A TAYLOR EXPANSION THEOREM FOR AN ELLIPTIC EXTENSION OF THE ASKEY–WILSON OPERATOR

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Abstract. We establish Taylor series expansions in rational (and elliptic) function bases using E. Rains' elliptic extension of the Askey–Wilson divided difference operator. The expansion theorem we consider extends M. E. H. Ismail's expansion for the Askey–Wilson monomial basis. Three immediate applications (essentially already due to Rains) include simple proofs of Frenkel and Turaev's elliptic extensions of Jackson's ${}_8\phi_7$ summation and of Bailey's ${}_{10}\phi_9$ transformation, and the computation of the connection coefficients of Spiridonov's elliptic extension of Rahman's biorthogonal rational functions. We adumbrate other examples including the nonterminating extension of Jackson's ${}_8\phi_7$ summation and a quadratic expansion.

1. Introduction

Taylor series expansion is a powerful and well-known tool in analysis for studying the local behaviour of a suitable function (being approximated by its partial Taylor sums). The explicit expansion of a function in terms of another given basis of the function space is on one hand an important concept in harmonic analysis, and on the other hand, from a more algebraic point of view, it is simply a fundamental technique for obtaining identities, which, for instance, is one of the main ideas of umbral calculus [16].

In [6], Ismail gave a Taylor expansion theorem involving the Askey–Wilson divided difference operator. He utilized it to give simple proofs of the q-Pfaff–Saalschütz summation, and of the more general Sears transformation (relating two Askey–Wilson polynomials). More q-Taylor expansions related to the Askey–Wilson operator were given in [7] and later in [9] (to cite just a few relevant papers).

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As a matter of fact, none of the expansions obtained in the aforementioned papers involved *well-poised* series. Such expansions and related difference operators were however considered (in more generality, namely in the setting of *multivariate elliptic hypergeometric series*) by Rains [11], [12], and were also investigated by Rosengren [13], [14].

The purpose of the present paper is two-fold. Although the elliptic Taylor expansion in Theorem 4.2 has not been stated explicitly before (to the author's knowledge), it is implicit from the (more general) work of Rains [11] who ad-hoc also gave corresponding applications. Regarding the much higher level of generality and complexity of results in [11], it appears (to the present author) that these results, even in their simplest noteworthiest cases can easily be missed by non-specialists (who are maybe not so much interested in the multivariate theory which requires a more elaborate setup). Therefore one of our intentions is to make these results easy accessible. The other aim is to announce some new applications concerning infinite Taylor expansions in the non-elliptic case involving well-poised basic series; see the final section.

2. Preliminaries

For the following material, we refer to Gasper and Rahman's text [5]. Throughout this paper, we assume q to be a fixed complex number satisfying $|q|<1$.

2.1. **Basic hypergeometric series.** For any complex number a and integer n the q-shifted factorial is defined by

$$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty},\qquad \text{where}\qquad (a;q)_\infty=\prod_{j\geq 0}(1-aq^j). \tag{2.1}$$

For products of q-shifted factorials we use the short notation

$$(a_1,a_2,\dots,a_m;q)_n=\prod_{k=1}^m(a_k;q)_n,$$

where n is an integer or infinity. A list of useful identities for manipulating the q-shifted factorials is given in [5, Appendix I].

We use

$${}_{s+1}\phi_s\left[\begin{matrix}a_1,a_2,\dots,a_{s+1}\\ b_1,b_2,\dots,b_s\end{matrix};q,z\right]:=\sum_{k=0}^\infty\frac{(a_1,a_2,\dots,a_{s+1};q)_k}{(q,b_1,\dots,b_s;q)_k}z^k \tag{2.2}$$

to denote the *basic hypergeometric* ${}_{s+1}\phi_s$ *series*. In (2.2), $a_1,\dots,a_{s+1}$ are called the *upper parameters*, $b_1,\dots,b_s$ the *lower parameters*, z is the *argument*, and q the *base* of the series. The ${}_{s+1}\phi_s$ series terminates if one of the upper parameters, say a_{s+1}, is of the form q^{-n} for a nonnegative integer n. If the ${}_{s+1}\phi_s$ series does not terminate, it converges when $|z|<1$.

