SOME q-SUPERCONGRUENCES MODULO THE SQUARE AND CUBE OF A CYCLOTOMIC POLYNOMIAL

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Abstract. Two q-supercongruences of truncated basic hypergeometric series containing two free parameters are established by employing specific identities for basic hypergeometric series. The results partly extend two q-supercongruences that were earlier conjectured by the same authors and involve q-supercongruences modulo the square and the cube of a cyclotomic polynomial. One of the newly proved q-supercongruences is even conjectured to hold modulo the fourth power of a cyclotomic polynomial.

1. Introduction

In 1914, Ramanujan [25] listed a number of representations of $1/\pi$, including

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}=\frac{4}{\pi}, \tag{1.1}$$

where $(a)_n=a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. Ramanujan's formulas gained unprecedented popularity in the 1980's when they were discovered to provide fast algorithms for calculating decimal digits of π. See, for instance, the monograph [2] by the Borwein brothers.

In 1997, Van Hamme [29] conjectured 13 intriguing p-adic analogues of Ramanujan-type formulas, such as

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}\equiv p(-1)^{(p-1)/2} \pmod{p^4}, \tag{1.2}$$

where $p>3$ is a prime. Van Hamme himself supplied proofs for three of them. Supercongruences like (1.2) are called Ramanujan-type supercongruences (see [33]). The proof of the supercongruence (1.2) was first given by Long [22]. As of today, all of Van Hamme's 13 supercongruences have been confirmed by various techniques (see [24, 28]).

In recent years, q-congruences and q-supercongruences have been established by different authors (see, for example, [5–13, 15–21, 23, 27, 30–32, 34]). In particular, the present

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authors [9] proved that, for any odd integer $d \geqslant 5$,

$$\sum_{k=0}^{n-1}[2dk+1]\frac{(q;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-3)k/2} \equiv \begin{cases} 0 \pmod{\Phi_n(q)^2}, & \text{if } n\equiv -1 \pmod{d}, \\ 0 \pmod{\Phi_n(q)^3}, & \text{if } n\equiv -1/2 \pmod{d}. \end{cases} \tag{1.3}$$

Here and in what follows, we adopt the standard q-notation: $[n] = 1 + q + \cdots + q^{n-1}$ is the q-*integer*; $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ is the q-*shifted factorial*, with the compact notation $(a_1, a_2, \dots, a_m; q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$ used for their products; and $\Phi_n(q)$ denotes the n-th *cyclotomic polynomial* in q, which may be defined as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

We should point out that the q-congruence (1.3) does not hold for $d = 3$. The present authors [9] also established the following companion of (1.3): for any odd integer $d \geqslant 3$ and integer $n > 1$,

$$\sum_{k=0}^{n-1}[2dk-1]\frac{(q^{-1};q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-1)k/2} \equiv \begin{cases} 0 \pmod{\Phi_n(q)^2}, & \text{if } n\equiv 1 \pmod{d}, \\ 0 \pmod{\Phi_n(q)^3}, & \text{if } n\equiv 1/2 \pmod{d}. \end{cases} \tag{1.4}$$

They also proposed the following conjectures [9, Conjectures 1 and 2], which are generalizations of (1.3) and (1.4).

Conjecture 1. *Let $d \geqslant 5$ be an odd integer. Then*

$$\sum_{k=0}^{n-1}[2dk+1]\frac{(q;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-3)k/2} \equiv \begin{cases} 0 \pmod{\Phi_n(q)^3}, & \text{if } n\equiv -1 \pmod{d}, \\ 0 \pmod{\Phi_n(q)^4}, & \text{if } n\equiv -1/2 \pmod{d}. \end{cases}$$

Conjecture 2. *Let $d \geqslant 5$ be an odd integer and let $n > 1$. Then*

$$\sum_{k=0}^{n-1}[2dk-1]\frac{(q^{-1};q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-1)k/2} \equiv \begin{cases} 0 \pmod{\Phi_n(q)^3}, & \text{if } n\equiv 1 \pmod{d}, \\ 0 \pmod{\Phi_n(q)^4}, & \text{if } n\equiv 1/2 \pmod{d}. \end{cases}$$

q-Supercongruences such as those above (modulo a third and even fourth power of a cyclotomic polynomial) are rather special. In fact, concrete results for truncated basic hypergeometric sums being congruent to 0 modulo a high power of a cyclotomic polynomial are very rare. See [8, 10–12, 14, 18] for recent papers featuring such results. The main goal of this paper is to add two complete two-parameter families of q-supercongruences to the list of such q-supercongruences (see Theorems 1 and 2).

