Recurrence formulas for Macdonald polynomials of type A

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Abstract

We consider products of two Macdonald polynomials of type A, indexed by dominant weights which are respectively a multiple of the first fundamental weight and a weight having zero component on the k-th fundamental weight. We give the explicit decomposition of any Macdonald polynomial of type A in terms of this basis.

2010 Mathematics Subject Classification: Primary 33D52; Secondary 05E05, 15A09.

1 Introduction

In the 1980's, I. G. Macdonald introduced a class of orthogonal polynomials which are Laurent polynomials in several variables and generalize the Weyl characters of compact simple Lie groups [6, 7, 8]. In the simplest situation, given a root system R, these polynomials are elements of the group algebra of the weight lattice of R, indexed by the dominant weights, and depending on two parameters (q, t).

When R is of type A_n, these Macdonald polynomials are in bijective correspondence with the symmetric functions $\mathcal{P}_\lambda(q, t)$ indexed by partitions, that were introduced by Macdonald some years before [4, 5]. In fact, they correspond to $\mathcal{P}_\lambda(q, t)(x_1, \ldots, x_{n+1})$, for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length n, with the $n + 1$ variables $(x_1, \ldots, x_{n+1})$ linked by the condition $x_1 \cdots x_{n+1} = 1$.

The purpose of this article is to extend the result of [3], given for the symmetric functions $\mathcal{P}_\lambda(q, t)$, to the framework of the root system A_n.

More precisely, in [3, Theorem 4.1] we obtained a recurrence formula giving the symmetric function $\mathcal{P}_{(\lambda_1,\ldots,\lambda_n)}(q,t)$ as a sum

$$\mathcal{P}_{(\lambda_1,\ldots,\lambda_n)} = \sum_{\theta\in\mathbb{N}^{n-1}} C_{\theta_1,\ldots,\theta_{n-1}} \mathcal{P}_{(\lambda_1+\theta_1,\ldots,\lambda_{n-1}+\theta_{n-1})}\, \mathcal{P}_{\lambda_n-|\theta|}, \tag{1.1}$$

with $|\theta| = \sum_{i=1}^{n-1} \theta_i$ and $\mathbb{N}$ the set of non-negative integers. This formula was obtained by inverting the "Pieri formula", which conversely expresses the product $\mathcal{P}_{(\lambda_1,\ldots,\lambda_{n-1})}\, \mathcal{P}_{\lambda_n}$ as a sum

$$\mathcal{P}_{(\lambda_1,\ldots,\lambda_{n-1})}\, \mathcal{P}_{\lambda_n} = \sum_{\theta\in\mathbb{N}^{n-1}} c_{\theta_1,\ldots,\theta_{n-1}}\, \mathcal{P}_{(\lambda_1+\theta_1,\ldots,\lambda_{n-1}+\theta_{n-1},\lambda_n-|\theta|)}.$$

Both expansions are identities between symmetric functions, valid for any number of variables.

These identities may also be written in terms of Macdonald polynomials of type A_n. For this purpose let $\{\omega_i, 1 \le i \le n\}$ be the n fundamental weights of the root system A_n. Let P_λ denote the Macdonald polynomial associated with the dominant weight $\lambda = \sum_{i=1}^{n} \lambda_i\omega_i$. The recurrence formula (1.1), written for $n+1$ variables $(x_1,\ldots,x_{n+1})$ linked by $x_1\cdots x_{n+1} = 1$, yields

$$P_\lambda = \sum_{\theta\in\mathbb{N}^{n-1}} C_{\theta_1,\ldots,\theta_{n-1}} P_{(\lambda_n-|\theta|)\omega_1}\, P_\mu, \tag{1.2}$$

with $\mu = \sum_{i=1}^{n-2}(\lambda_i+\theta_i-\theta_{i+1})\omega_i + (\lambda_{n-1}+\lambda_n+\theta_{n-1})\omega_{n-1}$. This alternative formulation is obvious and does not bring anything new.

However the method of [3], when applied in the A_n root system framework, allows to get a much stronger result. Indeed, let k be a fixed integer with $1 \le k \le n$. In this paper we shall write the Macdonald polynomial P_λ in terms of products $P_{r\omega_1}P_\mu$, with $\mu = \sum_{i=1}^{n} \mu_i\omega_i$ and $\mu_k = 0$. There are n such recurrence formulas, (1.2) being the particular case $k = n$ of the latter.

This paper is organized as follows. In Section 2 we introduce our notation for the root system A_n and recall general facts about the corresponding Macdonald polynomials. Their Pieri formula, which involves a specific infinite multidimensional matrix, is studied in Section 3, starting from the one given by Macdonald for the symmetric functions $\mathcal{P}_\lambda(q,t)$ [5, p. 340]. In Section 4 we invert the Pieri matrix by applying a particular multidimensional matrix inverse, given separately in the Appendix. This matrix inverse is equivalent to one previously obtained in [3, Section 2] by using operator methods. As result of inverting the Pieri formula we obtain recurrence formulas for A_n Macdonald polynomials. Finally, in Section 5 we detail the examples of the A_2 and A_3 cases and compare them to earlier results.

Acknowledgemnts. We thank the anonymous referees for helpful comments. The second author was partly supported by FWF Austrian Science Fund grants P17563-N13 and S9607.

2 Macdonald polynomials of type A

The standard references for Macdonald polynomials associated with root systems are [6, 7, 8].

Let us consider the space $\mathbb{R}^{n+1}$ endowed with the usual scalar product and the quotient space $V = \mathbb{R}^{n+1}/\mathbb{R}(1,\dots,1)$, where $\mathbb{R}(1,\dots,1)$ is the subspace spanned by the vector $(1,\dots,1)$. Let $\varepsilon_1,\dots,\varepsilon_{n+1}$ denote the images in V of the coordinate vectors of $\mathbb{R}^{n+1}$, linked by $\sum_{i=1}^{n+1}\varepsilon_i = 0$.

The root system of type A_n is formed by the vectors $\{\varepsilon_i - \varepsilon_j, i \neq j\}$. The positive roots are $\{\varepsilon_i - \varepsilon_j, i < j\}$ and the simple roots are $\varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le n$. The Weyl group is the symmetric group $W = S_{n+1}$ acting by permutation of the coordinates.

The weight lattice P is formed by integral linear combinations of the fundamental weights $\{\omega_i, 1 \le i \le n\}$, defined by $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$. Let $\omega_i = 0$ for $i = 0, n+1$. We denote by P^+ the set of dominant weights $\lambda = \sum_{i=1}^n \lambda_i\omega_i$, which are non-negative integral linear combinations of the fundamental weights.

There is the following correspondence between dominant weights and partitions. Given a dominant weight, if we write it as

$$\lambda = \sum_{i=1}^{n} \lambda_i\omega_i = \sum_{i=1}^{n+1} \mu_i\varepsilon_i,$$

the sequence $\mu = (\mu_1,\dots,\mu_{n+1})$ is a partition with length $\le n+1$. We have

$$\lambda_i = \mu_i - \mu_{i+1} \quad \text{and} \quad \mu_i = \mu_{n+1} + \sum_{j=i}^{n} \lambda_j.$$

Thus μ is defined up to μ_{n+1} and two partitions μ,ν correspond to the same weight λ if and only if $\mu_1 - \nu_1 = \cdots = \mu_{n+1} - \nu_{n+1}$. We denote by $\mathcal{C}_\lambda$ the family of partitions thus defined.

Let A denote the group algebra over $\mathbb{R}$ of the free Abelian group P. For each $\lambda \in P$ let e^λ denote the corresponding element of A, subject to the multiplication rule $e^\lambda e^\mu = e^{\lambda+\mu}$. The set $\{e^\lambda, \lambda \in P\}$ forms an $\mathbb{R}$-basis of A.

The Weyl group $W = S_{n+1}$ acts on P and on A. Let $W\lambda$ denote the orbit of $\lambda \in P$ and A^W the subspace of W-invariants in A. There are two important bases of A^W, both indexed by dominant weights. The first one is given by the orbit-sums

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu.$$

The second one is provided by the Weyl characters

$$\chi_\lambda = \delta^{-1} \sum_{w \in W} \det(w) e^{w(\lambda+\rho)},$$

with $\rho = \sum_{i=1}^n (n-i+1)\varepsilon_i$ and $\delta = \sum_{w\in W} \det(w) e^{w(\rho)}$. The Macdonald polynomials $\{P_\lambda, \lambda \in P^+\}$ form another basis, defined as the eigenvectors of a specific self-adjoint operator (which we do not describe here).

For $1 \le i \le n+1$ define $x_i = e^{\varepsilon_i}$, so that the variables x_i are linked by $x_1 \cdots x_{n+1} = 1$. Then δ is the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$. There is a correspondence between A^W and the symmetric polynomials restricted to $n+1$ variables $x = (x_1, \dots, x_{n+1})$ linked by the previous condition.

In terms of bases this correspondence may be described as follows. Let λ be any dominant weight and $x_1 \cdots x_{n+1} = 1$. All monomial symmetric functions $m_\mu(x_1, \dots, x_{n+1})$ with $\mu \in \mathcal{C}_\lambda$ are equal and their common value is the orbit-sum m_λ. Similarly, the Weyl character χ_λ is the common value of the Schur functions $s_\mu(x_1, \dots, x_{n+1})$, $\mu \in \mathcal{C}_\lambda$, whereas the Macdonald polynomial P_λ is the common value of the symmetric polynomials $\mathcal{P}_\mu(q,t)(x_1, \dots, x_{n+1})$, with $\mu \in \mathcal{C}_\lambda$ and $\mathcal{P}_\mu(q,t)$ the symmetric function studied in Chapter 6 of [5].

