}},\tag{5.30}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\dots,k_r\}$.

Remark 5.31. This cubic summation formula is a D_r extension of

$$\sum_{k=0}^{m} \frac{1-aq^{4k}}{1-a} \frac{(a;q)_k\,(b;q)_k\,(q/b;q)_{2k}\,(a^2bq^{3n};q^3)_k\,(q^{-3n};q^3)_k}{(q^3;q^3)_k\,(aq^3/b;q^3)_k\,(ab;q)_{2k}\,(q^{1-3n}/ab;q)_k\,(aq^{3n+1};q)_k}\, q^k = \frac{(aq;q)_{3n}}{(ab;q)_{3n}} \frac{(ab^2;q^3)_n}{(aq^3/b;q^3)_n}, \tag{5.32}$$

due to Gasper and Rahman [11, eq. (4.1), $c \to 1$], to which it reduces for $r=1$.

Proof of Theorem 5.29: Doing the replacements $q \to q^3$, $p \to q$ in Theorem 5.10 we see that the following pair of matrices are inverses of each other:

$$\begin{aligned} f_{\mathbf{nk}} = {} & \prod_{1\le i<j\le r} \frac{(1-q^{3k_i-3k_j}x_i/x_j)(1-x_ix_jq^{3k_i+3k_j})}{(1-x_i/x_j)(1-x_ix_j)} \prod_{i=1}^{r} \frac{(1-x_i^2q^{6k_i})}{(1-x_i^2)} \\ & \times \prod_{i,j=1}^{r} \frac{(q^{-3n_j}x_i/x_j;q^3)_{k_i}\,(x_ix_j;q^3)_{k_i}}{(q^3x_i/x_j;q^3)_{k_i}\,(x_ix_jq^{3+3n_j};q^3)_{k_i}} \prod_{i=1}^{r} \frac{(ax_iq^{|\mathbf{n}|};q)_{3k_i}}{(x_iq^{1-|\mathbf{n}|}/a;q)_{3k_i}} \cdot q^{3\sum_{i=1}^r i\,k_i} \end{aligned}$$

and

$$\begin{aligned} g_{\mathbf{kl}} = {} & \prod_{1\le i<j\le r} \frac{(1-q^{3l_i-3l_j}x_i/x_j)(1-x_ix_jq^{3l_i+3l_j})}{(1-x_i/x_j)(1-x_ix_j)} \\ & \times \prod_{i,j=1}^{r} \frac{(q^{-3k_j}x_i/x_j;q^3)_{l_i}\,(x_ix_jq^{3k_j};q^3)_{l_i}}{(q^3x_i/x_j;q^3)_{l_i}\,(x_ix_jq^3;q^3)_{l_i}} \prod_{i=1}^{r} \frac{(1-aq^{|\mathbf{l}|+3l_i}x_i)(1-aq^{|\mathbf{l}|-3l_i}/x_i)}{(1-ax_i)(1-a/x_i)} \\ & \times \prod_{i=1}^{r} \frac{(x_i/a;q)_{3k_i}}{(ax_iq;q)_{3k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{l}|}\,(a/x_i;q)_{|\mathbf{l}|}}{(aq^{1+3k_i}x_i;q)_{|\mathbf{l}|}\,(aq^{1-3k_i}/x_i;q)_{|\mathbf{l}|}} \cdot q^{3\sum_{i=1}^r i\,l_i}. \end{aligned}$$

Now (5.1) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^{r} \frac{(x_i/a^3;q^3)_{k_i}}{(a^3x_iq^3;q^3)_{k_i}}$$