We shall prove that the respective first cases of Conjectures 1 and 2 are true by establishing the following more general result.

Theorem 1. *Let d and r be odd integers satisfying $d \geqslant 3$, $r \leqslant d-4$ (in particular, r may be negative) and $\gcd(d,r)=1$. Let n be an integer such that $n \geqslant d-r$ and $n \equiv -r \pmod{d}$. Then*

$$\sum_{k=0}^{M}[2dk+r]\frac{(q^r;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-r-2)k/2} \equiv 0 \pmod{[n]\Phi_n(q)^2}, \tag{1.5}$$

where $M=(dn-n-r)/d$ or $n-1$.

We shall also prove the following q-supercongruences.

Theorem 2. *Let d and r be odd integers satisfying $d \geqslant 3$, $r \leqslant d-4$ (in particular, r may be negative) and $\gcd(d,r)=1$. Let n be an integer such that $n \geqslant (d-r)/2$ and $n \equiv -r/2 \pmod{d}$. Then*

$$\sum_{k=0}^{M}[2dk+r]\frac{(q^r;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-r-2)k/2} \equiv 0 \pmod{[n]\Phi_n(q)}, \tag{1.6}$$

where $M=(dn-2n-r)/d$ or $n-1$.

The following generalization of the respective second cases of Conjectures 1 and 2 should be true.

Conjecture 3. *The q-supercongruence (1.6) holds modulo $[n]\Phi_n(q)^3$ for $d \geqslant 5$.*

We shall prove Theorems 1 and 2 in Sections 2 and 3, respectively, by making use of Andrews' multiseries extension (2.2) of the Watson transformation [1, Theorem 4], along with Gasper's very-well-poised Karlsson–Minton type summation [3, Eq. (5.13)]. It should be pointed out that Andrews' transformation plays an important part in combinatorics and number theory (see [7] and the introduction of [12] for more such examples).

2. Proof of Theorem 1

We need a simple q-congruence modulo $\Phi_n(q)^2$, which was already used in [10, 12].

Lemma 1. *Let α, r be integers and n a positive integer. Then*

$$(q^{r-\alpha n},q^{r+\alpha n};q^d)_k \equiv (q^r;q^d)_k^2 \pmod{\Phi_n(q)^2}. \tag{2.1}$$

We will further utilize a powerful transformation formula due to Andrews [1, Theorem 4], which may be stated as follows:

$$\sum_{k\geqslant 0}\frac{(a,q\sqrt{a},-q\sqrt{a},b_1,c_1,\dots,b_m,c_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/b_1,aq/c_1,\dots,aq/b_m,aq/c_m,aq^{N+1};q)_k}\left(\frac{a^mq^{m+N}}{b_1c_1\cdots b_mc_m}\right)^k$$
$$=\frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N}\sum_{j_1,\dots,j_{m-1}\geqslant 0}\frac{(aq/b_1c_1;q)_{j_1}\cdots(aq/b_{m-1}c_{m-1};q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}}$$
$$\times\frac{(b_2,c_2;q)_{j_1}\dots(b_m,c_m;q)_{j_1+\dots+j_{m-1}}}{(aq/b_1,aq/c_1;q)_{j_1}\dots(aq/b_{m-1},aq/c_{m-1};q)_{j_1+\dots+j_{m-1}}}$$
$$\times\frac{(q^{-N};q)_{j_1+\dots+j_{m-1}}}{(b_mc_mq^{-N}/a;q)_{j_1+\dots+j_{m-1}}}\frac{(aq)^{j_{m-2}+\dots+(m-2)j_1}q^{j_1+\dots+j_{m-1}}}{(b_2c_2)^{j_1}\cdots(b_{m-1}c_{m-1})^{j_1+\dots+j_{m-2}}}. \tag{2.2}$$

This transformation is a multiseries generalization of Watson's ${}_8\phi_7$ transformation formula (listed in [4, Appendix (III.18)]; cf. [4, Chapter 1] for the notation of a basic hypergeometric ${}_r\phi_s$ series we are using),

$${}_8\phi_7\left[\begin{matrix}a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n}\\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix};q,\ \frac{a^2q^{n+2}}{bcde}\right]$$
$$=\frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix}aq/bc,\ d,\ e,\ q^{-n}\\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix};q,\ q\right], \tag{2.3}$$

to which it reduces for $m=2$.