Given a positive integer r and a dominant weight λ, the "Pieri formula" expands the product

$$P_{r\omega_1}\, P_\lambda = \sum_\rho c_\rho\, P_{\lambda+\rho},$$

in terms of Macdonald polynomials, where the range of ρ and the values of the coefficients c_ρ are to be determined.

Let Q denote the root lattice, spanned by the simple roots. For any vector τ, define

$$\Sigma(\tau) = C(\tau) \cap (\tau + Q)$$

with $C(\tau)$ the convex hull of the Weyl group orbit of τ. Since the orbit of $\omega_1 = \varepsilon_1$ is the set $\{\varepsilon_i = \omega_i - \omega_{i-1}, 1 \le i \le n+1\}$, it is clear that $\Sigma(r\omega_1)$ is formed by vectors

$$\sum_{i=1}^{n+1} \theta_i(\omega_i - \omega_{i-1}) = \sum_{i=1}^{n} (\theta_i - \theta_{i+1})\omega_i,$$

with $\theta = (\theta_1, \dots, \theta_{n+1}) \in \mathbb{N}^{n+1}$ and $|\theta| = \sum_{i=1}^{n+1} \theta_i = r$.

By general results [8, (5.3.8), p. 104], it is known that the sum on the right-hand side of the Pieri formula is restricted to vectors ρ such that $\rho \in \Sigma(r\omega_1)$ and $\lambda + \rho \in P^+$. In the next section we shall give a direct proof of this result and make the value of the coefficient c_ρ explicit.

3 Pieri formula

Let $0 < q < 1$. For any integer r, the classical q-shifted factorial $(u;q)_r$ is defined by

$$(u;q)_\infty = \prod_{j \ge 0} (1 - uq^j), \qquad (u;q)_r = (u;q)_\infty / (uq^r;q)_\infty.$$

Let $u = (u_1, \dots, u_m)$ be m indeterminates and $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{N}^m$. For clarity of display, throughout this paper, any time such a pair (u, θ) is given, we shall implicitly assume m auxiliary variables $v = (v_1, \dots, v_m)$ to be defined by $v_i = q^{\theta_i} u_i$.

Macdonald polynomials of type A_n satisfy the following Pieri formula.

Theorem 3.1. *Let $\lambda = \sum_{i=1}^n \lambda_i \omega_i$ be a dominant weight and $r \in \mathbb{N}$. For any $1 \le i \le n+1$ define*
$$u_i = q^{\sum_{j=i}^n \lambda_j} t^{-i},$$
and for $\theta \in \mathbb{N}^{n+1}$,
$$d_\theta(u_1, \ldots, u_{n+1}; r) = \frac{(q;q)_r}{(t;q)_r} \prod_{j=1}^{n+1} \frac{(t;q)_{\theta_j}}{(q;q)_{\theta_j}} \prod_{1 \le i < j \le n+1} \frac{(tv_i/v_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}} \, \frac{(qu_i/tv_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}}.$$
We have
$$P_{r\omega_1}\, P_\lambda = \sum_{\substack{\theta \in \mathbb{N}^{n+1} \\ |\theta| = r}} d_\theta(u_1, \ldots, u_{n+1}; r)\, P_{\lambda+\rho},$$
with $\rho = \sum_{i=1}^n (\theta_i - \theta_{i+1})\omega_i$.

Proof. In a first step, we write the Pieri formula for arbitrary $\mathcal{P}_\mu(q,t)$ with $\mu = (\mu_1, \ldots, \mu_n)$ being a partition having length $\le n$. We start from [5, p. 340, Eq. (6.24)(i)] and [5, p. 342, Example 2(a)]. Replacing g_r by $(t;q)_r/(q;q)_r\, \mathcal{P}_{(r)}$ we have
$$\mathcal{P}_{(r)}\, \mathcal{P}_\mu = \sum_{\kappa \supset \mu} \varphi_{\kappa/\mu}\, \mathcal{P}_\kappa,$$
where the skew-diagram $\kappa - \mu$ is a horizontal r-strip, i.e. has at most one node in each column. The Pieri coefficient $\varphi_{\kappa/\mu}$ is given by
$$\frac{(t;q)_r}{(q;q)_r}\, \varphi_{\kappa/\mu} = \prod_{1 \le i \le j \le l(\kappa)} \frac{f(q^{\kappa_i - \kappa_j} t^{j-i})}{f(q^{\kappa_i - \mu_j} t^{j-i})} \, \frac{f(q^{\mu_i - \mu_{j+1}} t^{j-i})}{f(q^{\mu_i - \kappa_{j+1}} t^{j-i})} = \prod_{1 \le i \le j \le l(\kappa)} \frac{w_{\kappa_j - \mu_j}(q^{\kappa_i - \kappa_j} t^{j-i})}{w_{\kappa_{j+1} - \mu_{j+1}}(q^{\mu_i - \kappa_{j+1}} t^{j-i})},$$
with $f(u) = (tu;q)_\infty/(qu;q)_\infty$ and $w_s(u) = (tu;q)_s/(qu;q)_s$.

Since $\kappa - \mu$ is a horizontal strip, the length $l(\kappa)$ of κ is at most equal to $n+1$, so we can write $\kappa = (\mu_1 + \theta_1, \ldots, \mu_n + \theta_n, \theta_{n+1})$, with $|\theta| = r$. Then
$$\begin{aligned} \frac{(t;q)_r}{(q;q)_r}\, \varphi_{\kappa/\mu} &= \prod_{1 \le i \le j \le l(\kappa)} w_{\theta_j}(q^{\kappa_i - \kappa_j} t^{j-i}) \prod_{1 \le i < j \le l(\kappa)+1} \left(w_{\theta_j}(q^{\mu_i - \kappa_j} t^{j-i-1}) \right)^{-1} \\ &= \prod_{j=1}^{n+1} \frac{(t;q)_{\theta_j}}{(q;q)_{\theta_j}} \prod_{1 \le i < j \le n+1} \frac{(tv_i/v_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}} \, \frac{(qu_i/tv_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}}, \end{aligned}$$
where for $1 \le i \le n+1$ we set $u_i = q^{\mu_i} t^{-i}$ and $v_i = q^{\kappa_i} t^{-i} = q^{\theta_i} u_i$.

In a second step we translate this result in terms of A_n Macdonald polynomials. Given the dominant weight λ, we choose $\mu = (\mu_1, \ldots, \mu_{n+1})$ to be the unique element of $\mathcal{C}_\lambda$ such that $\mu_{n+1} = 0$, i.e. with length $\le n$. For $1 \le i \le n$ we have $\mu_i = \sum_{j=i}^n \lambda_j$. As for the partition κ (with length $\le n+1$), it belongs to $\mathcal{C}_\sigma$ with $\sigma = \sum_{k=1}^n (\kappa_k - \kappa_{k+1})\omega_k = \sum_{k=1}^n (\lambda_k + \theta_k - \theta_{k+1})\omega_k$. Hence the statement. $\square$

Remark. On the right-hand side of the Pieri formula, the condition $\lambda + \rho \in P^+$ is necessarily satisfied as soon as $d_\theta(u_1, \ldots, u_{n+1}; r) \neq 0$. Using the correspondence between dominant weights and partitions, this may be verified on the Pieri formula

$$\mathcal{P}_{(r)}\, \mathcal{P}_\mu = \sum_{\kappa=(\mu_1+\theta_1,\ldots,\mu_n+\theta_n,\theta_{n+1})} \varphi_{\kappa/\mu}\, \mathcal{P}_\kappa.$$

We only have to show that $\varphi_{\kappa/\mu}$ necessarily vanishes when the multi-integer κ is not a partition. But then there is an index i such that $\kappa_i < \kappa_{i+1}$ so that the factor $(qu_i/tv_{i+1}; q)_{\theta_{i+1}}$ in $\varphi_{\kappa/\mu}$ writes out as

$$(1 - q^{1+\mu_i-\kappa_{i+1}}) \cdots (1 - q^{\mu_i-\mu_{i+1}}).$$

Due to $\kappa_i < \kappa_{i+1}$ this product would be $\neq 0$ only if $\mu_i < \mu_{i+1}$, which is impossible since μ is a partition.