and

$$\begin{aligned} b_{\mathbf{n}} = {} & \prod_{i=1}^{r} \frac{(x_i^2q^3;q^3)_{n_i}}{(x_iq^{1-|\mathbf{n}|}/a;q^3)_{n_i}\,(x_iq^{3-|\mathbf{n}|}/a;q^3)_{n_i}} \prod_{1\le i<j\le r} \frac{(x_ix_jq^3;q^3)_{n_i}}{(x_ix_jq^{3+3n_j};q^3)_{n_i}} \\ & \times \frac{(q^{1-2|\mathbf{n}|}/a^2;q^3)_{|\mathbf{n}|}\,(a^2q^{3-|\mathbf{n}|};q^3)_{|\mathbf{n}|}\,(a^2q^{1-|\mathbf{n}|};q^3)_{|\mathbf{n}|}}{\prod_{i=1}^{r}\left[(a^3x_iq^3;q^3)_{n_i}\,(aq^{1+|\mathbf{n}|-3n_i}/x_i;q^3)_{n_i}\right]} \end{aligned}$$

by Milne and Lilly's C_r ${}_8\phi_7$ summation [23, Theorem 6.13]. This implies the inverse relation (5.2) which is easily transformed into (5.30). ■

Remark 5.33. By reversing the multisum in (5.30) we may obtain another, differently looking, D_r extension of Gasper and Rahman's cubic summation formula (5.32).

Using the same polynomial argument as above, we derive some more D_r cubic summation theorems from Theorem 5.29. For sake of brevity, we write them in compact form as follows.

Theorem 5.34 (A D_r cubic sum) *Let $x_1, \ldots, x_r$, $c_1, \ldots, c_r$, and a be indeterminate, let $r \geq 1$, and suppose that none of the denominators in (5.35) vanish. Then,*

$$\sum_{\substack{k_i \geq 0 \\ i=1,\ldots,r}} \Bigg(\prod_{i=1}^r \left(\frac{1 - a x_i q^{3k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq r} \left(\frac{1 - q^{3k_i - 3k_j} x_i / x_j}{1 - x_i / x_j} \right)$$
$$\times \prod_{1 \leq i < j \leq r} (x_i x_j; q^3)^{-1}_{k_i + k_j} \prod_{i,j=1}^r \frac{(c_j x_i / x_j; q^3)_{k_i} \, (x_i x_j / c_j; q^3)_{k_i}}{(q^3 x_i / x_j; q^3)_{k_i}}$$
$$\times \prod_{i=1}^r \frac{(a x_i; q)_{|\mathbf{k}|} \, (a q / x_i; q)_{|\mathbf{k}| - 3k_i}}{(a x_i q / c_i; q)_{|\mathbf{k}|} \, (a c_i q / x_i; q)_{|\mathbf{k}|}} \cdot \frac{(1/a^2; q)_{|\mathbf{k}|} \, (a^2 q; q)_{2|\mathbf{k}|}}{\prod_{i=1}^r (a^3 x_i q^3; q^3)_{k_i}}$$
$$\times a^{2|\mathbf{k}|} \prod_{i=1}^r x_i^{-2k_i} \cdot q^{6 e_2(\mathbf{k}) - \binom{2|\mathbf{k}|+1}{2} + 3 \sum_{i=1}^r i \, k_i} \Bigg)$$
$$= \prod_{i=1}^r \frac{(a x_i q; q)_\infty \, (x_i / a c_i; q)_\infty \, (x_i / a^3; q^3)_\infty \, (a^3 x_i q^3 / c_i; q^3)_\infty}{(a x_i q / c_i; q)_\infty \, (x_i / a; q)_\infty \, (x_i / a^3 c_i; q^3)_\infty \, (a^3 x_i q^3; q^3)_\infty}, \qquad (5.35)$$

provided the series terminates $($*and where* $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\{k_1, \ldots, k_r\})$.

Remark 5.36. The aforementioned D_r cubic summation theorems are the cases $a^2 = q^N$, with N being an arbitrary integer, of Theorem 5.34, where the right side of (5.35) simplifies differently, depending on the sign of N and the residue class of N mod 6.

Appendix A

Here we provide two determinant lemmas which we needed in the proofs of our Theorems 3.1 and 4.1. Our lemmas are interesting generalizations of the classical Vandermonde determinant evaluation

$$\det_{1 \leq i,j \leq r} (x_i^{r-j}) = \prod_{1 \leq i < j \leq r} (x_i - x_j),$$

and the "symplectic" Vandermonde determinant evaluation

$$\det_{1 \leq i,j \leq r} (x_i^{r-j} - x_i^{r+j}) = \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{1 \leq i \leq j \leq r} (1 - x_i x_j),$$

respectively.