The classical theory of basic hypergeometric series contains numerous summation and transformation formulae involving ${}_{s+1}\phi_s$ series. Many of these summation theorems require that the parameters satisfy the condition of being either balanced and/or very-well-poised. An ${}_{s+1}\phi_s$ basic hypergeometric series is called *balanced* if $b_1\cdots b_s = a_1\cdots a_{s+1}q$ and $z=q$. An ${}_{s+1}\phi_s$ series is *well-poised* if $a_1 q = a_2 b_1 = \cdots = a_{s+1}b_s$. An ${}_{s+1}\phi_s$ basic hypergeometric series is called *very-well-poised* if it is well-poised and if $a_2 = -a_3 = q\sqrt{a_1}$. Note that the factor

$$\frac{1-a_1q^{2k}}{1-a_1}$$

appears in a very-well-poised series. The parameter a_1 is usually referred to as the *special parameter* of such a series.

One of the most important theorems in the theory of basic hypergeometric series is Jackson's [8] terminating very-well-poised balanced ${}_8\phi_7$ summation (cf. [5, Eq. (2.6.2)]):

$${}_8\phi_7\left[\begin{matrix} a,\, q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2q^{1+n}/bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcdq^{-n}/a, aq^{1+n} \end{matrix}; q, q\right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}. \tag{2.3}$$

This identity stands on the top of the classical hierarchy of summations for basic hypergeometric series. Special cases include the terminating and nonterminating very-well-poised ${}_6\phi_5$ summations, the q-Pfaff–Saalschütz summation, the q-Gauß summation, the q-Chu–Vandermonde summation and the termininating and nonterminating q-binomial theorem, see [5].

2.2. **Elliptic hypergeometric series.** Here, we refer to Chapter 11 of Gasper and Rahman's text [5]. Define a modified Jacobi theta function with argument x and nome p by

$$\theta(x;p) := (x;p)_\infty (p/x;p)_\infty\,, \qquad \theta(x_1,\dots,x_m;p) = \prod_{k=1}^m \theta(x_k;p), \tag{2.4}$$

where $x, x_1,\dots,x_m \neq 0$, $|p|<1$. We note the following useful properties of theta functions:

$$\theta(x;p) = -x\,\theta(1/x;p), \tag{2.5}$$

$$\theta(px;p) = -\frac{1}{x}\,\theta(x;p), \tag{2.6}$$

and Riemann's *addition formula*

$$\theta(xy, x/y, uv, u/v;p) - \theta(xv, x/v, uy, u/y;p) = \frac{u}{y}\,\theta(yv, y/v, xu, x/u;p) \tag{2.7}$$

(cf. [20, p. 451, Example 5]).

Further, define a *theta shifted factorial* analogue of the q-shifted factorial by

$$(a;q,p)_n = \begin{cases} \prod_{k=0}^{n-1}\theta(aq^k;p), & n=1,2,\dots, \\ 1, & n=0, \\ 1/\prod_{k=0}^{-n-1}\theta(aq^{n+k};p), & n=-1,-2,\dots, \end{cases} \tag{2.8}$$

and let

$$(a_1,a_2,\dots,a_m;q,p)_n = \prod_{k=1}^{m}(a_k;q,p)_n,$$

where $a,a_1,\dots,a_m \neq 0$. Notice that $\theta(x;0)=1-x$ and, hence, $(a;q,0)_n=(a;q)_n$ is a *q-shifted factorial* in base q. The parameters q and p in $(a;q,p)_n$ are called the *base* and *nome*, respectively, and $(a;q,p)_n$ is called the *q,p-shifted factorial*. Observe that

$$(pa;q,p)_n = (-1)^n a^{-n} q^{-\binom{n}{2}}\,(a;q,p)_n, \tag{2.9}$$

which follows from (2.6). A list of other useful identities for manipulating the q,p-shifted factorials is given in [5, Sec. 11.2].

We call a series $\sum c_n$ an *elliptic hypergeometric series* if $g(n)=c_{n+1}/c_n$ is an elliptic function of n with n considered as a complex variable; i.e., the function $g(x)$ is a doubly periodic meromorphic function of the complex variable x. Without loss of generality, by the theory of theta functions, we may assume that

$$g(x) = \frac{\theta(a_1q^x,a_2q^x,\dots,a_{s+1}q^x;p)}{\theta(q^{1+x},b_1q^x,\dots,b_sq^x;p)}\,z,$$

where the *elliptic balancing condition*, namely

$$a_1a_2\cdots a_{s+1} = qb_1b_2\cdots b_s,$$

holds. If we write $q=e^{2\pi i\sigma}$, $p=e^{2\pi i\tau}$, with complex σ, τ, then $g(x)$ is indeed periodic in x with periods σ^{-1} and $\tau\sigma^{-1}$.