From now on, we fix some integer $1 \leq k \leq n$. Substituting $r - |\theta|$ for θ_k, the Pieri formula may be written in the more explicit form

$$P_{r\omega_1}\, P_\lambda = \sum_{\substack{\theta=(\theta_1,\ldots,\theta_{k-1},0,\theta_{k+1},\ldots,\theta_{n+1})\in\mathbb{N}^n \\ |\theta|\leq r}} \hat{d}_\theta(u_1, \ldots, u_{n+1}; r)\, P_{\lambda+\rho},$$

with

$$\rho = \sum_{\substack{1\leq i\leq n \\ i\neq k-1,k}} (\theta_i - \theta_{i+1})\omega_i + \theta_{k-1}\,\omega_{k-1} + (r - |\theta|)(\omega_k - \omega_{k-1}) - \theta_{k+1}\,\omega_k,$$

and

$$\hat{d}_\theta(u_1, \ldots, u_{n+1}; r) = \frac{(q;q)_r}{(t;q)_r}\, \frac{(t;q)_{r-|\theta|}}{(q;q)_{r-|\theta|}} \prod_{\substack{j=1 \\ j\neq k}}^{n+1} \frac{(t;q)_{\theta_j}}{(q;q)_{\theta_j}}$$

$$\times \prod_{\substack{1\leq i<j\leq n+1 \\ j\neq k}} \frac{(tv_i/v_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}}\, \frac{(qu_i/tv_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}} \prod_{i=1}^{k-1} \frac{(tv_i/v_k;q)_{r-|\theta|}}{(qv_i/v_k;q)_{r-|\theta|}}\, \frac{(qu_i/tv_k;q)_{r-|\theta|}}{(u_i/v_k;q)_{r-|\theta|}}.$$

Here u_i, v_i $(1 \leq i \leq n+1)$ are as in Theorem 3.1, except $v_k = q^{r-|\theta|}u_k$. The sum is restricted to $|\theta| \leq r$ since $1/(q;q)_s = 0$ for $s < 0$.

In a second step, we concentrate on the situation $\lambda_k = 0$. Then each term on the right-hand side vanishes unless $\theta_{k+1} = 0$. Indeed, if $\lambda_k = 0$, one has $u_k = tu_{k+1}$ and $v_{k+1} = q^{\theta_{k+1}}u_{k+1}$. Hence for $i = k$ and $j = k+1$ the factor $(qu_i/tv_j;q)_{\theta_j}$ evaluates as

$$(qu_k/tv_{k+1};q)_{\theta_{k+1}} = (q^{1-\theta_{k+1}};q)_{\theta_{k+1}} = \delta_{\theta_{k+1},0}.$$

Therefore if $\lambda_k = 0$ the Pieri formula can be written as

$$P_{r\omega_1}\, P_\lambda = \sum_{\substack{\theta=(\theta_1,\ldots,\theta_{k-1},0,0,\theta_{k+2},\ldots,\theta_{n+1})\in\mathbb{N}^{n-1} \\ |\theta|\leq r}} \tilde{d}_\theta(u_1, \ldots, u_{k-1}, u_k, u_{k+2}, \ldots, u_{n+1}; k, r)\ P_{\lambda+\rho},$$

with

$$\rho = \sum_{\substack{1\le i\le n\\ i\neq k-1,k,k+1}} (\theta_i-\theta_{i+1})\omega_i + \theta_{k-1}\,\omega_{k-1} + (r-|\theta|)(\omega_k-\omega_{k-1}) - \theta_{k+2}\,\omega_{k+1},$$

and

$$\begin{aligned}
&\tilde d_\theta(u_1,\dots,u_{k-1},u_k,u_{k+2},\dots,u_{n+1};k,r) = \\
&\qquad \frac{(q;q)_r}{(t;q)_r}\,\frac{(t;q)_{r-|\theta|}}{(q;q)_{r-|\theta|}} \prod_{\substack{i=1\\ i\neq k,k+1}}^{n+1} \frac{(t;q)_{\theta_i}}{(q;q)_{\theta_i}} \prod_{\substack{1\le i<j\le n+1\\ i\neq k,k+1\\ j\neq k,k+1}} \frac{(tv_i/v_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}}\,\frac{(qu_i/tv_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}} \\
&\qquad\qquad \times \prod_{i=1}^{k-1} \frac{(tv_i/v_k;q)_{r-|\theta|}}{(qv_i/v_k;q)_{r-|\theta|}}\,\frac{(qu_i/tv_k;q)_{r-|\theta|}}{(u_i/v_k;q)_{r-|\theta|}} \prod_{j=k+2}^{n+1} \frac{(tv_k/v_j;q)_{\theta_j}}{(qv_k/v_j;q)_{\theta_j}}\,\frac{(qu_k/t^2v_j;q)_{\theta_j}}{(u_k/tv_j;q)_{\theta_j}}.
\end{aligned}$$

Here the notations are the same as before, including $v_k = q^{r-|\theta|}u_k$. For $j\ge k+2$ we have used

$$\frac{(tv_k/v_j;q)_{\theta_j}}{(qv_k/v_j;q)_{\theta_j}}\,\frac{(qu_k/tv_j;q)_{\theta_j}}{(u_k/v_j;q)_{\theta_j}}\,\frac{(tv_{k+1}/v_j;q)_{\theta_j}}{(qv_{k+1}/v_j;q)_{\theta_j}}\,\frac{(qu_{k+1}/tv_j;q)_{\theta_j}}{(u_{k+1}/v_j;q)_{\theta_j}} = \frac{(tv_k/v_j;q)_{\theta_j}}{(qv_k/v_j;q)_{\theta_j}}\,\frac{(qu_k/t^2v_j;q)_{\theta_j}}{(u_k/tv_j;q)_{\theta_j}},$$

which is a direct consequence of $v_{k+1}=u_{k+1}=u_k/t$.

In a third step, we perform some relabelling in order to remove the two 0's appearing in θ. For that purpose, for n indeterminates $(u_0,u_1,\dots,u_{n-1})$ and $\theta=(\theta_1,\dots,\theta_{n-1})\in\mathbb{N}^{n-1}$, we define

$$\begin{aligned}
&D_\theta(u_0,u_1,\dots,u_{n-1};k,r) = \\
&\quad (q/t)^{|\theta|}\,\frac{(t^2u_0;q)_{|\theta|}}{(qtu_0;q)_{|\theta|}} \prod_{i=1}^{n-1} \frac{(t;q)_{\theta_i}}{(q;q)_{\theta_i}}\,\frac{(q^{|\theta|+1}u_i;q)_{\theta_i}}{(q^{|\theta|}tu_i;q)_{\theta_i}} \prod_{1\le i<j\le n-1} \frac{(tv_i/v_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}}\,\frac{(qu_i/tv_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}} \\
&\qquad \times \prod_{i=1}^{k-1} \frac{(u_i/u_0;q)_{\theta_i}}{(qu_i/tu_0;q)_{\theta_i}}\,\frac{(qu_i/tu_0;q)_{\theta_i-r+|\theta|}}{(u_i/u_0;q)_{\theta_i-r+|\theta|}}\,\frac{(u_i/tu_0;q)_{\theta_i-r+|\theta|}}{(qu_i/t^2u_0;q)_{\theta_i-r+|\theta|}} \prod_{i=k}^{n-1} \frac{(tu_i/u_0;q)_{\theta_i}}{(qu_i/u_0;q)_{\theta_i}}.
\end{aligned}$$

Lemma. *If we write*

$$w_i = \begin{cases} q^{-r}t^{-2}, & i=0,\\ q^{-r}u_i/tu_k, & 1\le i\le k-1,\\ q^{-r}u_{i+2}/tu_k, & k\le i\le n-1,\end{cases}$$

we have

$$D_\theta(w_0,w_1,\dots,w_{n-1};k,r) = \tilde d_{(\theta_1,\dots,\theta_{k-1},0,0,\theta_k,\dots,\theta_{n-1})}(u_1,\dots,u_{k-1},u_k,u_{k+2},\dots,u_{n+1};k,r).$$

Proof. Merely by substitution, and using $v_k = q^{r-|\theta|}u_k$, we only have to prove

$$(q/t)^{|\theta|}\,\frac{(q^{-r};q)_{|\theta|}}{(q^{1-r}/t;q)_{|\theta|}}\prod_{j=k+2}^{n+1}\frac{(q^{|\theta|-r+1}u_j/tu_k;q)_{\theta_j}}{(q^{|\theta|-r}u_j/u_k;q)_{\theta_j}}\,\frac{(t^2u_j/u_k;q)_{\theta_j}}{(qtu_j/u_k;q)_{\theta_j}}$$

$$\times\prod_{i=1}^{k-1}\frac{(q^{|\theta|-r+1}u_i/tu_k;q)_{\theta_i}}{(q^{|\theta|-r}u_i/u_k;q)_{\theta_i}}\,\frac{(tu_i/u_k;q)_{\theta_i}}{(qu_i/u_k;q)_{\theta_i}}\,\frac{(qu_i/u_k;q)_{\theta_i-r+|\theta|}}{(tu_i/u_k;q)_{\theta_i-r+|\theta|}}\,\frac{(u_i/u_k;q)_{\theta_i-r+|\theta|}}{(qu_i/tu_k;q)_{\theta_i-r+|\theta|}}=$$

$$\frac{(q;q)_r}{(t;q)_r}\,\frac{(t;q)_{r-|\theta|}}{(q;q)_{r-|\theta|}}\prod_{i=1}^{k-1}\frac{(tv_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}{(qv_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}\,\frac{(qu_i/tq^{r-|\theta|}u_k;q)_{r-|\theta|}}{(u_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}$$

$$\times\prod_{j=k+2}^{n+1}\frac{(tq^{r-|\theta|}u_k/v_j;q)_{\theta_j}}{(q^{r-|\theta|+1}u_k/v_j;q)_{\theta_j}}\,\frac{(qu_k/t^2v_j;q)_{\theta_j}}{(u_k/tv_j;q)_{\theta_j}}.$$