Next, we require a very-well-poised Karlsson–Minton type summation due to Gasper [3, Eq. (5.13)] (see also [4, Ex. 2.33 (i)]):

$$\sum_{k=0}^{\infty}\frac{(a,q\sqrt{a},-q\sqrt{a},b,a/b,d,e_1,aq^{n_1+1}/e_1,\dots,e_m,aq^{n_m+1}/e_m;q)_k}{(q,\sqrt{a},-\sqrt{a},aq/b,bq,aq/d,aq/e_1,e_1q^{-n_1},\dots,aq/e_m,e_mq^{-n_m};q)_k}\left(\frac{q^{1-\nu}}{d}\right)^k$$
$$=\frac{(q,aq,aq/bd,bq/d;q)_\infty}{(bq,aq/b,aq/d,q/d;q)_\infty}\prod_{j=1}^{m}\frac{(aq/be_j,bq/e_j;q)_{n_j}}{(aq/e_j,q/e_j;q)_{n_j}}, \tag{2.4}$$

where $n_1,\dots,n_m$ are non-negative integers, $\nu=n_1+\dots+n_m$, and the convergence condition $|q^{1-\nu}/d|<1$ if the series does not terminate. We point out that an elliptic extension of the terminating $d=q^{-\nu}$ case of (2.4) can be found in [26, Eq. (1.7)].

In particular, we note that for $d=bq$ the right-hand side of (2.4) vanishes. Putting in addition $b=q^{-N}$ we get the following terminating summation formula:

$$\sum_{k=0}^{N}\frac{(a,q\sqrt{a},-q\sqrt{a},e_1,aq^{n_1+1}/e_1,\dots,e_m,aq^{n_m+1}/e_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/e_1,e_1q^{-n_1},\dots,aq/e_m,e_mq^{-n_m},aq^{N+1};q)_k}q^{(N-\nu)k}=0, \tag{2.5}$$

which is valid for $N>\nu=n_1+\dots+n_m$.

A suitable combination of (2.2) and (2.5) yields the following multi-series summation formula, derived in [12, Lemma 2] (whose proof we nevertheless give here, to make the paper self-contained):

Lemma 2. *Let $m \geqslant 2$. Let q, a and $e_1, \dots, e_{m+1}$ be arbitrary parameters with $e_{m+1} = e_1$, and let $n_1, \dots, n_m$ and N be non-negative integers such that $N > n_1 + \cdots + n_m$. Then*

$$0 = \sum_{j_1,\dots,j_{m-1}\geqslant 0} \frac{(e_1 q^{-n_1}/e_2;q)_{j_1} \cdots (e_{m-1}q^{-n_{m-1}}/e_m;q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}}$$
$$\times \frac{(aq^{n_2+1}/e_2, e_3;q)_{j_1} \dots (aq^{n_m+1}/e_m, e_{m+1};q)_{j_1+\cdots+j_{m-1}}}{(e_1 q^{-n_1}, aq/e_2;q)_{j_1} \dots (e_{m-1}q^{-n_{m-1}}, aq/e_m;q)_{j_1+\cdots+j_{m-1}}}$$
$$\times \frac{(q^{-N};q)_{j_1+\cdots+j_{m-1}}}{(e_1 q^{n_m-N+1}/e_m;q)_{j_1+\cdots+j_{m-1}}} \frac{(aq)^{j_{m-2}+\cdots+(m-2)j_1} q^{j_1+\cdots+j_{m-1}}}{(aq^{n_2+1}e_3/e_2)^{j_1}\cdots(aq^{n_{m-1}+1}e_m/e_{m-1})^{j_1+\cdots+j_{m-2}}}. \tag{2.6}$$