Lemma A.1 *Let $x_1,\dots,x_r$, $y_1,\dots,y_r$, a, and c be indeterminate. Then*

$$\det_{1\le i,j\le r}\left(x_i^{r+1-j}-a^{r+1-j}\frac{(x_i-c/\prod_{s=1}^r y_s)}{(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_i-y_s)}{(a-y_s)}\right)$$
$$=\frac{(a-c/\prod_{j=1}^r x_j)}{(a-c/\prod_{j=1}^r y_j)}\prod_{i=1}^r\frac{(a-x_i)}{(a-y_i)}\prod_{i=1}^r x_i\prod_{1\le i<j\le r}(x_i-x_j). \tag{A.2}$$

Proof: In the determinant on the left side of (A.2) we take x_i out of the i-th row, $i=1,\dots,r$, and a^{r-j} out of the j-th column, $j=1,\dots,r$, obtaining

$$a^{\binom r2}\prod_{i=1}^r x_i\det_{1\le i,j\le r}\left(\left(\frac{x_i}{a}\right)^{r-j}-\frac{a(x_i-c/\prod_{s=1}^r y_s)}{x_i(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_i-y_s)}{(a-y_s)}\right).$$

In the last determinant we subtract the r-th column from all other columns. We are left with entries $(x_i/a)^{r-j}-1$ for $i=1,\dots,r$ and $j=1,\dots,r-1$, but the r-th column remains unchanged,

$$1-\frac{a(x_i-c/\prod_{s=1}^r y_s)}{x_i(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_i-y_s)}{(a-y_s)}\quad\text{for}\quad i=1,\dots,r.$$

Next we expand the determinant along the last column, to get

$$\sum_{k=1}^r(-1)^{r+k}\left(1-\frac{a(x_k-c/\prod_{s=1}^r y_s)}{x_k(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_k-y_s)}{(a-y_s)}\right)\det_{\substack{1\le i\le r,i\ne k\\1\le j\le r-1}}\left(\left(\frac{x_i}{a}\right)^{r-j}-1\right).$$

In the minors we take $(x_i/a-1)$ out of the i-th row

$$\det_{\substack{1\le i\le r,i\ne k\\1\le j\le r-1}}\left(\left(\frac{x_i}{a}\right)^{r-j}-1\right)=\prod_{\substack{i=1\\i\ne k}}^r\left(\frac{x_i}{a}-1\right)\det_{\substack{1\le i\le r,i\ne k\\1\le j\le r-1}}\left(\sum_{s=0}^{r-1-j}\left(\frac{x_i}{a}\right)^s\right). \tag{A.3}$$

Now, the determinant on the right side of (A.3) can be reduced to the Vandermonde determinant

$$\det_{\substack{1\le i\le r,i\ne k\\1\le j\le r-1}}\left(\left(\frac{x_i}{a}\right)^{r-1-j}\right),$$

and therefore simplifies to

$$\prod_{\substack{1\le i<j\le r\\i,j\ne k}}\left(\frac{x_i}{a}-\frac{x_j}{a}\right).$$

Substituting our calculations, we arrive at

$$\det_{1\le i,j\le r}\left(x_i^{r+1-j}-a^{r+1-j}\frac{(x_i-c/\prod_{s=1}^r y_s)}{(a-c/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_i-y_s)}{(a-y_s)}\right)$$

$$
\begin{aligned}
&= \prod_{i=1}^{r}(a-x_i)\prod_{i=1}^{r}x_i\prod_{1\le i<j\le r}(x_i-x_j)\\
&\times\sum_{k=1}^{r}\left(1-\frac{a(x_k-c/\prod_{s=1}^{r}y_s)}{x_k(a-c/\prod_{s=1}^{r}y_s)}\prod_{s=1}^{r}\frac{(x_k-y_s)}{(a-y_s)}\right)(a-x_k)^{-1}\prod_{\substack{i=1\\i\neq k}}^{r}(x_k-x_i)^{-1}. \qquad \text{(A.4)}
\end{aligned}
$$