We have obviously

$$\frac{(q^{|\theta|-r+1}u_i/tu_k;q)_{\theta_i}}{(q^{|\theta|-r}u_i/u_k;q)_{\theta_i}}\,\frac{(u_i/u_k;q)_{\theta_i-r+|\theta|}}{(qu_i/tu_k;q)_{\theta_i-r+|\theta|}}=\frac{(qu_i/tq^{r-|\theta|}u_k;q)_{r-|\theta|}}{(u_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}.$$

Using the identities

$$\frac{(aq^{-n};q)_n}{(bq^{-n};q)_n}=\frac{(q/a;q)_n}{(q/b;q)_n}\,(a/b)^n,$$

$$\frac{(a;q)_n}{(b;q)_n}\,\frac{(b;q)_{n-k}}{(a;q)_{n-k}}=\frac{(q^{1-n}/a;q)_k}{(q^{1-n}/b;q)_k}\,(a/b)^k,$$

we get

$$\begin{aligned}\frac{(tu_i/u_k;q)_{\theta_i}}{(qu_i/u_k;q)_{\theta_i}}\,\frac{(qu_i/u_k;q)_{\theta_i-r+|\theta|}}{(tu_i/u_k;q)_{\theta_i-r+|\theta|}}&=\frac{(q^{1-\theta_i}u_k/tu_i;q)_{r-|\theta|}}{(q^{-\theta_i}u_k/u_i;q)_{r-|\theta|}}\,(t/q)^{r-|\theta|}\\&=\frac{(tv_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}{(qv_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}.\end{aligned}$$

Similarly we obtain

$$(t/q)^{\theta_j}\,\frac{(q^{|\theta|-r+1}u_j/tu_k;q)_{\theta_j}}{(q^{|\theta|-r}u_j/u_k;q)_{\theta_j}}=\frac{(tq^{r-|\theta|}u_k/v_j;q)_{\theta_j}}{(q^{r-|\theta|+1}u_k/v_j;q)_{\theta_j}}$$

$$(q/t)^{\theta_j}\,\frac{(t^2u_j/u_k;q)_{\theta_j}}{(qtu_j/u_k;q)_{\theta_j}}=\frac{(qu_k/t^2v_j;q)_{\theta_j}}{(u_k/tv_j;q)_{\theta_j}}.\qquad\square$$

Finally we have proved the following Pieri formula.

Theorem 3.2. *Let* $\lambda=\sum_{i=1}^n\lambda_i\omega_i$ *be a dominant weight and* $r\in\mathbb{N}$*. Assume* $\lambda_k=0$ *for some fixed* $1\le k\le n$*. Define*

$$u_i=\begin{cases}q^{-r}t^{-2}, & i=0,\\ q^{-r+\sum_{j=i}^{k-1}\lambda_j}t^{k-i-1}, & 1\le i\le k-1,\\ q^{-r-\sum_{j=k+1}^{i+1}\lambda_j}t^{k-i-3}, & k\le i\le n-1.\end{cases}$$

We have

$$P_{r\omega_1}\, P_\lambda = \sum_{\substack{\theta=(\theta_1,\ldots,\theta_{n-1})\in\mathbb{N}^{n-1}\\ |\theta|\le r}} D_\theta(u_0,u_1,\ldots,u_{n-1};k,r)\, P_{\lambda+\rho},$$

with

$$\rho=\sum_{i=1}^{k-2}(\theta_i-\theta_{i+1})\omega_i+\theta_{k-1}\,\omega_{k-1}+(r-|\theta|)(\omega_k-\omega_{k-1})-\theta_k\,\omega_{k+1}+\sum_{i=k+2}^{n}(\theta_{i-2}-\theta_{i-1})\omega_i.$$

Remark. For $k=1,2$ (resp. $k=n,\ n-1$) the first (resp. the last) sum in the above expression of ρ must be understood as zero. This convention will be kept in the next sections.

4 A recurrence formula

Given two multi-integers $\beta=(\beta_1,\ldots,\beta_{n-1})$, $\kappa=(\kappa_1,\ldots,\kappa_{n-1})\in\mathbb{Z}^{n-1}$, we write $\beta\ge\kappa$ for $\beta_i\ge\kappa_i$ $(1\le i\le n-1)$. We say that an infinite $(n-1)$-dimensional matrix $F=(f_{\beta\kappa})_{\beta,\kappa\in\mathbb{Z}^{n-1}}$ is lower-triangular if $f_{\beta\kappa}=0$ unless $\beta\ge\kappa$. When all $f_{\kappa\kappa}\neq 0$, there exists a unique lower-triangular matrix $G=(g_{\kappa\gamma})_{\kappa,\gamma\in\mathbb{Z}^{n-1}}$ such that

$$\sum_{\beta\ge\kappa\ge\gamma} f_{\beta\kappa}\, g_{\kappa\gamma}=\delta_{\beta\gamma},$$

for all $\beta,\gamma\in\mathbb{Z}^{n-1}$, where $\delta_{\beta\gamma}$ is the usual Kronecker symbol. We refer to F and G as mutually inverse.

Such a pair of infinite multidimensional inverse matrices is given in the Appendix, as a corollary of [3, Theorem 2.7] (and, in fact, equivalent to the latter). This result is essential for our purpose.

Given n indeterminates $(u_0,u_1,\ldots,u_{n-1})$, $\theta=(\theta_1,\ldots,\theta_{n-1})\in\mathbb{N}^{n-1}$, and $k,r\in\mathbb{N}$ with $1\le k\le n$, we define

$$\begin{aligned}
&C_{\theta_1,\ldots,\theta_{n-1}}(u_0,u_1,\ldots,u_{n-1};k,r)=\\
&\quad q^{|\theta|}\,\frac{(t^2u_0;q)_{|\theta|}}{(qtu_0;q)_{|\theta|}}\prod_{i=1}^{n-1}\frac{(q/t;q)_{\theta_i}}{(q;q)_{\theta_i}}\,\frac{(qu_i;q)_{\theta_i}}{(qtu_i;q)_{\theta_i}}\prod_{1\le i<j\le n-1}\frac{(qv_i/tv_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}}\,\frac{(tu_i/v_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}}\\
&\quad\times\prod_{i=1}^{k-1}\frac{(u_i/tu_0;q)_{\theta_i}}{(qu_i/t^2u_0;q)_{\theta_i}}\,\frac{(qtu_0/u_i;q)_r}{(t^2u_0/u_i;q)_r}\,\frac{(tu_0/u_i;q)_r}{(qu_0/u_i;q)_r}\prod_{i=k}^{n-1}\frac{(tu_i/u_0;q)_{\theta_i}}{(qu_i/u_0;q)_{\theta_i}}\\
&\quad\times\frac{1}{\Delta(v)}\det_{1\le i,j\le n-1}\left[v_i^{n-j-1}\left(1-t^{j-1}\frac{1-tv_i}{1-v_i}\prod_{s=1}^{n-1}\frac{v_i-u_s}{v_i-tu_s}\right)\right],
\end{aligned}$$

with $\Delta(v)$ the Vandermonde determinant $\prod_{1\le i<j\le n-1}(v_i-v_j)$. Here is our main result.

Theorem 4.1. *Let $\lambda = \sum_{i=1}^n \lambda_i\omega_i$ be a dominant weight. Assume $\lambda_k = 0$ for some fixed $1 \le k \le n$. For any positive integer $r \le \lambda_{k-1}$ the weight*

$$\lambda^{(r)} = \lambda + r(\omega_k - \omega_{k-1}) = \lambda + r\varepsilon_k$$

is dominant. Define

$$u_i = \begin{cases} q^{-r}t^{-2}, & i = 0,\\ q^{-r+\sum_{j=i}^{k-1}\lambda_j} t^{k-i-1}, & 1 \le i \le k-1,\\ q^{-r-\sum_{j=k+1}^{i+1}\lambda_j} t^{k-i-3}, & k \le i \le n-1. \end{cases}$$

We have

$$P_{\lambda^{(r)}} = \sum_{\substack{\theta=(\theta_1,\dots,\theta_{n-1})\in\mathbb{N}^{n-1}\\ |\theta|\le r}} C_\theta(u_0,u_1,\dots,u_{n-1};k,r)\, P_{(r-|\theta|)\omega_1}\, P_{\lambda+\rho},$$

with

$$\rho = \sum_{i=1}^{k-2}(\theta_i - \theta_{i+1})\omega_i + \theta_{k-1}\,\omega_{k-1} - \theta_k\,\omega_{k+1} + \sum_{i=k+2}^{n}(\theta_{i-2}-\theta_{i-1})\omega_i.$$

Remark. The weight $\lambda+\rho$ has no component on ω_k. Further, similarly as in Theorem 3.1 (see the Remark following the proof of that theorem), the condition $\lambda+\rho \in P^+$ is necessarily satisfied in Theorem 4.2 as soon as $C_\theta(u_0,u_1,\dots,u_{n-1};k,r) \neq 0$. We omit the details which involve a tedious case-by-case analysis.

