q-ANALOGUES OF TWO PRODUCT FORMULAS OF HYPERGEOMETRIC FUNCTIONS BY BAILEY

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Dedicated to Mourad E.H. Ismail

Abstract. We use Andrews’ q-analogues of Watson’s and Whipple’s \(3F_2\) summation theorems to deduce two formulas for products of specific basic hypergeometric functions. These constitute q-analogues of corresponding product formulas for ordinary hypergeometric functions given by Bailey. The first formula was obtained earlier by Jain and Srivastava by a different method.

1. Introduction

We refer to Slater’s text [9] for an introduction to hypergeometric series, and to Gasper and Rahman’s text [5] for an introduction to basic hypergeometric series, whose notations we follow. Throughout, we assume \(|q| < 1\) and \(|z| < 1\).

In [1], George Andrews proved the following two theorems:

**Theorem 1.**

\[
\begin{aligned}
\Phiq3 \left[ \begin{array}{c}
a, b, c^\frac{1}{2}, -c^\frac{1}{2} \\
(abq)^\frac{1}{2}, -(abq)^\frac{1}{2}, c
\end{array} ; q, q \right] &= a^n \frac{(aq, bq, cq/a, cq/b; q^2)_\infty}{(q, abq, cq, cq/ab; q^2)_\infty},
\end{aligned}
\]

where \(b = q^{-n}\) and \(n\) is a nonnegative integer.

**Theorem 2.**

\[
\Phiq3 \left[ \begin{array}{c}
a, q/a, c^\frac{1}{2}, -c^\frac{1}{2} \\
-q, e, cq/e
\end{array} ; q, q \right] = q^{\binom{n+1}{2}} \frac{(ea, eq/a, caq/e, cq^2/ae; q^2)_\infty}{(e, cq/e; q)_\infty},
\]

where \(a = q^{-n}\) and \(n\) is a nonnegative integer.

By a standard polynomial argument (1.2) also holds when \(a\) is a complex variable but \(c = q^{-2n}\) with \(n\) being a nonnegative integer. (This is the case we will make use of.)

Theorems 1 and 2 are q-analogues of Watson’s and of Whipple’s \(3F_2\) summation theorems, listed as Equations (III.23) and (III.24) in [9, p. 245], respectively.

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2. TWO PRODUCT FORMULAS FOR BASIC HYPERGEOMETRIC FUNCTIONS

We now have the following two product formulas which are derived using Theorems 1 and 2. The first one in Theorem 3 was already given earlier by Jain and Srivastava [7, Equation (4.9)] (as Slobodan Damjanović has kindly pointed out to the author, after seeing an earlier version of this note), who established the result by specializing a general reduction formula for double basic hypergeometric series. The second formula in Theorem 4 appears to be new.

**Theorem 3.**

\[
\phi_{1}^{a,-a \over a^2} ; q, z \phi_{1}^{b,-b \over b^2} ; q, -z = 4 \phi_{3}^{ab,-ab,abq,-abq \over a^2q,b^2q,a^2b^2} ; q^2, z^2.
\] (2.1)

**Theorem 4.**

\[
\phi_{1}^{a,q/a -q \over q} ; q, z \phi_{1}^{b,q/b -q \over q} ; q, -z = 4 \phi_{3}^{ab,q^2/ab,abq/bq/a -q^2,q,-q \over q^2} ; q^2, -q^2
\]

\[
- (a-b)(1-q/ab) \sum_{j=0}^{\infty} \frac{(q^{-j}/ab, q^{-j}/ab, q^2z^j)}{(q^2z^j)^j} (i\pi)(-i\pi)^j \phi_{3}^{abq,q^3/ab,abq^2/bq^2/a -q^2,q^3,-q^3} ; q^2, -q^2 \].
\] (2.2a)

**Sketch of proofs.** To prove Theorem 3, compare coefficients of \(z^n\). The resulting identity is equivalent to Theorem 1. The proof of Theorem 4 is similar. Comparison of coefficients of \(z^n\) gives an identity which is equivalent to Theorem 2 (where in the latter theorem the restriction \(a = q^{-n}\) is replaced by \(c = q^{-2n}\), as mentioned). The second identity in Equation (2.2) follows from splitting the sum over \(j\) into two parts depending on the parity of \(j\). (This is motivated by the particular numerator factors in the \(j\)-th summand.) The technical details – elementary manipulation of \(q\)-shifted factorials – are routine and thus omitted.

