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The major index generating function of standard Young tableaux of shapes of the form "staircase minus rectangle"

C. Krattenthaler and M. J. Schlosser

Abstract. A specialisation of a transformation formula for multi-dimensional elliptic hypergeometric series is used to provide compact, non-determinantal formulae for the generating function with respect to the major index of standard Young tableaux of skew shapes of the form "staircase minus rectangle".

1. Introduction

A *standard Young tableau* of skew shape λ/μ, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ are n-tuples of non-negative integers which are in non-increasing order and satisfy $\lambda_i \ge \mu_i$ for all i, is an arrangement of the numbers $1, 2, \dots, |\lambda - \mu|$ (where the last quantity denotes the sum of the respective differences of the integers, $\sum_{i=1}^{n}(\lambda_i - \mu_i)$) of the form

$$\begin{array}{ccccccccc}
 & & & & \pi_{1,\mu_1+1} & \cdots\cdots\cdots & & & \pi_{1,\lambda_1} \\
 & & \pi_{2,\mu_2+1} & \cdots & \pi_{2,\mu_1+1} & \cdots\cdots & & \pi_{2,\lambda_2} & \\
 & \iddots & & & \vdots & & \iddots & & \\
\pi_{n,\mu_n+1} & \cdots\cdots\cdots & & & & \pi_{n,\lambda_n} & & &
\end{array}$$

such that numbers along rows and columns are increasing. The *major index* of a standard Young tableau T, denoted by $\operatorname{maj}(T)$, is defined as the sum over all i such that $i+1$ appears in a lower row in T than i. It is well-known (see e.g. [**19**, Prop. 7.19.11 in combination with Theorem 7.16.1 and (7.10) with $x_i = q^{i-1}$, $i = 1, 2, \dots$, and $y_i = 0$ for $i \ge 2$]) that the generating function $\sum_T q^{\operatorname{maj}(T)}$, where the sum runs over all standard Young tableaux of shape λ/μ, equals

$$[|\lambda - \mu|]_q! \cdot \det_{1 \le i,j \le n}\left(\frac{1}{[\lambda_i - i - \mu_j + j]_q!}\right), \tag{1.1}$$

where $[m]_q! := [m]_q\,[m-1]_q \cdots [1]_q$ with $[\alpha]_q = 1 + q + q^2 + \dots + q^{\alpha-1} = \frac{1-q^\alpha}{1-q}$.

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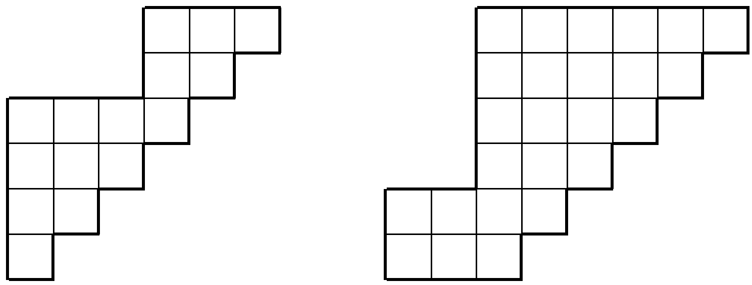

FIGURE 1. Two skew shapes of the form "staircase minus rectangle"

The purpose of this note is to provide formulae for this major index generating function for standard Young tableaux of shapes λ/μ, where λ is a staircase shape, i.e., $\lambda = (N, N-1, \dots, N-n+1)$ for some positive integers N and n, and μ is a rectangular shape, i.e., $\mu = (m, m, \dots, m, 0, \dots, 0)$, for some non-negative integer m and r repetitions of m (in the sequel we denote such partitions μ by (m^r), for short), which are (computationally) simpler than the determinantal formula (1.1). Figure 1 shows the Young diagrams of two such shapes according to standard English convention (cf. [**18**, p. 29]): the diagram on the left represents the shape $(6,5,4,3,2,1)/(3,3,0,0,0,0)$, and the diagram on the right represents the shape $(8,7,6,5,4,3)/(2,2,2,2,0,0)$. In particular, for $N = n$, the above announced formula reduces to a closed form product formula. To be precise, we show that the generating function $\sum_T q^{\mathrm{maj}(T)}$ for standard Young tableaux T of shape $(n, n-1, \dots, 1)/(m^r)$ (of which the left shape in Figure 1 is an example) equals

$$q^{\frac{1}{2}mr(r+m-2n)+\binom{n+1}{3}}(1+q)^{\binom{n}{2}-mr}\left[\tbinom{n+1}{2}-mr\right]_q! \times \prod_{i=1}^{n}\frac{[i-1]_{q^2}!}{[2i-1]_q!}\prod_{i=1}^{r}\frac{[i-1]_{q^2}!\,[n+m-r+2i-1]_q!}{[m+i-1]_{q^2}!\,[n-m-r+2i-1]_q!}, \tag{1.2}$$