Proof. We make use of the multidimensional matrix inverse given in the Appendix. Let $\beta = (\beta_1,\dots,\beta_{n-1})$, $\kappa = (\kappa_1,\dots,\kappa_{n-1})$, $\gamma = (\gamma_1,\dots,\gamma_{n-1}) \in \mathbb{Z}^{n-1}$. If we define

$$\begin{aligned} f_{\beta\kappa} &= C_{\beta_1-\kappa_1,\dots,\beta_{n-1}-\kappa_{n-1}}\big(q^{|\kappa|}u_0, q^{\kappa_1+|\kappa|}u_1,\dots,q^{\kappa_{n-1}+|\kappa|}u_{n-1};k,r-|\kappa|\big),\\ g_{\kappa\gamma} &= D_{\kappa_1-\gamma_1,\dots,\kappa_{n-1}-\gamma_{n-1}}\big(q^{|\gamma|}u_0, q^{\gamma_1+|\gamma|}u_1,\dots,q^{\gamma_{n-1}+|\gamma|}u_{n-1};k,r-|\gamma|\big), \end{aligned}$$

by this result, the infinite lower-triangular multidimensional matrices $(f_{\beta\kappa})_{\beta,\kappa\in\mathbb{Z}^{n-1}}$ and $(g_{\kappa\gamma})_{\kappa,\gamma\in\mathbb{Z}^{n-1}}$ are mutually inverse.

Now let us replace in Theorem 3.2 λ_i by $\lambda_i+\gamma_i-\gamma_{i+1}$ for $1\le i\le k-2$, λ_{k-1} by $\lambda_{k-1}+\gamma_{k-1}$, λ_{k+1} by $\lambda_{k+1}-\gamma_k$, λ_i by $\lambda_i+\gamma_{i-2}-\gamma_{i-1}$ for $k+2\le i\le n$, r by $r-|\gamma|$, respectively. Then u_0 is replaced by $q^{|\gamma|}u_0$, and u_i by $q^{\gamma_i+|\gamma|}u_i$ for $1\le i\le n-1$. In explicit terms, we are considering the identity

$$P_{(r-|\gamma|)\omega_1}\, P_{\lambda+\tilde\gamma} = \sum_{\substack{\theta=(\theta_1,\dots,\theta_{n-1})\in\mathbb{N}^{n-1}\\ |\theta|\le r}} D_\theta(q^{|\gamma|}u_0, q^{\gamma_1+|\gamma|}u_1,\dots,q^{\gamma_{n-1}+|\gamma|}u_{n-1};k,r-|\gamma|)\, P_{\lambda+\tilde\gamma+\rho},$$

with

$$u_i = \begin{cases} q^{-r}t^{-2}, & i = 0,\\ q^{-r+\sum_{j=i}^{k-1}\lambda_j} t^{k-i-1}, & 1 \le i \le k-1,\\ q^{-r-\sum_{j=k+1}^{i+1}\lambda_j} t^{k-i-3}, & k \le i \le n-1, \end{cases}$$

and

$$\rho = \sum_{i=1}^{k-2}(\theta_i - \theta_{i+1})\omega_i + \theta_{k-1}\,\omega_{k-1} + (r - |\theta|)(\omega_k - \omega_{k-1}) - \theta_k\,\omega_{k+1} + \sum_{i=k+2}^{n}(\theta_{i-2} - \theta_{i-1})\omega_i,$$

$$\tilde{\gamma} = \sum_{i=1}^{k-2}(\gamma_i - \gamma_{i+1})\omega_i + \gamma_{k-1}\,\omega_{k-1} - \gamma_k\,\omega_{k+1} + \sum_{i=k+2}^{n}(\gamma_{i-2} - \gamma_{i-1})\omega_i.$$

After substituting the summation indices $\theta_i \mapsto \kappa_i - \gamma_i$ for $1 \le i \le n-1$, we obtain exactly

$$\sum_{\kappa \in \mathbb{Z}^{n-1}} g_{\kappa\gamma} y_\kappa = w_\gamma \qquad (\gamma \in \mathbb{Z}^{n-1}),$$

with

$$y_\kappa = P_{\lambda+\tilde{\kappa}}, \qquad w_\gamma = P_{(r-|\gamma|)\omega_1}\,P_{\lambda+\tilde{\gamma}},$$

and

$$\tilde{\kappa} = \sum_{i=1}^{k-2}(\kappa_i - \kappa_{i+1})\omega_i + \kappa_{k-1}\,\omega_{k-1} + (r - |\kappa|)(\omega_k - \omega_{k-1}) - \kappa_k\,\omega_{k+1} + \sum_{i=k+2}^{n}(\kappa_{i-2} - \kappa_{i-1})\omega_i.$$

This immediately yields the inverse relation

$$\sum_{\beta \in \mathbb{Z}^{n-1}} f_{\beta\kappa} w_\beta = y_\kappa \qquad (\kappa \in \mathbb{Z}^{n-1}).$$

We conclude by setting $\kappa_i = 0$ for all $1 \le i \le n-1$. □

Finally, by the substitutions $r \to \lambda_k$ and $\lambda_{k-1} \to \lambda_{k-1} + \lambda_k$, we obtain the following very remarkable expansion.

Theorem 4.2. *Let $\lambda = \sum_{i=1}^{n} \lambda_i \omega_i$ be a dominant weight and $k \in \mathbb{N}$ fixed with $1 \le k \le n$. Define*

$$u_i = \begin{cases} q^{-\lambda_k} t^{-2}, & i = 0,\\ q^{\sum_{j=i}^{k-1}\lambda_j} t^{k-i-1}, & 1 \le i \le k-1,\\ q^{-\sum_{j=k}^{i+1}\lambda_j} t^{k-i-3}, & k \le i \le n-1, \end{cases}$$

and $\mu = \lambda - \lambda_k\,(\omega_k - \omega_{k-1}) = \lambda - \lambda_k\,\varepsilon_k$. We have

$$P_\lambda = \sum_{\substack{\theta=(\theta_1,\ldots,\theta_{n-1})\in\mathbb{N}^{n-1}\\ |\theta|\le\lambda_k}} C_\theta(u_0, u_1, \ldots, u_{n-1}; k, \lambda_k)\,P_{(\lambda_k - |\theta|)\omega_1}\,P_{\mu+\rho},$$

with

$$\rho = \sum_{i=1}^{k-2}(\theta_i - \theta_{i+1})\omega_i + \theta_{k-1}\,\omega_{k-1} - \theta_k\,\omega_{k+1} + \sum_{i=k+2}^{n}(\theta_{i-2} - \theta_{i-1})\omega_i.$$

Remark. Observe that the weights μ and $\mu+\rho$ have no component on ω_k.

The $k=n$ special case is worth writing out explicitly.

Corollary. *Let $\lambda=\sum_{i=1}^n \lambda_i\omega_i$ be a dominant weight. Define $u_0=q^{-\lambda_n}t^{-2}$ and $u_i=q^{\sum_{l=i}^{n-1}\lambda_l}t^{n-i-1}$ $(1\le i\le n-1)$. We have*

$$P_\lambda=\sum_{\substack{\theta=(\theta_1,\dots,\theta_{n-1})\in\mathbb{N}^{n-1}\\ |\theta|\le\lambda_n}} C_\theta(u_0,u_1,\dots,u_{n-1};n,\lambda_n)\,P_{(\lambda_n-|\theta|)\omega_1}\,P_\mu,$$

with $\mu=\sum_{i=1}^{n-2}(\lambda_i+\theta_i-\theta_{i+1})\omega_i+(\lambda_{n-1}+\lambda_n+\theta_{n-1})\omega_{n-1}$.

The reader may check that this is exactly Theorem 4.1 of [3] (with $n\mapsto n-1$), written for $x_1\cdots x_{n+1}=1$, up to the normalization $Q_\lambda=b_\lambda\,P_\lambda$ with

$$b_\lambda=\prod_{1\le i\le j\le n}\frac{(q^{\sum_{l=i}^{j-1}\lambda_l}t^{j-i+1};q)_{\lambda_j}}{(q^{1+\sum_{l=i}^{j-1}\lambda_l}t^{j-i};q)_{\lambda_j}}=\prod_{1\le i\le j\le n}\frac{(tu_i/u_j;q)_{\lambda_j}}{(qu_i/u_j;q)_{\lambda_j}},$$

where we set $u_n=1/t$.

5 Examples

In this section we write out the formulas in Theorem 4.2 explicitly for $n=2,3$.