Proof. By specializing the parameters in the multi-sum transformation (2.2) by $b_i \mapsto aq^{n_i+1}/e_i$, $c_i \mapsto e_{i+1}$, for $1 \le i \le m$ (where $e_{m+1} = e_1$), and dividing both sides of the identity by the prefactor of the multi-sum, we obtain that the series on the right-hand side of (2.6) equals

$$\frac{(e_m q^{-n_m}, aq/e_1;q)_N}{(aq, e_m q^{-n_m}/e_1;q)_N}$$
$$\times \sum_{k=0}^{N} \frac{(a, q\sqrt{a}, -q\sqrt{a}, e_1, aq^{n_1+1}/e_1, \dots, e_m, aq^{n_m+1}/e_m, q^{-N};q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/e_1, e_1 q^{-n_1}, \dots, aq/e_m, e_m q^{-n_m}, aq^{N+1};q)_k} q^{(N-\nu)k},$$

with $\nu = n_1 + \cdots + n_m$. Now the last sum vanishes by the special case of Gasper's summation stated in (2.5). □

Using [11, Lemma 2.1], we can prove the following result which is similar to [11, Lemma 2.2].

Lemma 3. *Let d, n be positive integers with $\gcd(d,n) = 1$. Let r be an integer. Then*

$$\sum_{k=0}^{m} [2dk+r] \frac{(q^r;q^d)_k^d}{(q^d;q^d)_k^d} q^{d(d-r-2)k/2} \equiv 0 \pmod{[n]},$$

$$\sum_{k=0}^{n-1} [2dk+r] \frac{(q^r;q^d)_k^d}{(q^d;q^d)_k^d} q^{d(d-r-2)k/2} \equiv 0 \pmod{[n]},$$

where $0 \leqslant m \leqslant n-1$ and $dm \equiv -r \pmod{n}$.

We have collected enough ingredients which enables us to prove Theorem 1.

Proof of Theorem 1. The q-congruence (1.5) modulo $[n]$ follows from Lemma 3 immediately. In what follows, we shall prove the modulus $\Phi_n(q)^3$ case of (1.5).

For $M=(dn-n-r)/d$, the left-hand side of (1.5) can be written as the following multiple of a terminating ${}_{d+5}\phi_{d+4}$ series:

$$[r]\sum_{k=0}^{(dn-n-r)/d}\frac{(q^r,q^{d+r/2},-q^{d+r/2},q^r,\dots,q^r,q^{(d+r)/2},q^{d+(d-1)n},q^{r-(d-1)n};q^d)_k}{(q^d,q^{r/2},-q^{r/2},q^d,\dots,q^d,q^{(d+r)/2},q^{r-(d-1)n},q^{d+(d-1)n};q^d)_k}q^{d(d-r-2)k/2}.$$

Here, the $q^r,\dots,q^r$ in the numerator means $d-1$ instances of q^r, and similarly, the $q^d,\dots,q^d$ in the denominator means $d-1$ instances of q^d. By Andrews' transformation (2.2), we may rewrite the above expression as

$$[r]\frac{(q^{d+r},q^{(r-d)/2-(d-1)n};q^d)_{(dn-n-r)/d}}{(q^{(d+r)/2},q^{r-(d-1)n};q^d)_{(dn-n-r)/d}}\sum_{j_1,\dots,j_{m-1}\geqslant 0}\frac{(q^{d-r};q^d)_{j_1}\cdots(q^{d-r};q^d)_{j_{m-1}}}{(q^d;q^d)_{j_1}\cdots(q^d;q^d)_{j_{m-1}}}$$
$$\times\frac{(q^r,q^r;q^d)_{j_1}\dots(q^r,q^r;q^d)_{j_1+\dots+j_{m-2}}(q^{(d+r)/2},q^{d+(d-1)n};q^d)_{j_1+\dots+j_{m-1}}}{(q^d,q^d;q^d)_{j_1}\dots(q^d,q^d;q^d)_{j_1+\dots+j_{m-1}}}$$
$$\times\frac{(q^{r-(d-1)n};q^d)_{j_1+\dots+j_{m-1}}}{(q^{(3d+r)/2};q^d)_{j_1+\dots+j_{m-1}}}q^{(d-r)(j_{m-2}+\dots+(m-2)j_1)+d(j_1+\dots+j_{m-1})},\tag{2.7}$$

where $m=(d+1)/2$.