a result which was originally (implicitly) obtained by DeWitt [**4**, Theorem V.3] using completely different means. She proves in fact the stronger result that a Schur s-function of a shape of the form $(n, n-1, \dots, 1)/(m^r)$ is the constant multiple of a particular Schur P-function. If this is combined with Kawanaka's product formula for the principal specialisation of Schur P-functions (see [**10**] and [**15**]), then one obtains the above formula. Furthermore, for $N = n+1$, we show that the generating function $\sum_T q^{\mathrm{maj}(T)}$ for standard Young tableaux T of shape $(n+1, n, \dots, 2)/(m^r)$ equals

$$(1+q)^{\binom{n}{2}-(m-1)r}q^{\frac{1}{2}mr(r+m-2n+2)+r(1-n-m)+\binom{n+1}{3}+\binom{n}{2}} \times \left[\tbinom{n+2}{2}-mr-1\right]_q!\prod_{i=1}^{n}\frac{[i-1]_{q^2}!}{[2i]_q!}\prod_{i=1}^{r}\frac{[i-1]_{q^2}!\,[n+m-r+2i-1]_q!}{[m+i-1]_{q^2}!\,[n-m-r+2i]_q!} \times \sum_{\ell=0}^{r}\frac{(-1)^r q^{2nl_1}}{(1-q^2)^r}\begin{bmatrix} r \\ \ell \end{bmatrix}_{q^2}\frac{(q^{-2n};q^2)_\ell\,(q^{n+m-r};q^2)_{r-\ell}\,(q^{n-m-r+1};q^2)_{r-\ell}}{(q^{n+m-r+1};q^2)_{r-\ell}}. \tag{1.3}$$

Here, the shifted q-factorials are defined by $(\alpha;q)_k := (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{k-1})$ for $k \geq 1$, and $(\alpha;q)_0 := 1$.

In general, if $N = n + s$, where s is a non-negative integer, we are able to express the major index generating function for standard Young tableaux of shape $(n+s, n+s-1, \ldots, s+1)/(m^r)$ as an $\lceil s/2 \rceil$-fold basic hypergeometric sum, see Theorem 1 in Section 2. If n is large compared to r and s, then this formula is computationally superior to the determinantal formula (1.1).

A notable feature of the proof of Theorem 1 that we give here is that we require a basic hypergeometric specialisation of a transformation formula for multi-dimensional elliptic hypergeometric series due to Rains and, independently, Coskun and Gustafson; see Section 3.

2. The main result

In this section we present our main result, a multi-dimensional basic hypergeometric series which gives the major index generating function for standard Young tableaux of a skew shape that is the difference between a staircase and a rectangle.

Theorem 1. *Let N, n, m, r be non-negative integers with $N \geq n$ and $N-r+1 \geq m$. If $N-n$ is even, the generating function $\sum_T q^{\operatorname{maj}(T)}$ for standard Young tableaux T of shape $(N, N-1, \ldots, N-n+1)/(m^r)$ equals*

$$
\begin{aligned}
(2.1)\quad & (-1)^{\binom{(N-n)/2}{2}+\frac{1}{2}r(N-n)}(1+q)^{\binom{n}{2}-\binom{(N-n)/2}{2}-mr}(1-q)^{-\binom{(N-n)/2}{2}-r(N-n)} \\
& \times q^{\frac{1}{2}mr(r+m-2n)+\frac{1}{2}r(N-n)\left(\frac{1}{2}(N-3n)-m+1\right)+\binom{n+1}{3}+(N-n)\left(\binom{n}{2}+\binom{(N-n)/2}{2}\right)} \\
& \times \frac{\left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q!}{\left[r+\frac{N-n-2}{2}\right]_{q^2}!^{(N-n)/2}\left[\frac{N+n-2}{2}\right]_{q^2}!^{(N-n)/2}} \frac{\prod_{i=1}^{(N+n)/2}[i-1]_{q^2}!}{\prod_{i=1}^{n}[N-n+2i-1]_q!} \\
& \times \prod_{i=1}^{r} \frac{\left[\frac{N-n}{2}+i-1\right]_{q^2}!\,[n+m-r+2i-1]_q!\,\left(q^{n+m-r+2i};q^2\right)_{(N-n)/2}}{[m+i-1]_{q^2}!\,[N-m-r+2i-1]_q!} \\
\times & \sum_{0\leq \ell_1<\ell_2<\cdots<\ell_{(N-n)/2}\leq r+\frac{N-n-2}{2}} q^{\sum_{i=1}^{(N-n)/2}(N+n-2(2i-1))l_i} \prod_{1\leq i<j\leq \frac{N-n}{2}} [\ell_j-\ell_i]_{q^2}^2 \\
& \cdot \prod_{i=1}^{\frac{N-n}{2}} \Bigg(\begin{bmatrix} \frac{N-n-2}{2}+r \\ \ell_i \end{bmatrix}_{q^2} \left(q^{2-N-n};q^2\right)_{\ell_i} \left(q^{n+m-r-2i+1};q^2\right)_{r+i-\ell_i-1} \\
& \qquad \cdot \frac{\left(q^{N-m-r-2i+2};q^2\right)_{r+i-\ell_i-1}}{\left(q^{N+m-r-2i+2};q^2\right)_{r+i-\ell_i-1}} \Bigg),
\end{aligned}
$$