5.1 The root system A_2

For $k=2$ we have $u_0=q^{-\lambda_2}/t^2$, $u_1=q^{\lambda_1}$, and

$$\begin{aligned}C_\theta(u_0,u_1;2,r)=q^\theta\,&\frac{(t^2u_0;q)_\theta}{(qtu_0;q)_\theta}\,\frac{(q/t;q)_\theta}{(q;q)_\theta}\,\frac{(qu_1;q)_\theta}{(qtu_1;q)_\theta}\,\frac{(u_1/tu_0;q)_\theta}{(qu_1/t^2u_0;q)_\theta}\\&\times\frac{(qtu_0/u_1;q)_r}{(t^2u_0/u_1;q)_r}\,\frac{(tu_0/u_1;q)_r}{(qu_0/u_1;q)_r}\left(1-\frac{1-tv_1}{1-v_1}\,\frac{v_1-u_1}{v_1-tu_1}\right).\end{aligned}$$

After some simplifications, we obtain

$$P_{\lambda_1\omega_1+\lambda_2\omega_2}=\sum_{\theta\in\mathbb{N}}C_\theta^{(2)}(\lambda)\,P_{(\lambda_2-\theta)\omega_1}\,P_{(\lambda_1+\lambda_2+\theta)\omega_1},$$

with

$$\begin{aligned}C_\theta^{(2)}(\lambda)&=C_\theta(u_0,u_1;2,\lambda_2)\\&=t^\theta\,\frac{(q^{\lambda_2-\theta+1};q)_\theta}{(tq^{\lambda_2-\theta};q)_\theta}\,\frac{(1/t;q)_\theta}{(q;q)_\theta}\,\frac{(q^{\lambda_1+1};q)_\theta}{(tq^{\lambda_1+1};q)_\theta}\,\frac{(tq^{\lambda_1};q)_{\lambda_2+\theta}}{(q^{\lambda_1+1};q)_{\lambda_2+\theta}}\,\frac{(tq^{\lambda_1+1};q)_{\lambda_2}}{(t^2q^{\lambda_1};q)_{\lambda_2}}\,\frac{1-q^{\lambda_1+2\theta}}{1-q^{\lambda_1+\theta}}.\end{aligned}$$

This result may be compared with the Jing–Józefiak classical result [1], more precisely its restriction to three variables (x_1, x_2, x_3) subject to $x_1x_2x_3 = 1$. Namely, given a partition (μ_1, μ_2), the Macdonald symmetric function $\mathcal{P}_{(\mu_1,\mu_2)}(q,t)$ is given by

$$\mathcal{P}_{(\mu_1,\mu_2)} = \sum_{\theta\in\mathbb{N}} \mathcal{C}_\theta(\mu)\, \mathcal{P}_{(\mu_2-\theta)}\, \mathcal{P}_{(\mu_1+\theta)},$$

with

$$\begin{aligned}\mathcal{C}_\theta(\mu) = \frac{(tq^{\mu_1-\mu_2+1};q)_{\mu_2}}{(t^2q^{\mu_1-\mu_2};q)_{\mu_2}}\,\frac{(q^{\mu_2-\theta+1};q)_\theta}{(tq^{\mu_2-\theta};q)_\theta}\,\frac{(tq^{\mu_1-\mu_2};q)_{\mu_2+\theta}}{(q^{\mu_1-\mu_2+1};q)_{\mu_2+\theta}}& \\ \times\, t^\theta\,\frac{(1/t;q)_\theta}{(q;q)_\theta}\,\frac{(q^{\mu_1-\mu_2+1};q)_\theta}{(tq^{\mu_1-\mu_2+1};q)_\theta}\,\frac{1-q^{\mu_1-\mu_2+2\theta}}{1-q^{\mu_1-\mu_2+\theta}}&.\end{aligned}$$

Our formula is equivalent to the main result of [1] by the correspondence $\lambda_1 = \mu_1 - \mu_2$, $\lambda_2 = \mu_2$ between dominant weights and partitions, recalled in Section 2.

For $k = 1$ we have $u_0 = q^{-\lambda_1}/t^2$, $u_1 = q^{-\lambda_1-\lambda_2}/t^3$, and

$$C_\theta(u_0, u_1; 1, r) = q^\theta\,\frac{(t^2u_0;q)_\theta}{(qtu_0;q)_\theta}\,\frac{(q/t;q)_\theta}{(q;q)_\theta}\,\frac{(qu_1;q)_\theta}{(qtu_1;q)_\theta}\,\frac{(tu_1/u_0;q)_\theta}{(qu_1/u_0;q)_\theta}\left(1-\frac{1-tv_1}{1-v_1}\,\frac{v_1-u_1}{v_1-tu_1}\right).$$

After some simplifications, we obtain

$$P_{\lambda_1\omega_1+\lambda_2\omega_2} = \sum_{\theta\in\mathbb{N}} C_\theta^{(1)}(\lambda)\, P_{(\lambda_1-\theta)\omega_1}\, P_{(\lambda_2-\theta)\omega_2},$$

with

$$\begin{aligned}C_\theta^{(1)}(\lambda) &= C_\theta(u_0, u_1; 1, \lambda_1)\\ &= t^\theta\,\frac{(1/t;q)_\theta}{(q;q)_\theta}\,\frac{(q^{\lambda_1};1/q)_\theta}{(tq^{\lambda_1-1};1/q)_\theta}\,\frac{(q^{\lambda_2};1/q)_\theta}{(tq^{\lambda_2-1};1/q)_\theta}\,\frac{(t^3q^{\lambda_1+\lambda_2-1};1/q)_\theta}{(t^2q^{\lambda_1+\lambda_2-1};1/q)_\theta}\,\frac{1-t^3q^{\lambda_1+\lambda_2-2\theta}}{1-t^3q^{\lambda_1+\lambda_2-\theta}}.\end{aligned}$$

We thus recover exactly Perelomov, Ragoucy and Zaugg's result given in [9, Theorem 1(a)].

5.2 The root system A_3

For $k = 1, 2, 3$ our formulas in Theorem 4.2 write respectively as

$$\begin{aligned}P_{\lambda_1\omega_1+\lambda_2\omega_2+\lambda_3\omega_3} &= \sum_{(i,j)\in\mathbb{N}^2} C_{ij}^{(1)}(\lambda)\, P_{(\lambda_1-i-j)\omega_1}\, P_{(\lambda_2-i)\omega_2+(\lambda_3+i-j)\omega_3},\\ &= \sum_{(i,j)\in\mathbb{N}^2} C_{ij}^{(2)}(\lambda)\, P_{(\lambda_2-i-j)\omega_1}\, P_{(\lambda_1+\lambda_2+i)\omega_1+(\lambda_3-j)\omega_3},\\ &= \sum_{(i,j)\in\mathbb{N}^2} C_{ij}^{(3)}(\lambda)\, P_{(\lambda_3-i-j)\omega_1}\, P_{(\lambda_1+i-j)\omega_1+(\lambda_2+\lambda_3+j)\omega_2}.\end{aligned}$$

In order to make these expansions explicit, we need to evaluate the determinant of the 2 by 2 matrix A given by

$$A_{kl} = v_k^{2-l}\left(1 - t^{l-1}\frac{1-tv_k}{1-v_k}\,\frac{v_k-u_1}{v_k-tu_1}\,\frac{v_k-u_2}{v_k-tu_2}\right),$$

with $v_1 = q^i u_1, v_2 = q^j u_2$.

More precisely we need to compute the quotient of this determinant by the Vandermonde determinant $v_1 - v_2 = q^i u_1 - q^j u_2$. There is no evidence this quotient may be written in canonical form. Inspired by the explicit result of [2, Theorem 1] (see below), we write this quotient of determinants as

$$\begin{aligned}\frac{\det\, A}{q^iu_1-q^ju_2} = \frac{(t-1)^2}{(t-q^i)(t-q^j)}\Bigg(&\frac{1-q^{2i}u_1}{1-q^iu_1}\,\frac{1-q^{2j}u_2}{1-q^ju_2}\left(1+t^{-1}\frac{1-q^i}{1-q^iu_1/tu_2}\,\frac{1-q^j}{1-q^ju_2/tu_1}\right)\\ &- (q^iu_1+q^ju_2)\frac{1-q^i}{1-q^iu_1}\,\frac{1-q^j}{1-q^ju_2}\,\frac{1-q^i/t}{1-q^iu_1/tu_2}\,\frac{1-q^j/t}{1-q^ju_2/tu_1}\Bigg).\end{aligned}$$

The above identity (which is not trivial) may be easily verified by using any formal calculus software.

Next, for $(i,j) \in \mathbb{N}^2$ we define

$$\begin{aligned}&\nabla_{ij}(u_0,u_1,u_2) =\\ &q^{i+j}\,\frac{(t^2u_0;q)_{i+j}}{(qtu_0;q)_{i+j}}\,\frac{(1/t;q)_i}{(q;q)_i}\,\frac{(u_1;q)_i}{(qtu_1;q)_i}\,\frac{(1/t;q)_j}{(q;q)_j}\,\frac{(u_2;q)_j}{(qtu_2;q)_j}\,\frac{(q^{i-j+1}u_1/tu_2;q)_j}{(q^{i-j+1}u_1/u_2;q)_j}\,\frac{(tq^{-j}u_1/u_2;q)_j}{(q^{-j}u_1/u_2;q)_j}\\ &\qquad\times\Bigg(\frac{1-q^{2i}u_1}{1-u_1}\,\frac{1-q^{2j}u_2}{1-u_2}\left(1+t^{-1}\frac{1-q^i}{1-q^iu_1/tu_2}\,\frac{1-q^j}{1-q^ju_2/tu_1}\right)\\ &\qquad\qquad- (q^iu_1+q^ju_2)\frac{1-q^i}{1-u_1}\,\frac{1-q^j}{1-u_2}\,\frac{1-q^i/t}{1-q^iu_1/tu_2}\,\frac{1-q^j/t}{1-q^ju_2/tu_1}\Bigg).\end{aligned}$$