We are done if we can show that the sum in (A.4) equals

$$
\frac{(a-c/\prod_{j=1}^{r}x_j)}{(a-c/\prod_{j=1}^{r}y_j)}\frac{1}{\prod\limits_{i=1}^{r}(a-y_i)}.
$$

This is accomplished by splitting the sum and applying the partial fraction decomposition

$$
\prod_{i=1}^{r}\frac{(t-a_i)}{(t-b_i)} = 1+\sum_{j=1}^{r}\frac{\prod\limits_{i=1}^{r}(b_j-a_i)}{(t-b_j)\prod\limits_{\substack{i=1\\i\neq j}}^{r}(b_j-b_i)}, \qquad \text{(A.5)}
$$

and the equivalent formula

$$
\prod_{i=1}^{r}\frac{(t-a_i)}{(t-b_i)} = \prod_{i=1}^{r}\frac{a_i}{b_i}+\sum_{j=1}^{r}\frac{t\prod\limits_{i=1}^{r}(b_j-a_i)}{(t-b_j)\,b_j\prod\limits_{\substack{i=1\\i\neq j}}^{r}(b_j-b_i)}, \qquad \text{(A.6)}
$$

(which can be obtained from (A.5) by the replacements $t\to 1/t$, $a_i\to 1/a_i$, $b_i\to 1/b_i$, for $i=1,\dots,r$) appropriately to its parts. Namely, we write the sum on the right side of (A.4) as

$$
\begin{aligned}
&\sum_{k=1}^{r}\left(1-\frac{a(x_k-c/\prod_{s=1}^{r}y_s)}{x_k(a-c/\prod_{s=1}^{r}y_s)}\prod_{s=1}^{r}\frac{(x_k-y_s)}{(a-y_s)}\right)\frac{1}{(a-x_k)}\frac{1}{\prod\limits_{\substack{i=1\\i\neq k}}^{r}(x_k-x_i)}\\
&\quad=\sum_{k=1}^{r}\frac{1}{(a-x_k)\prod\limits_{\substack{i=1\\i\neq k}}^{r}(x_k-x_i)}\\
&\quad-\frac{a}{(a-c/\prod_{j=1}^{r}y_j)\prod\limits_{i=1}^{r}(a-y_i)}\sum_{k=1}^{r}\frac{\prod\limits_{i=1}^{r}(x_k-y_i)}{(a-x_k)\prod\limits_{\substack{i=1\\i\neq k}}^{r}(x_k-x_i)}\\
&\quad+\frac{c/\prod_{j=1}^{r}y_j}{(a-c/\prod_{j=1}^{r}y_j)\prod\limits_{i=1}^{r}(a-y_i)}\sum_{k=1}^{r}\frac{a\prod\limits_{i=1}^{r}(x_k-y_i)}{(a-x_k)x_k\prod\limits_{\substack{i=1\\i\neq k}}^{r}(x_k-x_i)}. \qquad \text{(A.7)}
\end{aligned}
$$

The first expression can be summed by the partial fraction decomposition (A.6) with $a_i = 0$, $t \to 1/t$, and $b_i \to 1/b_i$, for $i = 1, \dots, r$, and reduces to

$$\sum_{k=1}^{r} \frac{1}{(a - x_k) \prod_{\substack{i=1 \\ i \neq k}}^{r} (x_k - x_i)} = \frac{1}{\prod_{i=1}^{r} (a - x_i)}, \tag{A.8}$$

the second by the partial fraction decomposition (A.5),

$$\begin{aligned} &\frac{a}{(a - c/\prod_{j=1}^{r} y_j) \prod_{i=1}^{r} (a - y_i)} \sum_{k=1}^{r} \frac{\prod_{i=1}^{r} (x_k - y_i)}{(a - x_k) \prod_{\substack{i=1 \\ i \neq k}}^{r} (x_k - x_i)} \\ &\qquad = \frac{a}{(a - c/\prod_{j=1}^{r} y_j) \prod_{i=1}^{r} (a - y_i)} \left(\prod_{i=1}^{r} \frac{(a - y_i)}{(a - x_i)} - 1 \right), \end{aligned} \tag{A.9}$$