while, if $N-n$ is odd, it equals

(2.2)
$$
\begin{aligned}
&(-1)^{\binom{(N-n+1)/2}{2}+\frac12 r(N-n+1)}(1+q)^{\binom n2-\binom{(N-n+1)/2}{2}-mr}(1-q)^{-\binom{(N-n+1)/2}{2}-r(N-n)}\\
&\times q^{\frac12 mr(r+m-2n+2)+\frac12 r(N-n+1)\left(\frac12(N-3n+1)-m\right)+\binom{n+1}{3}+(N-n)\binom n2+(N-n+1)\binom{(N-n+1)/2}{2}}\\
&\quad\times\frac{\left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q!}{\left[r+\frac{N-n-1}{2}\right]_{q^2}!^{(N-n+1)/2}\left[\frac{N+n-1}{2}\right]_{q^2}!^{(N-n+1)/2}}\frac{\prod_{i=1}^{(N+n+1)/2}[i-1]_{q^2}!}{\prod_{i=1}^{n}[N-n+2i-1]_q!}\\
&\quad\times\prod_{i=1}^{r}\frac{\left[\frac{N-n+1}{2}+i-1\right]_{q^2}!\,[n+m-r+2i-1]_q!\,\left(q^{n+m-r+2i+1};q^2\right)_{(N-n-1)/2}}{[m+i-1]_{q^2}!\,[N-m-r+2i-1]_q!}\\
&\quad\times\sum_{0\le\ell_1<\ell_2<\dots<\ell_{(N-n+1)/2}\le r+\frac{N-n-1}{2}}q^{\sum_{i=1}^{(N-n+1)/2}(N+n+1-2(2i-1))l_i}\\
&\qquad\qquad\cdot\prod_{1\le i<j\le\frac{N-n+1}{2}}[\ell_j-\ell_i]_{q^2}^2\prod_{i=1}^{\frac{N-n+1}{2}}\left(\begin{bmatrix}\frac{N-n-1}{2}+r\\ \ell_i\end{bmatrix}_{q^2}\right.\\
&\qquad\qquad\left.\cdot\frac{\left(q^{1-N-n};q^2\right)_{\ell_i}\left(q^{n+m-r-2i+2};q^2\right)_{r+i-\ell_i-1}\left(q^{N-m-r-2i+2};q^2\right)_{r+i-\ell_i-1}}{\left(q^{N+m-r-2i+2};q^2\right)_{r+i-\ell_i-1}}\right).
\end{aligned}
$$

Proof. According to Formula (1.1), the major index generating function for standard Young tableaux which we want to compute is equal to

$$
\left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q!\det_{1\le i,j\le n}\left(\begin{cases}\dfrac{1}{[N+1-2i-m+j]_q!} & j\le r\\ \dfrac{1}{[N+1-2i+j]_q!} & j>r\end{cases}\right).
$$

We now do a Laplace expansion with respect to the first r columns. In this way we obtain

$$
\begin{aligned}
&\left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q!\\
&\quad\times\sum_{1\le k_1<\dots<k_r\le n}(-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i}\det_{1\le i,j\le r}\left(\frac{1}{[N+1-2k_i-m+j]_q!}\right)\\
&\qquad\qquad\cdot\det_{\substack{1\le i\le n,\,i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{[N+1-2i+j]_q!}\right).
\end{aligned}
$$

Using the simple determinant evaluation

$$
\det_{1\le i,j\le s}\left(\frac{1}{[X_i+j]_q!}\right)=q^{2\binom{s+1}{3}+\sum_{i=1}^s(i-1)X_i}\prod_{i=1}^{s}\frac{1}{[X_i+s]_q!}\prod_{1\le i<j\le s}[X_i-X_j]
$$