It is readily verified that we have

$$\frac{C_{ij}(u_0,u_1,u_2;1,r)}{\nabla_{ij}(u_0,u_1,u_2)} = \frac{(tu_1/u_0;q)_i}{(qu_1/u_0;q)_i}\,\frac{(tu_2/u_0;q)_j}{(qu_2/u_0;q)_j},$$

$$\frac{C_{ij}(u_0,u_1,u_2;2,r)}{\nabla_{ij}(u_0,u_1,u_2)} = \frac{(u_1/tu_0;q)_i}{(qu_1/t^2u_0;q)_i}\,\frac{(qtu_0/u_1;q)_r}{(t^2u_0/u_1;q)_r}\,\frac{(tu_0/u_1;q)_r}{(qu_0/u_1;q)_r}\,\frac{(tu_2/u_0;q)_j}{(qu_2/u_0;q)_j},$$

$$\begin{aligned}\frac{C_{ij}(u_0,u_1,u_2;3,r)}{\nabla_{ij}(u_0,u_1,u_2)} = &\frac{(u_1/tu_0;q)_i}{(qu_1/t^2u_0;q)_i}\,\frac{(qtu_0/u_1;q)_r}{(t^2u_0/u_1;q)_r}\,\frac{(tu_0/u_1;q)_r}{(qu_0/u_1;q)_r}\\ &\times\frac{(u_2/tu_0;q)_j}{(qu_2/t^2u_0;q)_j}\,\frac{(qtu_0/u_2;q)_r}{(t^2u_0/u_2;q)_r}\,\frac{(tu_0/u_2;q)_r}{(qu_0/u_2;q)_r}.\end{aligned}$$

Now, by Theorem 4.2 the respective recurrence coefficients are determined to be

$$\begin{aligned}
C_{ij}^{(1)}(\lambda) &= C_{ij}(q^{-\lambda_1}/t^2, q^{-\lambda_1-\lambda_2}/t^3, q^{-\lambda_1-\lambda_2-\lambda_3}/t^4; 1, \lambda_1),\\
C_{ij}^{(2)}(\lambda) &= C_{ij}(q^{-\lambda_2}/t^2, q^{\lambda_1}, q^{-\lambda_2-\lambda_3}/t^3; 2, \lambda_2),\\
C_{ij}^{(3)}(\lambda) &= C_{ij}(q^{-\lambda_3}/t^2, q^{\lambda_1+\lambda_2}t, q^{\lambda_2}; 3, \lambda_3).
\end{aligned}$$

The cases $k = 1, 2$ are new. For $k = 3$ we recover the first author's earlier result in [2, Theorem 1], more precisely the restriction of this result to four variables (x_1, x_2, x_3, x_4) subject to $x_1x_2x_3x_4 = 1$. Namely given a partition (μ_1, μ_2, μ_3) and $u = q^{\mu_1-\mu_2}$, $v = q^{\mu_2-\mu_3}$, the Macdonald symmetric function $\mathcal{P}_{(\mu_1,\mu_2,\mu_3)}(q, t)$ is given by

$$\mathcal{P}_{(\mu_1,\mu_2,\mu_3)} = \sum_{(i,j)\in\mathbb{N}^2} \mathcal{C}_{ij}(\mu)\, \mathcal{P}_{(\mu_3-i-j)}\, \mathcal{P}_{(\mu_1+i,\mu_2+j)},$$

with

$$\begin{aligned}
\mathcal{C}_{ij}(\mu) = t^{i+j}\, &\frac{(1/t;q)_i}{(q;q)_i} \frac{(1/t;q)_j}{(q;q)_j} \frac{(tuv;q)_i}{(qt^2uv;q)_i} \frac{(v;q)_j}{(qtv;q)_j} \frac{(q^{-j}t^2u;q)_i}{(q^{-j}tu;q)_i} \frac{(qu;q)_i}{(qtu;q)_i}\\
\times &\frac{(t;q)_{\mu_1-\mu_2+i-j}}{(q;q)_{\mu_1-\mu_2+i-j}} \frac{(t;q)_{\mu_2+j}}{(q;q)_{\mu_2+j}} \frac{(t;q)_{\mu_3-i-j}}{(q;q)_{\mu_3-i-j}} \frac{(q;q)_{\mu_1-\mu_2}}{(t;q)_{\mu_1-\mu_2}} \frac{(q;q)_{\mu_2-\mu_3}}{(t;q)_{\mu_2-\mu_3}} \frac{(q;q)_{\mu_3}}{(t;q)_{\mu_3}}\\
\times &\frac{(q^{i-j}t^2u;q)_{\mu_2+j}}{(q^{i-j+1}tu;q)_{\mu_2+j}} \frac{(qtu;q)_{\mu_2-\mu_3}}{(t^2u;q)_{\mu_2-\mu_3}} \frac{(qt^2uv;q)_{\mu_3}}{(t^3uv;q)_{\mu_3}} \frac{(qtv;q)_{\mu_3}}{(t^2v;q)_{\mu_3}} \frac{1-q^{2i}tuv}{1-tuv} \frac{1-q^{2j}v}{1-v}\\
\times &\left(1 + u\,\frac{1-q^i}{1-q^iu}\, \frac{1-q^{-j}}{1-q^{-j}t^2u} \Big(t - v(q^itu + q^j) \frac{t-q^i}{1-q^{2i}tuv} \frac{t-q^j}{1-q^{2j}v}\Big)\right).
\end{aligned}$$

The reader may check our formula is indeed equivalent to [2, Theorem 1] by using the correspondence $\lambda_1 = \mu_1 - \mu_2$, $\lambda_2 = \mu_2 - \mu_3$, $\lambda_3 = \mu_3$ between dominant weights and partitions.

6 Final remark

The Macdonald polynomial P_λ, $\lambda = \sum_{i=1}^n \lambda_i\omega_i$, is in bijective correspondence with the symmetric function $\mathcal{P}_\mu(x_1, \ldots, x_{n+1})$ with $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i = \sum_{j=i}^n \lambda_j$, subject to the condition $x_1 \cdots x_{n+1} = 1$. Therefore the n recurrence relations that we have obtained for P_λ may be expressed in terms of $\mathcal{P}_\mu(x_1, \ldots, x_{n+1})$, subject to $x_1 \cdots x_{n+1} = 1$.

One may wonder whether this restriction can be removed. Equivalently, being given some fixed integer $1 \le k \le n$, is it possible to expand the symmetric function $\mathcal{P}_\mu$ in terms of products $\mathcal{P}_{(r)}\mathcal{P}_\rho$ for partitions $\rho = (\rho_1, \ldots, \rho_n)$ satisfying $\rho_k = \rho_{k+1}$?

Such a development has been obtained in [3] for $k = n$, in which case $\rho_n = \rho_{n+1} = 0$. However this method cannot be used for other values of k.

Actually the Pieri expansion of $\mathcal{P}_{(r)}\mathcal{P}_\rho$ involves symmetric functions $\mathcal{P}_\sigma$ with $\sigma - \rho$ a horizontal r-strip. Hence some of these partitions σ have length $l(\sigma) = n + 1$. The only exception occurs for $k = n$ since in that case $\rho_n = 0$ entails $l(\sigma) \le n$.

Therefore, except for $k = n$, the Pieri multiplication does not conserve the space generated by $\{\mathcal{P}_\kappa, l(\kappa) \leq n\}$, and it is not possible to define a Pieri matrix to invert.

This difficulty does not arise in the A_n framework. Then the Pieri matrix can be defined, because the condition $x_1 \cdots x_{n+1} = 1$ and the property [5, (4.17), p. 325]

$$\mathcal{P}_{(\sigma_1,\ldots,\sigma_{n+1})}(x_1,\ldots,x_{n+1}) = (x_1 \cdots x_{n+1})^{\sigma_{n+1}}\, \mathcal{P}_{(\sigma_1-\sigma_{n+1},\ldots,\sigma_n-\sigma_{n+1},0)}(x_1,\ldots,x_{n+1})$$

allow to deal with partitions of length $n+1$.

Appendix: A multidimensional matrix inverse

The following result (equivalent to one previously given in [3]) is crucial to obtain the recursion formula in Section 4.

Lemma. *Let $t, u_0, u_1, \ldots, u_n$ be indeterminates and $r, k \in \mathbb{N}$ with $1 \leq k \leq n+1$. Define*