and the third can be summed by (A.6),

$$\begin{aligned} &\frac{c/\prod_{j=1}^{r} y_j}{(a - c\prod_{j=1}^{r} y_j) \prod_{i=1}^{r} (a - y_i)} \sum_{k=1}^{r} \frac{a \prod_{i=1}^{r} (x_k - y_i)}{(a - x_k)\, x_k \prod_{\substack{i=1 \\ i \neq k}}^{r} (x_k - x_i)} \\ &\qquad = \frac{c/\prod_{j=1}^{r} y_j}{(a - c/\prod_{j=1}^{r} y_j) \prod_{i=1}^{r} (a - y_i)} \left(\prod_{i=1}^{r} \frac{(a - y_i)}{(a - x_i)} - \prod_{i=1}^{r} \frac{y_i}{x_i} \right). \end{aligned} \tag{A.10}$$

Simplifying (A.7) by means of (A.8), (A.9), and (A.10), we get

$$\begin{aligned} &\sum_{k=1}^{r} \left(1 - \frac{a(x_k - c/\prod_{s=1}^{r} y_s)}{x_k(a - c/\prod_{s=1}^{r} y_s)} \prod_{s=1}^{r} \frac{(x_k - y_s)}{(a - y_s)} \right) (a - x_k)^{-1} \prod_{\substack{i=1 \\ i \neq k}}^{r} (x_k - x_i)^{-1} \\ &= \prod_{i=1}^{r} \frac{1}{(a - x_i)} - \frac{a}{(a - c/\prod_{j=1}^{r} y_j) \prod_{i=1}^{r} (a - y_i)} \left(\prod_{i=1}^{r} \frac{(a - y_i)}{(a - x_i)} - 1 \right) \\ &\qquad + \frac{c/\prod_{j=1}^{r} y_j}{(a - c/\prod_{j=1}^{r} y_j) \prod_{i=1}^{r} (a - y_i)} \left(\prod_{i=1}^{r} \frac{(a - y_i)}{(a - x_i)} - \prod_{i=1}^{r} \frac{y_i}{x_i} \right) \\ &\qquad\qquad = \frac{(a - c/\prod_{j=1}^{r} x_j)}{(a - c/\prod_{j=1}^{r} y_j)} \frac{1}{\prod_{i=1}^{r} (a - y_i)}, \end{aligned}$$

which completes the proof of Lemma A.1. ∎

Lemma A.11 *Let $x_1,\dots,x_r$, $y_1,\dots,y_r$, and c be indeterminate. Then*

$$\det_{1\le i,j\le r}\left(x_i^{r+1-j}-x_i^j\frac{(x_i-c/\prod_{s=1}^r y_s)}{(1-x_ic/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_i-y_s)}{(1-x_iy_s)}\right)$$
$$=\prod_{i=1}^r\frac{(1-y_ic/\prod_{j=1}^r y_j)}{(1-x_ic/\prod_{j=1}^r y_j)}\prod_{i=1}^r(1-x_i^2)\prod_{i=1}^r x_i$$
$$\times\prod_{i,j=1}^r(1-x_iy_j)^{-1}\prod_{1\le i<j\le r}[(x_i-x_j)(1-x_ix_j)(1-y_iy_j)]. \tag{A.12}$$

Proof: Here we use a completely different method than in the proof of Lemma A.1. In the determinant on the left side of (A.12) we take $x_i(x_ic-\prod_{s=1}^r y_s)^{-1}\prod_{s=1}^r(1-x_iy_s)^{-1}$ out of the i-th row, $i=1,\dots,r$, obtaining

$$\det_{1\le i,j\le r}\left(x_i^{r+1-j}-x_i^j\frac{(x_i-c/\prod_{s=1}^r y_s)}{(1-x_ic/\prod_{s=1}^r y_s)}\prod_{s=1}^r\frac{(x_i-y_s)}{(1-x_iy_s)}\right)$$
$$=\prod_{i=1}^r\frac{x_i}{(x_ic-\prod_{j=1}^r y_j)}\prod_{i,j=1}^r(1-x_iy_j)^{-1}\cdot\Delta(c,\mathbf{x},\mathbf{y}),$$