It is easy to see that the q-shifted factorial $(q^{d+r};q^d)_{(dn-n-r)/d}$ contains the factor $1-q^{(d-1)n}$ which is a multiple of $1-q^n$. Moreover, since none of $(r-d)/2$, $(d+r)/2$ and $(d+r)/2+dn-n-r-d$ are multiples of n, the q-shifted factorials

$$(q^{(r-d)/2-(d-1)n};q^d)_{(dn-n-r)/d}\quad\text{and}\quad(q^{(d+r)/2};q^d)_{(dn-n-r)/d}$$

have the same number (0 or 1) of factors of the form $1-q^{\alpha n}$ ($\alpha\in\mathbb{Z}$). Besides, the q-shifted factorial $(q^{r-(d-1)n};q^d)_{(dn-n-r)/d}$ is relatively prime to $\Phi_n(q)$. Thus we conclude that the fraction before the multi-sum in (2.7) is congruent to 0 modulo $\Phi_n(q)$.

Note that the non-zero terms in the multi-summation in (2.7) are those indexed by $(j_1,\dots,j_{m-1})$ that satisfy the inequality $j_1+\dots+j_{m-1}\leqslant(dn-n-r)/d$ because the factor $(q^{r-(d-1)n};q^d)_{j_1+\dots+j_{m-1}}$ appears in the numerator. None of the factors appearing in the denominator of the multi-sum of (2.7) contain a factor of the form $1-q^{\alpha n}$ (and are therefore relatively prime to $\Phi_n(q)$), except for $(q^{(3d+r)/2};q^d)_{j_1+\dots+j_{m-1}}$ when

$$(dn-d-n-r)/(2d)\leqslant j_1+\dots+j_{m-1}\leqslant(dn-n-r)/d.$$

Since

$$\frac{(q^{(d+r)/2};q^d)_{j_1+\dots+j_{m-1}}}{(q^{(3d+r)/2};q^d)_{j_1+\dots+j_{m-1}}}=\frac{1-q^{(d+r)/2}}{1-q^{(d+r)/2+(j_1+\dots+j_{m-1})d}},$$

the denominator of the above fraction contains a factor of the form $1-q^{\alpha n}$ if and only if $j_1+\dots+j_{m-1}=(dn-d-n-r)/(2d)$ (in this case, the denominator contains the factor $1-q^{(d-1)n/2}$). Writing $n=ad-r$ (with $a\geqslant 1$), we have $j_1+\dots+j_{m-1}=a(d-1)/2-(r+1)/2$. Noticing that $m-1=(d-1)/2$ and $r\leqslant d-4$, there must exist an i such that $j_i\geqslant a$. Then $(q^{d-r};q^d)_{j_i}$ has the factor $1-q^{d-r+d(a-1)}=1-q^n$ which is divisible by $\Phi_n(q)$. Hence the denominator of the reduced form of the multi-sum in (2.7) is relatively prime to $\Phi_n(q)$. It remains to show that the multi-sum in (2.7), without the previous fraction, is congruent to 0 modulo $\Phi_n(q)^2$.

By repeated applications of Lemma 1, the multi-sum in (2.7) (without the previous fraction), modulo $\Phi_n(q)^2$, is congruent to

$$\sum_{j_1,\dots,j_{m-1}\geqslant 0} q^{(d-r)(j_{m-2}+\cdots+(m-2)j_1)+d(j_1+\cdots+j_{m-1})}\frac{(q^{d-r};q^d)_{j_1}\cdots(q^{d-r};q^d)_{j_{m-1}}}{(q^d;q^d)_{j_1}\cdots(q^d;q^d)_{j_{m-1}}}$$
$$\times\frac{(q^{r+(m+1)n},q^{r-(m+1)n};q^d)_{j_1}\dots(q^{r+(2m-2)n},q^{r-(2m-2)n};q^d)_{j_1+\cdots+j_{m-2}}}{(q^{d-mn},q^{d+mn};q^d)_{j_1}\dots(q^{d-(2m-3)n},q^{d+(2m-3)n};q^d)_{j_1+\cdots+j_{m-2}}}$$
$$\times\frac{(q^{d+(d-1)n},q^{(d+r)/2};q^d)_{j_1+\cdots+j_{m-1}}(q^{r-(d-1)n};q^d)_{j_1+\cdots+j_{m-1}}}{(q^{d-(2m-2)n},q^{d+(2m-2)n};q^d)_{j_1+\cdots+j_{m-1}}(q^{(3d+r)/2};q^d)_{j_1+\cdots+j_{m-1}}},$$