(which is readily established by writing $[\alpha]_q=(1-q^\alpha)/(1-q)$ for every q-integer, factoring out common denominators of rows, and reduction to a Vandermonde determinant), the above determinants can be evaluated, so that we arrive at a multi-dimensional series of basic hypergeometric type:

$$
\begin{aligned}
(2.3)\quad & \left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q! \\
&\times \sum_{1\le k_1<\cdots<k_r\le n} (-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i} q^{2\binom{r+1}{3}+2\binom{n-r+1}{3}} \\
&\qquad\cdot q^{\sum_{i=1}^r (i-1)(N+1-2k_i-m)+\sum_{i=1}^{n-r}(i-1)(N+1+r)} \\
&\qquad\cdot q^{-2\sum_{i=1}^n (i-1)i+2\sum_{i=1}^r (k_i-1)k_i+2\sum_{i=1}^r i\left(\binom{k_{i+1}}{2}-\binom{k_i+1}{2}\right)} \\
&\qquad\cdot \prod_{i=1}^n \frac{1}{[N+n+1-2i]_q!} \prod_{i=1}^r \frac{[N+n+1-2k_i]_q!}{[N+1-2k_i-m+r]_q!} \\
&\qquad\cdot \prod_{1\le i<j\le r} \left[(N+1-2k_i-m)-(N+1-2k_j-m)\right]_q \\
&\qquad\cdot \prod_{\substack{1\le i<j\le n\\ i,j\notin\{k_1,\dots,k_r\}}} \left[(N+1-2i+r)-(N+1-2j+r)\right]_q \\
&= \left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q! \prod_{i=1}^n \frac{1}{[N+n+1-2i]_q!} \prod_{1\le i<j\le n} [2j-2i]_q \\
&\times \sum_{1\le k_1<\cdots<k_r\le n} (-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i} q^{2\binom{r+1}{3}+2\binom{n-r+1}{3}+(N+1-m)\binom{r}{2}+(N+1+r)\binom{n-r}{2}} \\
&\qquad\cdot q^{-4\binom{n+1}{3}+2r\binom{n+1}{2}+2\sum_{i=1}^r \binom{k_i}{2}-2\sum_{i=1}^r (2i-1)k_i} \\
&\cdot \prod_{i=1}^r \frac{[N+n+1-2k_i]_q!}{[N+1-2k_i-m+r]_q!} \prod_{i=1}^r \frac{1}{[2k_i-2]_q!!\,[2n-2k_i]_q!!} \prod_{1\le i<j\le r} [2k_j-2k_i]_q^2 \\
&= (-1)^{\binom{r}{2}} (1+q)^{\binom{n}{2}-(n-1)r} (1-q)^{-r(r-1)} \\
&\qquad\times q^{2\binom{r+1}{3}+2\binom{n-r+1}{3}+(N+1-m)\binom{r}{2}+(N+1+r)\binom{n-r}{2}-4\binom{n+1}{3}+2r\binom{n+1}{2}-2r^2} \\
&\qquad\times \left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q! \\
&\qquad\times \prod_{i=1}^n \frac{[i-1]_{q^2}!}{[N+n+1-2i]_q!} \prod_{i=1}^r \frac{[N+n-1]_q!}{[n-1]_{q^2}!\,[N-m+r-1]_q!} \\
&\qquad\times \sum_{0\le k_1<\cdots<k_r\le n-1} q^{-2\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \left(1-q^{-2(k_i-k_j)}\right)^2 \\
&\qquad\qquad\cdot \prod_{i=1}^r \frac{(q^{N-m+r-1};q^{-2})_{k_i}\,(q^{N-m+r-2};q^{-2})_{k_i}\,(q^{2n-2};q^{-2})_{k_i}}{(q^{N+n-1};q^{-2})_{k_i}\,(q^{N+n-2};q^{-2})_{k_i}\,(q^{-2};q^{-2})_{k_i}},
\end{aligned}
$$

where the double q-factorials are defined by $[2\alpha]_q!! = [2\alpha]_q\,[2\alpha-2]_q\cdots[2]_q$, and, by convention, $k_{r+1}=n+1$.

At this point, we should note that we may change the range of summation to

$$
(2.4)\qquad 0\le k_1<\cdots<k_r\le \left\lfloor \tfrac12(N-m+r-1)\right\rfloor .
$$

Indeed, the summand in (2.3) vanishes for $k_r > n-1$ because of the term $(q^{2n-2};q^{-2})_{k_i}$, unless this incident is "neutralised" by vanishing terms in the denominator. However, since $n \ge r$, we have

$$N - m + r - 1 \le N + n - 2, \tag{2.5}$$

except if $m = 0$ and $n = r$. In the latter case, one sees directly that, due to cancellation of terms, the extension to (2.4) does not change the sum, whereas in the former case the inequality (2.5) guarantees that the summand still vanishes for $n - 1 < k_i \le \frac{1}{2}(N - m + r - 1)$.