The general form of an elliptic hypergeometric series is thus

$${}_{s+1}E_s\left[\begin{matrix}a_1,\dots,a_{s+1}\\ b_1,\dots,b_s\end{matrix};q,p;z\right] := \sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{s+1};q,p)_k}{(q,b_1\dots,b_s;q,p)_k}z^k,$$

provided $a_1a_2\cdots a_{s+1}=qb_1b_2\cdots b_s$. Here $a_1,\dots,a_{s+1}$ are the upper parameters, $b_1,\dots,b_s$ the lower parameters, q is the base, p the nome, and z is the argument of the series. For convergence reasons, one usually requires $a_{s+1}=q^{-n}$ (n being a nonnegative integer), so that the sum is in fact finite.

Very-well-poised elliptic hypergeometric series are defined as

$$
\begin{aligned}
&{}_{s+1}V_s(a_1;a_6,\dots,a_{s+1};q,p;z)\\
&\quad:={}_{s+1}E_s\left[\begin{matrix}a_1,\ qa_1^{\frac12},-qa_1^{\frac12},qa_1^{\frac12}/p^{\frac12},-qa_1^{\frac12}p^{\frac12},a_6,\dots,a_{s+1}\\ a_1^{\frac12},-a_1^{\frac12},a_1^{\frac12}p^{\frac12},-a_1^{\frac12}/p^{\frac12},a_1q/a_6,\dots,a_1q/a_{s+1}\end{matrix};q,p;-z\right]\\
&\quad=\sum_{k=0}^{\infty}\frac{\theta(a_1q^{2k};p)}{\theta(a_1;p)}\frac{(a_1,a_6,\dots,a_{s+1};q,p)_k}{(q,a_1q/a_6,\dots,a_1q/a_{s+1};q,p)_k}(qz)^k,
\end{aligned}\tag{2.10}
$$

where

$$q^2a_6^2a_7^2\cdots a_{s+1}^2=(a_1q)^{s-5}.$$

It is convenient to abbreviate

$${}_{s+1}V_s(a_1;a_6,\dots,a_{s+1};q,p):={}_{s+1}V_s(a_1;a_6,\dots,a_{s+1};q,p;1).$$

Note that in (2.10) we have used

$$\frac{\theta(aq^{2k};p)}{\theta(a;p)}=\frac{(qa^{\frac12},-qa^{\frac12},qa^{\frac12}/p^{\frac12},-qa^{\frac12}p^{\frac12};q,p)_k}{(a^{\frac12},-a^{\frac12},a^{\frac12}p^{\frac12},-a^{\frac12}/p^{\frac12};q,p)_k}(-q)^{-k},$$

which shows that in the elliptic case the number of pairs of numerator and denominator paramters involved in the construction of the *very-well-poised term* is *four* (whereas in the basic case this number is *two*, in the ordinary case only *one*).

The above definitions for ${}_{s+1}E_s$ and ${}_{s+1}V_s$ series are due to Spiridonov [17], see [5, Ch. 11].

In their study of elliptic $6j$ symbols (which are elliptic solutions of the Yang–Baxter equation found by Baxter [2] and Date et al. [3]), Frenkel and Turaev [4] discovered the following ${}_{12}V_{11}$ transformation:

$$
\begin{aligned}
{}_{12}V_{11}(a;b,c,d,e,f,\lambda aq^{n+1}/ef,q^{-n};q,p)&=\frac{(aq,aq/ef,\lambda q/e,\lambda q/f;q,p)_n}{(aq/e,aq/f,\lambda q/ef,\lambda q;q,p)_n}\\
&\quad\times{}_{12}V_{11}(\lambda;\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{n+1}/ef,q^{-n};q,p),
\end{aligned}\tag{2.11}
$$

where $\lambda=a^2q/bcd$. This is an extension of Bailey's very-well-poised ${}_{10}\phi_9$ transformation [5, Eq. (2.9.1)], to which it reduces when $p=0$.