where $m=(d+1)/2$. However, this sum vanishes in light of the $m=(d+1)/2$, $q\mapsto q^d$, $a=q^r$, $e_1=q^{(d+r)/2}$, $e_m=q^{r-(2m-2)n}$, $e_i=q^{r-(m+i-2)n}$, $n_1=(dn-d+n+r)/(2d)$, $n_m=0$, $n_i=(n+r-d)/d$, $2\leqslant i\leqslant m-1$, $N=(dn-n-r)/d$ case of Lemma 2. (It is easy to verify that $N-n_1-\cdots-n_m=d(d-r-2)/2>0$.) This proves that (1.5) holds modulo $\Phi_n(q)^3$ for $M=(dn-n-r)/d$.

Since $(q^r;q^d)_k/(q^d;q^d)_k$ is congruent to 0 modulo $\Phi_n(q)$ for $(dn-n-r)/d<k\leqslant n-1$, we conclude that (1.5) also holds modulo $\Phi_n(q)^3$ for $M=n-1$. $\square$

3. Proof of Theorem 2

We first give a simple lemma on a property of certain arithmetic progressions.

Lemma 4. *Let d and r be odd integers satisfying $d\geqslant 3$, $r\leqslant d-4$ and $\gcd(d,r)=1$. Let n be an integer such that $n\geqslant (d-r)/2$ and $n\equiv -r/2 \pmod{d}$. Then there are no multiples of n in the arithmetic progression*

$$\frac{d+r}{2},\ \frac{d+r}{2}+d,\dots,\frac{d+r}{2}+dn-2n-r-d. \tag{3.1}$$

Proof. By the condition $\gcd(d,r)=1$, we have $\gcd((d+r)/2,(d-r)/2)=1$. Suppose that

$$(d+r)/2+ad=bn \tag{3.2}$$

for some integers a and b with $a\geqslant 0$. Then $(d+r)/2+ad>(r-d)/2\geqslant -n$ and so $b\geqslant 0$. Since $n\equiv (d-r)/2 \pmod{d}$, we deduce from (3.2) that $b\equiv -1 \pmod{d}$ and thereby $b\geqslant d-1$. But we have

$$\frac{d+r}{2}+dn-2n-r-d=dn-2n+\frac{d-r}{2}-d\leqslant (d-1)n-d,$$

thus implying that no number in the arithmetic progression (3.1) is a multiple of n. $\square$

Proof of Theorem 2. As before, the q-congruence (1.6) modulo $[n]$ can be deduced from Lemma 3. It remains to prove the modulus $\Phi_n(q)^2$ case of (1.6).

For $M=(dn-2n-r)/d$, the left-hand side of (1.6) can be written as the following multiple of a terminating ${}_{d+5}\phi_{d+4}$ series (this time we changed the position of $q^{(d+r)/2}$):

$$[r]\sum_{k=0}^{(dn-2n-r)/d}\frac{(q^r,q^{d+r/2},-q^{d+r/2},q^{(d+r)/2},q^r,\dots,q^r,q^{d+(d-2)n},q^{r-(d-2)n};q^d)_k}{(q^d,q^{r/2},-q^{r/2},q^{(d+r)/2},q^d,\dots,q^d,q^{r-(d-2)n},q^{d+(d-2)n};q^d)_k}q^{d(d-r-2)k/2}.$$

Here, the $q^r,\dots,q^r$ in the numerator stands for $d-1$ instances of q^r, and similarly, the $q^d,\dots,q^d$ in the denominator stands for $d-1$ instances of q^d. By Andrews' transformation (2.2), we may rewrite the above expression as