This last observation makes it possible to apply the transformation formula in Corollary 4 in Section 3. In order to do so, we have to distinguish between $N - n$ being even or odd. If $N - n$ is even, then we have

$$\frac{(q^{2n-2};q^{-2})_{k_i}}{(q^{N+n-2};q^{-2})_{k_i}} = \frac{\left(q^{N+n-2-2k_i};q^{-2}\right)_{(N-n)/2}}{(q^{N+n-2};q^{-2})_{(N-n)/2}}.$$

We then apply (3.5) with q replaced by q^{-2}, $s = \frac{N-n}{2}$, $b = q^{N-m+r-1}$, $c = q^{N-m+r-2}$, $d = q^{N+n-2}$, $f = q^{N+n-1}$ to the sum in (2.3). After considerable (but routine) manipulation of the arising expression, the result turns out to equal (2.1).

If $N - n$ is odd, then we have

$$\frac{(q^{2n-2};q^{-2})_{k_i}}{(q^{N+n-1};q^{-2})_{k_i}} = \frac{\left(q^{N+n-1-2k_i};q^{-2}\right)_{(N-n+1)/2}}{(q^{N+n-1};q^{-2})_{(N-n+1)/2}}.$$

Here, we apply (3.5) with q replaced by q^{-2}, $s = \frac{N-n+1}{2}$, $b = q^{N-m+r-1}$, $c = q^{N-m+r-2}$, $d = q^{N+n-1}$, $f = q^{N+n-2}$ to the sum in (2.3). Once again, after lengthy, but routine, manipulation of the arising expression, we arrive at (2.2). □

3. A transformation formula for elliptic hypergeometric series and its consequences

In this section we derive the transformation formula for multi-dimensional basic hypergeometric series which is used crucially in the proof of Theorem 1 in the previous section. It arises by specialising and taking appropriate limits in a transformation formula for multi-dimensional elliptic hypergeometric series. In order to state this formula, we first need to introduce "elliptic notation." Given a complex number p with $|p| < 1$, we define a (rescaled) theta function $\theta(x;p)$ by

$$\theta(x;p) = \prod_{j=0}^{\infty}(1 - p^j x)(1 - p^{j+1}/x).$$

Furthermore, out of these "bricks," we build elliptic shifted factorials. Namely, fixing another complex parameter, q say, and a non-negative integer m, we set

$$(a;q,p)_m = \theta(a;p)\,\theta(aq;p)\cdots\theta(aq^{m-1};p),$$

where the right-hand side is understood as 1 if $m = 0$. For convenience, we also employ the short notation

$$(a_1, a_2, \dots, a_k; q, p)_m = (a_1;q,p)_m\,(a_2;q,p)_m \cdots (a_k;q,p)_m.$$

The following result is a special case of a multi-dimensional ${}_{12}V_{11}$ transformation formula conjectured by Warnaar (let $x = q$ in [**20**, Conj. 6.1]), which has subsequently been proven (in more generality) by Rains [**13**, Theorem 4.9] and, independently, by Coskun and Gustafson [**3**]. In the present form the identity has been stated in [**16**, Theorem 3.2] (with a small typo corrected).

THEOREM 2. *Let a, b, c, d, e, f be indeterminates, let m be a nonnegative integer, and $r \geq 1$. Then we have*

$$\begin{aligned}
&\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\, \theta(aq^{k_i+k_j};p)^2 \\
&\quad\times \prod_{i=1}^r \frac{\theta(aq^{2k_i};p)(a,b,c,d,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{\theta(a;p)(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{r-1-m}/\lambda,aq^{1+m};q,p)_{k_i}} \\
&= \prod_{i=1}^r \frac{(b,c,d,ef/a;q,p)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q,p)_{i-1}} \\
&\qquad\times \prod_{i=1}^r \frac{(aq;q,p)_m\,(aq/ef;q,p)_{m+1-r}\,(\lambda q/e,\lambda q/f;q,p)_{m-i+1}}{(\lambda q;q,p)_m\,(\lambda q/ef;q,p)_{m+1-r}\,(aq/e,aq/f;q,p)_{m-i+1}} \\
&\quad\times \sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\, \theta(\lambda q^{k_i+k_j};p)^2 \\
&\quad\times \prod_{i=1}^r \frac{\theta(\lambda q^{2k_i};p)\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{\theta(\lambda;p)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{r-1-m}/a,\lambda q^{1+m};q,p)_{k_i}},
\end{aligned} \tag{3.1}$$

where $\lambda = a^2q^{2-r}/bcd$.

We now show that, by suitable specialisation, the above elliptic hypergeometric transformation formula reduces to the following transformation formula for multidimensional basic hypergeometric series of different dimensions.