The ${}_{12}V_{11}$ transformation in (2.11) appeared as a consequence of the tetrahedral symmetry of the elliptic $6j$ symbols. Frenkel and Turaev's transformation contains as a special case the following summation formula,

$${}_{10}V_9(a;b,c,d,e,q^{-n};q,p)=\frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_n}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_n},\tag{2.12}$$

where $a^2q^{n+1}=bcde$. The ${}_{10}V_9$ summation is an elliptic analogue of Jackson's ${}_8\phi_7$ summation formula (2.3). A striking feature of elliptic hypergeometric series is that already the simplest identities involve many parameters. The fundamental identity at the "bottom" of the hierarchy of identities for elliptic hypergeometric series is the ${}_{10}V_9$ summation. When keeping the nome p arbitrary (while $|p|<1$) there is no way to specialize (for the sake of obtaining lower order identities) any of the free parameters of an elliptic hypergeometric series in form of a limit tending to zero or infinity, due to the issue of convergence. For the same reason, elliptic hypergeometric series are only well-defined as complex functions if they are terminating (i.e., the sums are finite). See Gasper and Rahman's text [5, Ch. 11] for more details.

3. The Askey–Wilson operator

We will be considering meromorphic functions $f(z)$ symmetric in z and $1/z$. Writing $z=e^{i\theta}$ (note that θ need not be real), we may consider f to be a function in $x=\cos\theta=(z+1/z)/2$ and write $f[x]:=f(z)$.

Let $\mathcal{D}_q$ denote the *Askey–Wilson operator* acting on $x=\cos\theta$. It is defined as follows:

$$\mathcal{D}_q f[x]=\frac{f(q^{\frac12}z)-f(q^{-\frac12}z)}{\iota(q^{\frac12}z)-\iota(q^{-\frac12}z)}, \tag{3.1}$$

where $\iota[x]=x$ (i.e., $\iota(z)=(z+1/z)/2$). If follows from (3.1) that

$$\mathcal{D}_q f[x]=\frac{f(q^{\frac12}z)-f(q^{-\frac12}z)}{i(q^{\frac12}-q^{-\frac12})\sin\theta}. \tag{3.2}$$

The operator $\mathcal{D}_q$ was introduced in [1] and is a q-analogue of the differentiation operator. In particular, since

$$\mathcal{D}_q T_n[x]=\frac{q^{\frac n2}-q^{-\frac n2}}{q^{\frac12}-q^{-\frac12}}U_{n-1}[x], \tag{3.3}$$

where $T_n[\cos\theta]=\cos n\theta$ and $U_n[\cos\theta]=\sin(n+1)\theta/\sin\theta$ are the Chebyshev polynomials of the first and second kind, one easily sees that $\mathcal{D}_q$ maps polynomials to polynomials, lowering the degree by one.

In the calculus of the Askey–Wilson operator the so-called "Askey–Wilson monomials" $\phi_n(x;a)=(az,a/z;q)_n$ form a natural basis for polynomials or power series in x. One readily computes

$$\mathcal{D}_q(az,a/z;q)_n=-\frac{2a(1-q^n)}{(1-q)}(aq^{\frac12}z,aq^{\frac12}/z;q)_{n-1}. \tag{3.4}$$

We recall the following Taylor theorem for polynomials $f[x]$, proved by Ismail [6]:

Theorem 3.1. *If $f[x]$ is a polynomial in x of degree n, then*

$$f[x]=\sum_{k=0}^{n} f_k\phi_k(x;a),$$

where

$$f_k = \frac{(q-1)^k}{(2a)^k(q;q)_k} q^{-k(k-1)/4} \big(\mathcal{D}_q^k f\big)[x_k], \qquad x_k := \frac{1}{2}(aq^{\frac{k}{2}} + q^{-\frac{k}{2}}/a).$$