Theorem 3 is a \(q\)-analogue of Bailey’s formula in [2, p. 246, Equation (2.11)]:

\[
\phi_{1}^{a \over 2a} ; z \phi_{1}^{b \over 2b} ; -z = 2 \phi_{3}^{1/2(a+b),1/2(a+b+1),1 \over a+1/2,b+1/2,a+b; -4z}.
\] (2.3)

To obtain (2.3) from Theorem 3, replace \((a,b,z)\) by \((q^a,q^b,1-q)z/2\), and let \(q \to 1\).

Similarly, Theorem 4 is a \(q\)-analogue of Bailey’s formula in [2, p. 245, Equation (2.08)]:

\[
\phi_{0}^{a,1-a \over -} ; z \phi_{0}^{b,1-b \over -} ; -z = 4 \phi_{1}^{1/2(a+b),1/2(1-a+b),1/2(a+b),1/2(2-a-b) \over 1/2} ; 4z^2 \]

\[
- (a-b)(a+b-1)z
\]

\[
\times 4 \phi_{1}^{1/2(2+a-b),1/2(2-a+b),1/2(1+a+b),1/2(3-a-b) \over 3/2} ; 4z^2 \].
\] (2.4)
To obtain (2.4) from Theorem 4, replace \((a, b, z)\) by \((q^a, q^b, 2z/(1 - q))\) and let \(q \to 1\).

3. Related Results in the Literature

A different product formula for basic hypergeometric functions was established by Srivastava [10, Eq. (21)] (see also [11, Eq. (3.13)]):

\[
\begin{align*}
2\phi_1 \left[ \frac{a, b}{-ab}; q, z \right] 
&= 4\phi_3 \left[ \frac{a^2, b^2, ab, abq}{a^2b^2, -ab, -abq}; q^2, z^2 \right].
\end{align*}
\] (3.1)

This formula is a \(q\)-extension of Bailey’s formula in [2, p. 245, Equation (2.08)] (or, equivalently, of an identity recorded by Ramanujan [8, Ch. 13, Entry 24]).

Finally, we mention that in 1941 F.H. Jackson [6] had derived the identity

\[
\begin{align*}
2\phi_1 \left[ \frac{a^2, b^2}{a^2b^2q}; q^2, z \right] 
&= 4\phi_3 \left[ \frac{a^2, b^2, ab, -ab}{a^2b^2, abq^2, -abq^2}; q, z \right],
\end{align*}
\] (3.2)

which is a \(q\)-analogue of Clausen’s formula of 1828,

\[
\left( \frac{F_1 \left[ \frac{a, b}{a + b + \frac{1}{2}; z} \right]}{3F_2 \left[ \frac{2a, 2b, a + b}{2a + 2b, a + b + \frac{1}{2}; z} \right]} \right)^2 = \frac{a^2, b^2, ab, -ab}{a^2b^2, abq^2, -abq^2}; q, z \right].
\] (3.3)

Another \(q\)-analogue of Clausen’s formula was delivered by Gasper in [4]. While it has the advantage that it expresses a square of a basic hypergeometric series as a basic hypergeometric series, it only holds provided the series terminate:

\[
\left( 4\phi_3 \left[ \frac{a, b, aby, ab/y}{abq^2, -abq^2, -ab}; q, q \right] \right)^2 = 5\phi_4 \left[ \frac{a^2, b^2, ab, aby, ab/y}{a^2b^2, abq^2, -abq^2, -ab}; q, q \right].
\] (3.4)

See [5, Sec. 8.8] for a nonterminating extension of (3.4) and related identities.

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References


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