COROLLARY 3. *For all non-negative integers m, r and s, we have*

$$\begin{aligned}
&\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2 \\
&\qquad\qquad\cdot \prod_{i=1}^r \frac{(dq^{k_i};q)_s\,(b;q)_{k_i}\,(q^{-m};q)_{k_i}}{(q;q)_{k_i}\,(f;q)_{k_i}} \\
&= \frac{q^{\binom{r+s}{3}+\binom{r+1}{3}+s\binom{r}{2}-m\binom{r+s}{2}}}{f^{\binom{r}{2}}\,(q;q)_{r+s-1}^{s-1}} \prod_{i=1}^r \frac{(b;q)_{i-1}\,(bq^{s+r+i-m-1}/f;q)_{m-r+1}}{(q^{i-m}/f;q)_{m-i+1}} \\
&\quad\times \prod_{i=1}^{r+s-1} \frac{(q;q)_{i-1}\,(q;q)_m}{(q;q)_{m-i}} \prod_{i=r}^{r+s-1} \frac{(dq^{1-r}/b;q)_i}{(q;q)_{r+s-i-1}\,(d;q)_{i-r}\,(fq^{1-r-s}/b;q)_i} \\
&\qquad\times \sum_{0\le \ell_1<\ell_2<\cdots<\ell_s\le r+s-1} q^{\sum_{i=1}^s (2i-1)\ell_i} \prod_{1\le i<j\le s} (1-q^{\ell_i-\ell_j})^2 \\
&\qquad\qquad\cdot \prod_{i=1}^s \frac{(d;q)_{\ell_i}\,(fq^{1-r-s}/b;q)_{\ell_i}\,(q^{1-r-s};q)_{\ell_i}}{(q;q)_{\ell_i}\,(dq^{1-r}/b;q)_{\ell_i}\,(q^{-m};q)_{\ell_i}}.
\end{aligned} \tag{3.2}$$

PROOF. In (3.1), we let $p=0$, $d\to aq/d$, $f\to aq/f$, and then $a\to 0$. After having performed these substitutions and limits, we arrive at

$$\sum_{0\le k_1<k_2<\dots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2 \cdot \prod_{i=1}^r \frac{(b;q)_{k_i}\,(c;q)_{k_i}\,(e;q)_{k_i}\,(q^{-m};q)_{k_i}}{(q;q)_{k_i}\,(d;q)_{k_i}\,(f;q)_{k_i}\,(bceq^{2r-m-1}/df)_{k_i}}$$
$$= \prod_{i=1}^r \frac{(b;q)_{i-1}\,(c;q)_{i-1}\,(eq/f;q)_{i-1}}{(dq^{1-r}/c;q)_{i-1}\,(dq^{1-r}/b;q)_{i-1}\,(bceq^r/df;q)_{i-1}} \times \prod_{i=1}^r \frac{(f/e;q)_{m+1-r}\,(dfq^{1-r}/bc;q)_{m-i+1}}{(dfq^{1-r}/bce;q)_{m+1-r}\,(f;q)_{m-i+1}}$$
$$\times \sum_{0\le k_1<k_2<\dots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2 \cdot \prod_{i=1}^r \frac{(dq^{1-r}/c;q)_{k_i}\,(dq^{1-r}/b;q)_{k_i}\,(e;q)_{k_i}\,(q^{-m};q)_{k_i}}{(q;q)_{k_i}\,(d;q)_{k_i}\,(dfq^{1-r}/bc)_{k_i}\,(eq^{r-m}/f)_{k_i}}. \tag{3.3}$$

In this multi-dimensional Whipple transformation we let $e\to\infty$. This yields the identity

$$\sum_{0\le k_1<k_2<\dots<k_r\le m} q^{\sum_{i=1}^r (m-2r+2i)k_i} \left(\frac{df}{bc}\right)^{\sum_{i=1}^r k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2 \cdot \prod_{i=1}^r \frac{(b;q)_{k_i}\,(c;q)_{k_i}\,(q^{-m};q)_{k_i}}{(q;q)_{k_i}\,(d;q)_{k_i}\,(f;q)_{k_i}}$$
$$= \left(\frac{dq^{1-r}}{bc}\right)^{\binom r2} \prod_{i=1}^r \frac{(b;q)_{i-1}\,(c;q)_{i-1}}{(dq^{1-r}/c;q)_{i-1}\,(dq^{1-r}/b;q)_{i-1}} \cdot \frac{(dfq^{1-r}/bc;q)_{m-i+1}}{(f;q)_{m-i+1}}$$
$$\times \sum_{0\le k_1<k_2<\dots<k_r\le m} q^{\sum_{i=1}^r (m-r+2i-1)k_i} f^{\sum_{i=1}^r k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2 \prod_{i=1}^r \frac{(dq^{1-r}/c;q)_{k_i}\,(dq^{1-r}/b;q)_{k_i}\,(q^{-m};q)_{k_i}}{(q;q)_{k_i}\,(d;q)_{k_i}\,(dfq^{1-r}/bc;q)_{k_i}}.$$