As was shown in [6], the application of Theorem 3.1 to $f(z) = (bz, b/z; q)_n$ immediately gives the q-Pfaff–Saalschütz summation (cf. [5, Eq. (1.7.2)]), in the form

$$\frac{(bz, b/z; q)_n}{(ba, b/a; q)_n} = {}_3\phi_2\left[\begin{matrix} az, a/z, q^{-n} \\ ab, q^{1-n}a/b \end{matrix}; q, q\right],$$

while its application to the Askey–Wilson polynomials,

$$\omega_n(x; a, b, c, d; q) := {}_4\phi_3\left[\begin{matrix} az, a/z, abcdq^{n-1}, q^{-n} \\ ab, ac, ad \end{matrix}; q, q\right],$$

gives a connection coefficient identity which, by specialization, can be reduced to the Sears transformation (cf. [5, Eq. (3.2.1)]), in the form

$$\omega_n(x; a, b, c, d; q) = \frac{a^n(bc, bd; q)_n}{b^n(ac, ad; q)_n} \, \omega_n(x; b, a, c, d; q).$$

Ismail and Stanton [7] extended the above polynomial Taylor theorem to hold for entire functions of exponential growth, resulting in infinite Taylor expansions. Marco and Parcet [9] extended this yet further to hold for arbitrary q-differentiable functions, resulting in infinite Taylor expansions with explicit remainder term. Among other results they were able to recover the nonterminating q-Pfaff–Saalschütz summation (cf. [5, Appendix (II.24)]).

4. A well-poised and elliptic Askey–Wilson operator

Since

$$\begin{aligned}
&\mathcal{D}_q \frac{(az, a/z; q)_n}{(cz, c/z; q)_n} \\
&\quad = \frac{2}{(q^{\frac12} - q^{-\frac12})(z - 1/z)} \left[\frac{(aq^{\frac12}z, aq^{-\frac12}/z; q)_n}{(cq^{\frac12}z, cq^{-\frac12}/z; q)_n} - \frac{(aq^{-\frac12}z, aq^{\frac12}/z; q)_n}{(cq^{-\frac12}z, cq^{\frac12}/z; q)_n} \right] \\
&\quad = \frac{2}{(q^{\frac12} - q^{-\frac12})(z - 1/z)} \frac{(aq^{\frac12}z, aq^{\frac12}/z; q)_{n-1}}{(cq^{\frac12}z, cq^{\frac12}/z; q)_{n-1}} \\
&\qquad \times \left[\frac{(1 - azq^{n-\frac12})(1 - aq^{-\frac12}/z)}{(1 - czq^{n-\frac12})(1 - cq^{-\frac12}/z)} - \frac{(1 - azq^{-\frac12})(1 - aq^{n-\frac12}/z)}{(1 - czq^{-\frac12})(1 - cq^{n-\frac12}/z)} \right] \\
&= \frac{(-1)2a(1 - c/a)(1 - acq^{n-1})(1 - q^n)}{(1 - czq^{-\frac12})(1 - czq^{\frac12})(1 - cq^{-\frac12}/z)(1 - cq^{\frac12}/z)(1 - q)} \frac{(aq^{\frac12}z, aq^{\frac12}/z; q)_{n-1}}{(cq^{\frac32}z, cq^{\frac32}/z; q)_{n-1}},
\end{aligned}$$

it makes sense to define a c-generalized Askey–Wilson operator acting on x (or z) by

$$\mathcal{D}_{c,q} = (1-czq^{-\frac{1}{2}})(1-czq^{\frac{1}{2}})(1-cq^{-\frac{1}{2}}/z)(1-cq^{\frac{1}{2}}/z)\,\mathcal{D}_q,$$

which acts "degree-lowering" on the "rational monomials"

$$\frac{(az,a/z;q)_n}{(cz,c/z;q)_n}$$

in the form

$$\mathcal{D}_{c,q}\frac{(az,a/z;q)_n}{(cz,c/z;q)_n} \;=\; \frac{(-1)2a(1-c/a)(1-acq^{n-1})(1-q^n)}{(1-q)}\frac{(aq^{\frac{1}{2}}z,aq^{\frac{1}{2}}/z;q)_{n-1}}{(cq^{\frac{3}{2}}z,cq^{\frac{3}{2}}/z;q)_{n-1}}.$$

Clearly,

$$\mathcal{D}_{0,q} = \mathcal{D}_q.$$

More generally, for parameters c,q,p with $|q|,|p|<1$, we define an elliptic extension of the Askey–Wilson operator, acting on functions symmetric in $z^{\pm1}$, by

$$\mathcal{D}_{c,q,p}f(z) = 2q^{\frac{1}{2}}z\,\frac{\theta(czq^{-\frac{1}{2}},czq^{\frac{1}{2}},cq^{-\frac{1}{2}}/z,cq^{\frac{1}{2}}/z;p)}{\theta(q,z^2;p)}\Big(f(q^{\frac{1}{2}}z)-f(q^{-\frac{1}{2}}z)\Big). \tag{4.1}$$