$$\begin{aligned}[r]\frac{(q^{d+r},q^{-(d-2)n};q^d)_{(dn-2n-r)/d}}{(q^d,q^{r-(d-2)n};q^d)_{(dn-2n-r)/d}}&\sum_{j_1,\dots,j_{m-1}\geqslant 0}\frac{(q^{(d-r)/2};q^d)_{j_1}(q^{d-r};q^d)_{j_2}\cdots(q^{d-r};q^d)_{j_{m-1}}}{(q^d;q^d)_{j_1}(q^d;q^d)_{j_2}\cdots(q^d;q^d)_{j_{m-1}}}\\
&\times\frac{(q^r,q^r;q^d)_{j_1}\dots(q^r,q^r;q^d)_{j_1+\cdots+j_{m-2}}(q^r,q^{d+(d-2)n};q^d)_{j_1+\cdots+j_{m-1}}}{(q^{(d+r)/2},q^d;q^d)_{j_1}(q^d,q^d;q^d)_{j_1+j_2}\dots(q^d,q^d;q^d)_{j_1+\cdots+j_{m-1}}}\\
&\times\frac{(q^{r-(d-2)n};q^d)_{j_1+\cdots+j_{m-1}}}{(q^{d+r};q^d)_{j_1+\cdots+j_{m-1}}}q^{(d-r)(j_{m-2}+\cdots+(m-2)j_1)+d(j_1+\cdots+j_{m-1})},
\end{aligned}\tag{3.3}$$

where $m=(d+1)/2$.

It is easily seen that the q-shifted factorial $(q^{d+r};q^d)_{(dn-2n-r)/d}$ has the factor $1-q^{(d-2)n}$ which is a multiple of $1-q^n$. Clearly, the q-shifted factorial $(q^{-(d-2)n};q^d)_{(dn-2n-r)/d}$ has the factor $1-q^{-(d-1)n}$ (again being a multiple of $1-q^n$) since $(dn-2n-r)/d\geqslant 1$ holds according to the conditions $d\geqslant 3$, $r\leqslant d-4$, and $n\geqslant (d-r)/2$. This indicates that the q-factorial $(q^{d+r},q^{-(d-2)n};q^d)_{(dn-2n-r)/d}$ in the numerator of the fraction before the multi-sum in (3.3) is divisible by $\Phi_n(q)^2$. Further, it is not difficult to see that the q-factorial $(q^d,q^{r-(d-2)n};q^d)_{(dn-2n-r)/d}$ in the denominator is relatively prime to $\Phi_n(q)$.

Like the proof of Theorem 1, the non-zero terms in the multi-sum in (3.3) are those indexed by $(j_1,\dots,j_{m-1})$ satisfying the inequality $j_1+\cdots+j_{m-1}\leqslant (dn-2n-r)/d$ because of the appearance of the factor $(q^{r-(d-2)n};q^d)_{j_1+\cdots+j_{m-1}}$ in the numerator. By Lemma 4, the q-shifted factorial $(q^{(d+r)/2},q^d)_{j_1}$ in the denominator does not contain a factor of the form $1-q^{\alpha n}$ for $j_1\leqslant (dn-2n-r)/d$ (and are therefore relatively prime to $\Phi_n(q)$). In addition, none of the other factors appearing in the denominator of the multi-sum of (3.3) contain a factor of the form $1-q^{\alpha n}$, except for $(q^{d+r};q^d)_{j_1+\cdots+j_{m-1}}$ when $j_1+\cdots+j_{m-1}=(dn-2n-r)/d$ (in this case the denominator contains the factor $1-q^{(d-2)n}$).

Letting $n=ad+(d-r)/2$ (with $a\geqslant 0$), we get $j_1+\cdots+j_{m-1}=a(d-2)+(d-r)/2-1$. If $j_1\geqslant a+1$, then $(q^{(d-r)/2};q^d)_{j_1}$ contains the factor $1-q^{(d-r)/2+ad}=1-q^n$. If $j_1\leqslant a$, then $j_2+\cdots+j_{m-1}\geqslant a(d-3)+(d-r)/2-1$. Since $m-2=(d-3)/2$, $d\geqslant 3$, and $r\leqslant d-4$, there must be an i with $2\leqslant i\leqslant m-1$ and $j_i\geqslant 2a+1$. Then $(q^{d-r};q^d)_{j_i}$ contains the factor $1-q^{d-r+2ad}=1-q^{2n}$ which is a multiple of $\Phi_n(q)$. Therefore, the denominator of the reduced form of the multi-sum in (3.3) is relatively prime to $\Phi_n(q)$. This proves that (3.3) is congruent to 0 modulo $\Phi_n(q)^2$.

For $M = n-1$, since $(q^r; q^d)_k/(q^d; q^d)_k$ is congruent to 0 modulo $\Phi_n(q)$ for $(dn-2n-r)/d < k \leqslant n-1$, we conclude that (1.6) is also true modulo $\Phi_n(q)^2$ in this case. $\square$

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