This identity becomes more elegant if we replace q by $1/q$, b by $1/b$, c by $1/c$, d by $1/d$, and f by $1/f$, namely

(3.4)
$$\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r(2i-1)k_i}\prod_{1\le i<j\le r}(1-q^{k_i-k_j})^2\prod_{i=1}^r\frac{(b;q)_{k_i}\,(c;q)_{k_i}\,(q^{-m};q)_{k_i}}{(q;q)_{k_i}\,(d;q)_{k_i}\,(f;q)_{k_i}}$$
$$=\left(\frac{bc}{dq^{1-r}}\right)^{r(m-r+1)}\prod_{i=1}^r\frac{(b;q)_{i-1}\,(c;q)_{i-1}}{(dq^{1-r}/c;q)_{i-1}\,(dq^{1-r}/b;q)_{i-1}}\cdot\frac{(dfq^{1-r}/bc;q)_{m-i+1}}{(f;q)_{m-i+1}}$$
$$\times\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r(2i-1)k_i}\prod_{1\le i<j\le r}(1-q^{k_i-k_j})^2$$
$$\cdot\prod_{i=1}^r\frac{(dq^{1-r}/c;q)_{k_i}\,(dq^{1-r}/b;q)_{k_i}\,(q^{-m};q)_{k_i}}{(q;q)_{k_i}\,(d;q)_{k_i}\,(dfq^{1-r}/bc;q)_{k_i}}.$$

Now we let $c=dq^s$ in this transformation formula, where s is a non-negative integer. Due to the term $(q^{-r-s+1};q)_{k_i}$, on the right-hand side we are now summing over all $k_1,k_2,\dots,k_r$ with $0\le k_1<k_2<\cdots<k_r\le r+s-1$. Let

$$\{\ell_1,\ell_2,\dots,\ell_s\}=\{0,1,\dots,r+s-1\}\backslash\{k_1,k_2,\dots,k_r\},$$

with $\ell_1<\ell_2<\cdots<\ell_s$. After some simplification, we obtain (3.2). $\square$

Remarks. (1) The multiple Whipple transformation (3.3) also follows from letting $t=q^2$ in [**2**, Eq. (5.10)], followed by the principal specialisation formula for the Macdonald polynomial $P_\lambda(q,t)$.

(2) The transformation formula (3.2) is one of the rare examples of a transformation formula between multi-dimensional (basic) hypergeometric series of different dimension. Other examples that we are aware of are [**6**, Sec. 8], [**7**], [**8**], [**9**, Thm. 2.2], [**12**, Conjecture in Sec. 1], [**14**, Thm. 3.1], and [**21**, Thm. 4.1].

For our purpose, we shall have need of a more flexible statement of the transformation formula in Corollary 3. More precisely, in our application in the proof of Theorem 1, we have a pair $(q^M;q^{-2})_{k_i}\,(q^{M+1};q^{-2})_{k_i}$ of Pochhammer symbols in the numerator of our multi-dimensional sum (see (2.3)), and consequently we would like a statement of (3.2) in which the roles of $(b;q)_{k_i}$ and $(q^{-m};q)_{k_i}$ are interchangeable. Indeed, by a standard polynomial argument (see e.g. [**1**, Sec. 5.1] or [**17**, Sec. 2.3.4]), one is able to show the following "symmetric" form of Corollary 3.

Corollary 4. *Let one of b and c be of the form q^{-m}, where m is a non-negative integer. Then we have*

(3.5)
$$\sum_{0\le k_1<k_2<\cdots<k_r} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2 \prod_{i=1}^r \frac{(dq^{k_i};q)_s\,(b;q)_{k_i}\,(c;q)_{k_i}}{(q;q)_{k_i}\,(f;q)_{k_i}}$$
$$= (-1)^{\binom{r+s}{2}} \frac{q^{\binom{r+1}{3}+s\binom{r}{2}}}{f^{\binom{r}{2}}\,(q;q)_{r+s-1}^{s-1}} \prod_{i=1}^r \frac{(b;q)_{i-1}\,(q/f;q)_\infty\,(bcq^{s+r+i-1}/f;q)_\infty}{(cq^i/f;q)_\infty\,(bq^{s+i}/f;q)_\infty}$$
$$\times \prod_{i=1}^{r+s-1} (q;q)_{i-1}\,(c;q)_i \prod_{i=r}^{r+s-1} \frac{(dq^{1-r}/b;q)_i}{(q;q)_{r+s-i-1}\,(d;q)_{i-r}\,(fq^{1-r-s}/b;q)_i}$$
$$\times \sum_{0\le \ell_1<\ell_2<\cdots<\ell_s\le r+s-1} q^{\sum_{i=1}^s (2i-1)\ell_i} \prod_{1\le i<j\le s} (1-q^{\ell_i-\ell_j})^2$$
$$\cdot \prod_{i=1}^s \frac{(d;q)_{\ell_i}\,(fq^{1-r-s}/b;q)_{\ell_i}\,(q^{1-r-s};q)_{\ell_i}}{(q;q)_{\ell_i}\,(dq^{1-r}/b;q)_{\ell_i}\,(c;q)_{\ell_i}}.$$

Sketch of Proof. The assertion is obvious if $c=q^{-m}$, where m is a non-negative integer, because in that case the identity in (3.5) is directly equivalent to (3.2).