Note that

$$\mathcal{D}_{c,q,0} = \mathcal{D}_{c,q}.$$

In particular, using (2.7), we have

$$\mathcal{D}_{c,q,p}\frac{(az,a/z;q,p)_n}{(cz,c/z;q,p)_n} = \frac{(-1)2a\,\theta(c/a,acq^{n-1},q^n;p)}{\theta(q;p)}\frac{(aq^{\frac{1}{2}}z,aq^{\frac{1}{2}}/z;q,p)_{n-1}}{(cq^{\frac{3}{2}}z,cq^{\frac{3}{2}}/z;q,p)_{n-1}}. \tag{4.2}$$

The operator $\mathcal{D}_{c,q,p}$ is a special case of some multivariable difference operator introduced by Rains in [11]. Already in the single variable case Rains' operator involves two more parameters than $\mathcal{D}_{c,q,p}$. (Rains' difference operators generate a representation of the Sklyanin algebra, as observed in [11] and made explicit in [13] and [14, Sec. 6].) Rains' operator can be specialized to act degree-lowering (as the above $\mathcal{D}_{c,q,p}$ does), degree-preserving or degree-raising on abelian functions. Rains used his multivariable difference operators in [11] to construct BC_n-symmetric biorthogonal abelian functions that generalize Koornwinder's orthogonal polynomials. He further used his operator in [12] to derive BC_n-symmetric extensions of Frenkel and Turaev's ${}_{10}V_9$ summation and ${}_{12}V_{11}$ transformation.

For the current presentation, as we are mainly concerned with Taylor expansions, we find it indeed sufficient to consider the above operator $\mathcal{D}_{c,q,p}$ (which exhibits a very nice degree-lowering action in (4.2)) rather than the more general operator considered by Rains (in one dimension).

We describe the spaces of functions we will be dealing with. For a complex number c we define

$$W_c^m := \operatorname{span}\left\{\frac{g_n(z)}{(cz,c/z;q,p)_n},\ 0\le n\le m\right\},$$

where $g_n(z)$ runs over all functions being holomorphic for $z\neq 0$ with $g_n(z)=g_n(1/z)$ and

$$g_n(pz)=\frac{1}{p^n z^{2n}}g_n(z).$$

In classical terminology, $g_n(z)$ is an even theta function of order $2n$ and zero characteristics. Rains [12] refers to such functions as BC_1 theta functions of degree n, whereas in [15] we referred to them as D_n theta functions. It is well-known that the space V^n of even theta functions of order $2n$ and zero characteristics has dimension $n+1$ (see e.g. Weber [19, p. 49]).

Note that W_c^m consists of certain abelian functions. (For $p\to 0$ these degenerate to certain rational functions we may call "well-poised".)

Lemma 4.1. *For any arbitrary but fixed complex number a (satisfying $a\neq cq^jp^k$, for $j=0,\dots,m-1$, and $k\in\mathbb{Z}$, and $a\neq q^jp^k/c$, for $j=2-2m,\dots,1-m$, and $k\in\mathbb{Z}$), the set*

$$\left\{\frac{(az,a/z;q,p)_n}{(cz,c/z;q,p)_n},0\le n\le m\right\}$$

forms a basis for W_c^m.

Proof. This is equivalent to the fact that the set

$$\{(az,a/z;q,p)_n(cq^nz,cq^n/z;q,p)_{m-n},\ 0\le n\le m\} \tag{4.3}$$

forms a basis for V^m, the space of even theta functions of order $2m$ and zero characteristics, a fact easily proved by induction on m. For $m=0$ the statement is trivial. Now assume that it holds for a fixed $m\ge 0$. Since the $m+1$ products in (4.3) are linearly independent, it follows (by multiplication with the common factor $\theta(cq^mz,cq^m/z;p)$) that the $m+1$ products

$$(az,a/z;q,p)_n(cq^nz,cq^n/z;q,p)_{m+1-n},\quad 0\le n\le m,$$

are also linearly independent. It thus remains to be shown that $(az,a/z;q,p)_{m+1}$ is not a linear combination of $\{(az,a/z;q,p)_n(cq^nz,cq^n/z;q,p)_{m+1-n},\ 0\le n\le m\}$. Suppose

$$(az,a/z;q,p)_{m+1}=\sum_{n=0}^{m}\alpha_n(az,a/z;q,p)_n(cq^nz,cq^n/z;q,p)_{m+1-n}.$$

Letting $z=cq^m$ gives $(acq^m,aq^{-m}/c;q,p)_{m+1}=0$, which is a contradiction. □

Note that, in view of (4.2), the elliptic Askey–Wilson operator maps functions in W_c^m to functions in $W_{cq^{\frac{3}{2}}}^{m-1}$.