$$\begin{aligned}
f_{\beta\kappa} = {} & q^{|\beta|-|\kappa|}\, \frac{(t^2u_0;q)_{|\beta|}}{(qtu_0;q)_{|\beta|}}\, \frac{(qtu_0;q)_{|\kappa|}}{(t^2u_0;q)_{|\kappa|}} \prod_{i=1}^{n} \frac{(q/t;q)_{\beta_i-\kappa_i}}{(q;q)_{\beta_i-\kappa_i}}\, \frac{(q^{\kappa_i+|\kappa|+1}u_i;q)_{\beta_i-\kappa_i}}{(q^{\kappa_i+|\kappa|+1}tu_i;q)_{\beta_i-\kappa_i}} \\
& \times \prod_{i=1}^{k-1} \frac{(u_i/tu_0;q)_{\beta_i}}{(qu_i/t^2u_0;q)_{\beta_i}}\, \frac{(qu_i/tu_0;q)_{\kappa_i}}{(u_i/u_0;q)_{\kappa_i}}\, \frac{(u_i/u_0;q)_{|\kappa|-r+\kappa_i}}{(qu_i/tu_0;q)_{|\kappa|-r+\kappa_i}}\, \frac{(qu_i/t^2u_0;q)_{|\kappa|-r+\kappa_i}}{(u_i/tu_0;q)_{|\kappa|-r+\kappa_i}} \\
& \times \prod_{i=k}^{n} \frac{(tu_i/u_0;q)_{\beta_i}}{(qu_i/u_0;q)_{\beta_i}}\, \frac{(qu_i/u_0;q)_{\kappa_i}}{(tu_i/u_0;q)_{\kappa_i}} \\
& \times \prod_{1\leq i<j\leq n} \frac{(q^{\beta_i-\beta_j+1}u_i/tu_j;q)_{\beta_j-\kappa_j}}{(q^{\beta_i-\beta_j+1}u_i/u_j;q)_{\beta_j-\kappa_j}}\, \frac{(q^{\kappa_i-\beta_j}tu_i/u_j;q)_{\beta_j-\kappa_j}}{(q^{\kappa_i-\beta_j}u_i/u_j;q)_{\beta_j-\kappa_j}}\, \left(q^{\beta_i}u_i - q^{\beta_j}u_j\right)^{-1} \\
& \times \det_{1\leq i,j\leq n}\left[\left(q^{\beta_i}u_i\right)^{n-j}\left(1 - t^{j-1}\frac{(1-q^{\beta_i+|\kappa|}tu_i)}{(1-q^{\beta_i+|\kappa|}u_i)}\prod_{s=1}^{n}\frac{(q^{\beta_i}u_i - q^{\kappa_s}u_s)}{(q^{\beta_i}u_i - q^{\kappa_s}tu_s)}\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
g_{\kappa\gamma} = {} & \left(\frac{q}{t}\right)^{|\kappa|-|\gamma|} \frac{(t^2u_0;q)_{|\kappa|}}{(qtu_0;q)_{|\kappa|}}\, \frac{(qtu_0;q)_{|\gamma|}}{(t^2u_0;q)_{|\gamma|}} \prod_{i=1}^{n} \frac{(t;q)_{\kappa_i-\gamma_i}}{(q;q)_{\kappa_i-\gamma_i}}\, \frac{(q^{\gamma_i+|\kappa|+1}u_i;q)_{\kappa_i-\gamma_i}}{(q^{\gamma_i+|\kappa|}tu_i;q)_{\kappa_i-\gamma_i}} \\
& \times \prod_{i=1}^{k-1} \frac{(u_i/u_0;q)_{\kappa_i}}{(qu_i/tu_0;q)_{\kappa_i}}\, \frac{(qu_i/t^2u_0;q)_{\gamma_i}}{(u_i/tu_0;q)_{\gamma_i}}\, \frac{(qu_i/tu_0;q)_{|\kappa|-r+\kappa_i}}{(u_i/u_0;q)_{|\kappa|-r+\kappa_i}}\, \frac{(u_i/tu_0;q)_{|\kappa|-r+\kappa_i}}{(qu_i/t^2u_0;q)_{|\kappa|-r+\kappa_i}} \\
& \times \prod_{i=k}^{n} \frac{(tu_i/u_0;q)_{\kappa_i}}{(qu_i/u_0;q)_{\kappa_i}}\, \frac{(qu_i/u_0;q)_{\gamma_i}}{(tu_i/u_0;q)_{\gamma_i}} \\
& \times \prod_{1\leq i<j\leq n} \frac{(q^{\kappa_i-\kappa_j}tu_i/u_j;q)_{\kappa_j-\gamma_j}}{(q^{\kappa_i-\kappa_j+1}u_i/u_j;q)_{\kappa_j-\gamma_j}}\, \frac{(q^{\gamma_i-\kappa_j+1}u_i/tu_j;q)_{\kappa_j-\gamma_j}}{(q^{\gamma_i-\kappa_j}u_i/u_j;q)_{\kappa_j-\gamma_j}}.
\end{aligned}$$

Then the infinite lower-triangular n-dimensional matrices $(f_{\beta\kappa})_{\beta,\kappa\in\mathbf{Z}^n}$ and $(g_{\kappa\gamma})_{\kappa,\gamma\in\mathbf{Z}^n}$ are mutually inverse.

Proof. Given two non-zero sequences (ξ_κ) and (ζ_κ), and a pair of matrices $(f_{\beta\kappa})$ and $(g_{\kappa\gamma})$ which are mutually inverse, it is easily checked (using the trivial relation $\frac{\xi_\beta}{\xi_\gamma}\delta_{\beta\gamma} = \delta_{\beta\gamma}$) that the matrices $(f_{\beta\kappa}\,\xi_\beta/\zeta_\kappa)$ and $(g_{\kappa\gamma}\,\zeta_\kappa/\xi_\gamma)$ are mutually inverse.

We choose

$$\xi_\kappa = \Big(\frac{q}{t}\Big)^{|\kappa|} \frac{(t^2u_0;q)_{|\kappa|}}{(qtu_0;q)_{|\kappa|}} \prod_{i=1}^{k-1} \frac{(u_i/tu_0;q)_{\kappa_i}}{(qu_i/t^2u_0;q)_{\kappa_i}} \prod_{i=k}^{n} \frac{(tu_i/u_0;q)_{\kappa_i}}{(qu_i/u_0;q)_{\kappa_i}} \times \prod_{1\le i<j\le n} \frac{(qu_i/u_j;q)_{\kappa_i-\kappa_j}}{(tu_i/u_j;q)_{\kappa_i-\kappa_j}} \frac{(u_i/u_j;q)_{\kappa_i-\kappa_j}}{(qu_i/tu_j;q)_{\kappa_i-\kappa_j}},$$

$$\zeta_\kappa = \Big(\frac{q}{t}\Big)^{|\kappa|} \frac{(t^2u_0;q)_{|\kappa|}}{(qtu_0;q)_{|\kappa|}} \prod_{i=1}^{k-1} \frac{(u_i/u_0;q)_{\kappa_i}}{(qu_i/tu_0;q)_{\kappa_i}} \frac{(qu_i/tu_0;q)_{|\kappa|-r+\kappa_i}}{(u_i/u_0;q)_{|\kappa|-r+\kappa_i}} \frac{(u_i/tu_0;q)_{|\kappa|-r+\kappa_i}}{(qu_i/t^2u_0;q)_{|\kappa|-r+\kappa_i}} \times \prod_{i=k}^{n} \frac{(tu_i/u_0;q)_{\kappa_i}}{(qu_i/u_0;q)_{\kappa_i}} \prod_{1\le i<j\le n} \frac{(qu_i/u_j;q)_{\kappa_i-\kappa_j}}{(tu_i/u_j;q)_{\kappa_i-\kappa_j}} \frac{(u_i/u_j;q)_{\kappa_i-\kappa_j}}{(qu_i/tu_j;q)_{\kappa_i-\kappa_j}},$$

together with the pair of mutually inverse matrices $(f_{\beta\kappa})$ and $(g_{\kappa\gamma})$ as defined in [3, Theorem 2.7].

Several elementary manipulations of q-shifted factorials eventually lead to the result in the desired form. To give a sample, (concentrating only on the products over $\prod_{1\le i<j\le n}$ of q-shifted factorials) we use the simplification

$$\begin{aligned}
&\prod_{1\le i<j\le n} \frac{(q^{\kappa_i-\kappa_j+1}u_i/tu_j;q)_{\beta_i-\kappa_i}}{(q^{\kappa_i-\kappa_j+1}u_i/u_j;q)_{\beta_i-\kappa_i}} \frac{(q^{\kappa_i-\beta_j}tu_i/u_j;q)_{\beta_i-\kappa_i}}{(q^{\kappa_i-\beta_j}u_i/u_j;q)_{\beta_i-\kappa_i}} \\
&\times \prod_{1\le i<j\le n} \frac{(qu_i/u_j;q)_{\beta_i-\beta_j}}{(tu_i/u_j;q)_{\beta_i-\beta_j}} \frac{(u_i/u_j;q)_{\beta_i-\beta_j}}{(qu_i/tu_j;q)_{\beta_i-\beta_j}} \frac{(tu_i/u_j;q)_{\kappa_i-\kappa_j}}{(qu_i/u_j;q)_{\kappa_i-\kappa_j}} \frac{(qu_i/tu_j;q)_{\kappa_i-\kappa_j}}{(u_i/u_j;q)_{\kappa_i-\kappa_j}} = \\
&\prod_{1\le i<j\le n} \frac{(qu_i/tu_j;q)_{\beta_i-\kappa_j}}{(qu_i/u_j;q)_{\beta_i-\kappa_j}} \frac{(u_i/u_j;q)_{\kappa_i-\beta_j}}{(tu_i/u_j;q)_{\kappa_i-\beta_j}} \frac{(qu_i/u_j;q)_{\beta_i-\beta_j}}{(qu_i/tu_j;q)_{\beta_i-\beta_j}} \frac{(tu_i/u_j;q)_{\kappa_i-\kappa_j}}{(u_i/u_j;q)_{\kappa_i-\kappa_j}} = \\
&\prod_{1\le i<j\le n} \frac{(q^{\beta_i-\beta_j+1}u_i/tu_j;q)_{\beta_j-\kappa_j}}{(q^{\beta_i-\beta_j+1}u_i/u_j;q)_{\beta_j-\kappa_j}} \frac{(q^{\kappa_i-\beta_j}tu_i/u_j;q)_{\beta_j-\kappa_j}}{(q^{\kappa_i-\beta_j}u_i/u_j;q)_{\beta_j-\kappa_j}}
\end{aligned}$$

in the computation of $f_{\beta\kappa}$ in the Lemma. □

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