On the other hand, let us now suppose that b is of the form $b=q^{-\beta}$, for some non-negative integer β. On the right-hand side of (3.2), we write
$$\frac{(q;q)_m}{(q;q)_{m-i}} = (q^{m-i+1};q)_i = (-1)^i q^{im-\binom{i}{2}} (q^{-m};q)_i$$
and
$$\frac{(bq^{s+r+i-m-1}/f;q)_{m-r+1}}{(q^{i-m}/f;q)_{m-i+1}} = \frac{(bq^{s+r+i-m-1}/f;q)_\infty\,(q/f;q)_\infty}{(bq^{s+i}/f;q)_\infty\,(q^{i-m}/f;q)_\infty}.$$
Since we are currently assuming that $b=q^{-\beta}$ for some non-negative integer β, the above relation can be rewritten in the form
$$\frac{(bq^{s+r+i-m-1}/f;q)_{m-r+1}}{(q^{i-m}/f;q)_{m-i+1}} = \frac{(q/f)_{s+i-\beta-1}}{(q^{i-m}/f;q)_{s+r-\beta-1}}. \tag{3.6}$$
Consequently, the expression in (3.6) is rational in $c=q^{-m}$, and therefore as well the expressions on both sides of (3.5). Under a fixed choice of the parameters $r,s,b=q^{-\beta}$, one also sees that the degrees in c of numerator and denominator of these rational functions are bounded. Now comes the "polynomial argument": Identity (3.5) holds for infinitely many c's (namely for all c's of the form $c=q^{-m}$, where m is a non-negative integer), hence it must hold for arbitrary c, that is, for c considered as an indeterminate. This finishes the proof of this corollary. □

For the convenience of the reader, we state the special cases of (3.5) where $s=0$ and $s=1$ explicitly below. Namely, if we let $s=0$, then (3.5) reduces to

$$\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2 \prod_{i=1}^r \frac{(b;q)_{k_i}\,(q^{-m};q)_{k_i}}{(q;q)_{k_i}\,(f;q)_{k_i}} \tag{3.7}$$
$$= (-1)^{\binom{r}{2}} q^{\binom{r}{2}(m-r+1)} b^{r(m-r+1)} \prod_{i=1}^r \frac{(b;q)_{i-1}\,(q;q)_m\,(fq^{1-i}/b;q)_{m-r+1}\,(q;q)_{i-1}}{(f;q)_{m-i+1}\,(q;q)_{m-i+1}},$$

where m is a non-negative integer, while for $s = 1$ Identity (3.5) reduces to

(3.8)

$$\sum_{0\le k_1<k_2<\dots<k_r} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2 \prod_{i=1}^r \frac{(1-dq^{k_i})\,(b;q)_{k_i}\,(c;q)_{k_i}}{(q;q)_{k_i}\,(f;q)_{k_i}}$$
$$= (-1)^{\binom{r+1}{2}} \frac{q^{\binom{r+1}{3}+\binom{r}{2}}}{f^{\binom{r}{2}}} \prod_{i=1}^r \frac{(q;q)_{i-1}\,(b;q)_{i-1}\,(c;q)_i\,(q/f;q)_\infty\,(bcq^{r+i}/f;q)_\infty}{(cq^i/f;q)_\infty\,(bq^{i+1}/f;q)_\infty}$$
$$\times \sum_{\ell=0}^r q^\ell \frac{(d;q)_\ell\,(dq^{1-r+\ell}/b;q)_{r-\ell}\,(q^{-r};q)_\ell}{(q;q)_\ell\,(fq^{-r+\ell}/b;q)_{r-\ell}\,(c;q)_\ell},$$

where one of b or c is of the form q^{-m}, where m is a non-negative integer. We point out that (3.7) has been explicitly stated earlier in [**11**, Theorem 6], where two different proofs were provided: one proceeded by showing that this formula comes from the principal specialisation of the obvious expansion of a rectangular Schur function in two sets of variables, while the other derived it by specialising a multi-dimensional q-beta integral formula due to Evans and Kadell.

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Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

URL: `http://www.mat.univie.ac.at/~kratt`

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

URL: `http://www.mat.univie.ac.at/~schlosse`