We now define

$$\mathcal{D}_{c,q,p}^{(k)} = \mathcal{D}_{cq^{\frac{3}{2}},q,p}^{(k-1)} \, \mathcal{D}_{c,q,p}, \tag{4.4}$$

with $\mathcal{D}_{c,q,p}^{(0)} = \varepsilon$, the identity operator. We have the following elliptic expansion theorem which extends Theorem 3.1:

Theorem 4.2. *Let f be in W_c^n, then*

$$f(z) = \sum_{k=0}^{n} f_k \, \frac{(az, a/z; q, p)_k}{(cz, c/z; q, p)_k}, \tag{4.5}$$

where

$$f_k = \frac{(-1)^k q^{-k(k-1)/4} \, \theta(q;p)^k}{(2a)^k (q, c/a, acq^{k-1}; q, p)_k} \big[\mathcal{D}_{c,q,p}^{(k)} f\big]_{z=aq^{\frac{k}{2}}}.$$

Proof. First of all, due to Lemma 4.1 it is clear that the expansion (4.5) exists, so we just need to compute the coefficients f_k. Formula (4.2) yields (together with (4.4))

$$\begin{aligned}
&\left[\mathcal{D}_{c,q,p}^{(k)} \frac{(az, a/z; q, p)_n}{(cz, c/z; q, p)_n}\right]_{z=aq^{\frac{k}{2}}} \\
&= (-1)^k (2a)^k q^{\binom{k}{2}/2} \frac{(q;q,p)_n (c/a, acq^{n-1}; q, p)_k}{(q;q,p)_{n-k} \, \theta(q;p)^k} \left[\frac{(aq^{\frac{k}{2}} z, aq^{\frac{k}{2}}/z; q, p)_{n-k}}{(cq^{\frac{3k}{2}} z, cq^{\frac{3k}{2}}/z; q, p)_{n-k}}\right]_{z=aq^{\frac{k}{2}}} \\
&= (-1)^k (2a)^k q^{\binom{k}{2}/2} \frac{(q, c/a, acq^{k-1}; q, p)_k}{\theta(q;p)^k} \delta_{nk}.
\end{aligned}$$

The theorem now follows by applying $\mathcal{D}_{c,q,p}^{(j)}$ to both sides of (4.5) and then setting $z = aq^{\frac{j}{2}}$. ∎

Example 4.3. Let

$$f(z) = \frac{(bz, b/z; q, p)_n}{(cz, c/z; q, p)_n}.$$

Application of Theorem 4.2 in conjuction with (4.2) gives

$$\begin{aligned}
f_k &= \frac{(-1)^k q^{-k(k-1)/4} \, \theta(q;p)^k}{(2a)^k (q, c/a, acq^{k-1}; q, p)_k} \\
&\quad \times (-1)^k (2b)^k q^{\binom{k}{2}/2} \frac{(q;q,p)_n (c/b, bcq^{n-1}; q, p)_k}{(q;q,p)_{n-k} \, \theta(q;p)^k} \frac{(abq^k, b/a; q, p)_{n-k}}{(acq^{2k}, cq^k/a; q, p)_{n-k}} \\
&= \frac{(ab, b/a; q, p)_n}{(ac, c/a; q, p)_n} \frac{\theta(acq^{2k-1}; p)}{\theta(acq^{-1}; p)} \frac{(acq^{-1}, c/b, bcq^{n-1}, q^{-n}; q, p)_k}{(q, ab, aq^{1-n}/b, acq^n; q, p)_k} q^k,
\end{aligned}$$

thus yielding Frenkel and Turaev's ${}_{10}V_9$ summation (2.12), in the form

$$\frac{(ac,c/a,bz,b/z;q,p)_n}{(ab,b/a,cz,c/z;q,p)_n}={}_{10}V_9(acq^{-1};az,a/z,c/b,bcq^{n-1},q^{-n};q,p).$$