where $\Delta(c,\mathbf{x},\mathbf{y})$ is the determinant

$$\det_{1\le i,j\le r}\left(x_i^{r-j}(x_ic-\prod_{s=1}^r y_s)\prod_{s=1}^r(1-x_iy_s)\right.$$
$$\left.-x_i^{j-1}(c-x_i\prod_{s=1}^r y_s)\prod_{s=1}^r(x_i-y_s)\right). \tag{A.13}$$

Thus, in order to establish the lemma, we have to show that

$$\Delta(c,\mathbf{x},\mathbf{y})=\prod_{i=1}^r(y_ic-\prod_{j=1}^r y_j)\prod_{i=1}^r(1-x_i^2)\prod_{1\le i<j\le r}[(x_i-x_j)(1-x_ix_j)(1-y_iy_j)].$$

We will do this by identifying all factors using a polynomial argument.

We see that $\Delta(c,\mathbf{x},\mathbf{y})$ is a polynomial in c,x_i,y_i ($i=1,\dots,r$) of maximal degree $(7r^2-r)/2$. Now observe that if $x_{i_1}=x_{i_2}$, for $i_1\ne i_2$, two rows in the determinant (A.13) are equal, hence $\prod_{1\le i<j\le r}(x_i-x_j)$ must divide $\Delta(c,\mathbf{x},\mathbf{y})$. Next suppose $x_{i_1}=1/x_{i_2}$ for some $i_1\ne i_2$. In this case the i_1-th row is $-x_{i_2}^{-2r}$ times the i_2-th row which implies that $\prod_{1\le i<j\le r}(1-x_ix_j)$ also divides $\Delta(c,\mathbf{x},\mathbf{y})$. If $x_i=1$ or $x_i=-1$ then all entries of the i-th row are zero, so $\prod_{i=1}^r(1-x_i^2)$ divides $\Delta(c,\mathbf{x},\mathbf{y})$.

The remaining factors of $\Delta(c,\mathbf{x},\mathbf{y})$ are a bit more delicate to establish. For each special case we will succeed in specifying nontrivial linear combinations of the columns that vanish. Suppose $y_k=1/y_l$ for some $k\ne l$. Taking $-(1-x_iy_k)(1-x_i/y_k)\prod_{s\ne k,l}y_s$ out of the i-th row of (A.13), for all $i=1,\dots,r$, we obtain the determinant

$$\det_{1\le i,j\le r}\Bigg(x_i^{r-j}(1-x_ic/\prod_{s\ne k,l}y_s)\prod_{s\ne k,l}(1-x_iy_s)\\ -x_i^{j-1}(x_i-c/\prod_{s\ne k,l}y_s)\prod_{s\ne k,l}(x_i-y_s)\Bigg). \tag{A.14}$$

We expand the entries of this determinant in terms of the elementary symmetric functions (see [19, p.19])

$$e_m(y_1,\dots,\hat y_k,\dots,\hat y_l,\dots,y_r,c/\textstyle\prod_{s\ne k,l}y_s), \tag{A.15}$$

of order m with $r-1$ arguments, $\hat y_k$ and $\hat y_l$ indicating that the variables y_k,y_l are omitted. Namely, if we write $e_m(\mathbf{y}^{(k,l)})$ for the elementary symmetric function (A.15) for short, (A.14) can be written as

$$\det_{1\le i,j\le r}\left(\sum_{m=0}^{r-1}(-1)^m e_m(\mathbf{y}^{(k,l)})\left(x_i^{r-j+m}-x_i^{j-1+r-1-m}\right)\right). \tag{A.16}$$

To prove that this determinant vanishes we show that the columns of (A.16) are linearly dependent. As the coefficients for the linear combination we choose $(-1)^{j-1}e_{j-1}(\mathbf{y}^{(k,l)})$, for $j=1,\dots,r$. Then we have

$$\sum_{j=1}^{r}(-1)^{j-1}e_{j-1}(\mathbf{y}^{(k,l)})\sum_{m=0}^{r-1}(-1)^m e_m(\mathbf{y}^{(k,l)})\left(x_i^{r-j+m}-x_i^{j-1+r-1-m}\right)\\ =\sum_{j=0}^{r-1}\sum_{m=0}^{r-1}(-1)^{j+m}e_j(\mathbf{y}^{(k,l)})e_m(\mathbf{y}^{(k,l)})\left(x_i^{r-j-1+m}-x_i^{j+r-1-m}\right)=0. \tag{A.17}$$

That the sum equals 0 is because it is a double sum in j and m with terms that are skew symmetric in j and m. Hence we have proved that $\prod_{1\le i<j\le r}(1-y_iy_j)$ divides $\Delta(c,\mathbf{x},\mathbf{y})$.

Now suppose $c=\prod_{s\ne k}y_s$ for some $k=1,\dots,r$. If we take $-(1-x_iy_k)(1-x_i/y_k)\prod_{s=1}^r y_s$ out of the i-th row of (A.13) for all $i=1,\dots,r$, we obtain the determinant

$$\det_{1\le i,j\le r}\left(x_i^{r-j}\prod_{s\ne k}(1-x_iy_s)-x_i^{j-1}\prod_{s\ne k}(x_i-y_s)\right). \tag{A.18}$$

We expand the entries of this determinant in terms of the elementary symmetric functions

$$e_m(y_1,\dots,\hat y_k,\dots,y_r), \tag{A.19}$$

of order m with $r-1$ arguments, $\hat y_k$ indicating that the variable y_k is omitted. Namely, if we write $e_m(\mathbf{y}^{(k)})$ for the elementary symmetric function (A.19) for short, (A.18) can be written as

$$\det_{1\le i,j\le r}\left(\sum_{m=0}^{r-1}(-1)^m e_m(\mathbf{y}^{(k)})\left(x_i^{r-j+m}-x_i^{j-1+r-1-m}\right)\right). \tag{A.20}$$

To prove that this determinant vanishes we show that the columns of (A.20) are linearly dependent. Here the coefficients $(-1)^{j-1}e_{j-1}(\mathbf{y}^{(k)})$ for $j=1,\dots,r$ do the job (compare with (A.17)). Hence $\prod_{1\le i\le r}(c-\prod_{s\ne i}y_s)$ divides $\Delta(c,\mathbf{x},\mathbf{y})$.

Now suppose $y_k=0$ for some $k=1,\dots,r$. If we take $(-x_ic)$ out of the i-th row of (A.13) for all $i=1,\dots,r$, we obtain the determinant (A.18), and we can proceed as above. I.e., we have also shown that $\prod_{1\le i\le r}y_i$ divides $\Delta(c,\mathbf{x},\mathbf{y})$.

Collecting all factors of $\Delta(c,\mathbf{x},\mathbf{y})$ that we have identified so far, we now know that

$$\begin{aligned}\Delta(c,\mathbf{x},\mathbf{y}) \;=\; &\prod_{i=1}^{r}(y_ic-\prod\nolimits_{j=1}^{r}y_j)\prod_{i=1}^{r}(1-x_i^2)\\ &\times\prod_{1\le i<j\le r}[(x_i-x_j)(1-x_ix_j)(1-y_iy_j)]\cdot p(c,\mathbf{x},\mathbf{y}),\end{aligned} \tag{A.21}$$

where $p(c,\mathbf{x},\mathbf{y})$ is some polynomial in c,x_i,y_i ($i=1,\dots,r$). But the degree of the factors we already identified amounts to $(7r^2-r)/2$, which is the same degree as that of $\Delta(c,\mathbf{x},\mathbf{y})$. Thus the polynomial p has to be a constant which is easily seen to be 1, since the coefficients of $c^0\prod_{i=1}^{r}(x_i^{r-i}y_i^r)$ in $\Delta(c,\mathbf{x},\mathbf{y})$ and in the product on the right side of (A.21) both equal $(-1)^r$. ■

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