Example 4.4. Let

$$R_n(z;b,c,d,e,f;q,p)={}_{12}V_{11}(bcq^{-1};bz,b/z,d,e,f,bc^3q^{n-1}/def,q^{-n};q,p),$$

which is Spiridonov's [18] elliptic extension of Rahman's family of biorthogonal rational functions. We have

$$\begin{aligned}\mathcal{D}^{(k)}_{c,q,p}&R_n(z;b,c,d,e,f;q,p)\\&=\frac{(-1)^k(2b)^kq^{k(k+3)/4}\,(bc;q,p)_{2k}(c/b,d,e,f,bc^3q^{n-1}/def,q^{-n};q,p)_k}{\theta(q;p)^k\,(bc/d,bc/e,bc/f,bcq^n,defq^{1-n}/c^2;q,p)_k}\\&\qquad\times R_{n-k}(z;bq^{\frac{k}{2}},cq^{\frac{3k}{2}},dq^k,eq^k,fq^k;q,p).\end{aligned}$$

Application of Theorem 4.2 now yields, after some computation, the connection coefficient identity

$$\begin{aligned}R_n&(z;b,c,d,e,f;q,p)\\&=\sum_{k=0}^n\frac{(az,a/z;q,p)_k}{(cz,c/z;q,p)_k}\frac{b^kq^k(bc;q,p)_{2k}(c/b,d,e,f,bc^3q^{n-1}/def,q^{-n};q,p)_k}{a^k\,(q,bc/d,bc/e,bc/f,bcq^n,defq^{1-n}/c^2;q,p)_k}\\&\qquad\times R_{n-k}(aq^{\frac{k}{2}};bq^{\frac{k}{2}},cq^{\frac{3k}{2}},dq^k,eq^k,fq^k;q,p).\end{aligned}\tag{4.6}$$

(Observe that the left-hand side of (4.6) is independent of a.)

Now note that

$$R_m(c/f;b,c,d,e,f;q,p)=\frac{(bc,bc/de,c^2/df,c^2/ef;q,p)_m}{(bc/d,bc/e,c^2/f,c^2/def;q,p)_m},\tag{4.7}$$

due to Frenkel and Turaev's ${}_{10}V_9$ summation. Letting $a\to c/f$ in (4.6) gives, after some simplification,

$$\begin{aligned}R_n&(z;b,c,d,e,f;q,p)\\&=\frac{(bc,bc/de,c^2/df,c^2/ef;q,p)_n}{(bc/d,bc/e,c^2/f,c^2/def;q,p)_n}R_n(z;c/f,c,d,e,c/b;q,p),\end{aligned}\tag{4.8}$$

which is equivalent to Frenkel and Turaev's ${}_{12}V_{11}$ transformation in (2.11).

5. Outlook: Well-poised basic expansions

As an outlook we sketch some details of our further investigations. These concern infinite convergent expansions in the basic $p=0$ case.

For instance, using an extension of the well-poised basic Taylor expansion theorem involving a remainder term, we obtain, by using a symmetry argument, the following expansion:

$$\frac{(cz/d,c/dz,cz/e,c/ez;q)_\infty}{(cz,c/z,c^2z/bde,c^2/bdez;q)_\infty}=\frac{(cz/de,c/dez;q)_\infty}{(c^2z/bde,c^2/bdez;q)_\infty}\sum_{k\ge0}f_k\,\frac{(bz,b/z;q)_k}{(cz,c/z;q)_k}\\
+\frac{(bz,b/z;q)_\infty}{(cz,c/z;q)_\infty}\sum_{k\ge0}g_k\,\frac{(cz/de,c/dez;q)_k}{(c^2z/bde,c^2/bdez;q)_k}.$$

After the explicit computation of the coefficients f_k and g_k one recovers the nonterminating ${}_8\phi_7$ summation (cf. [5, Appendix (II.25)]), in the form

$$\frac{(cz/d,c/dz,cz/e,c/ez;q)_\infty}{(cz,c/z,c^2z/bde,c^2/bdez;q)_\infty}\\
=\frac{(cz/de,c/dez;q)_\infty}{(c^2z/bde,c^2/bdez;q)_\infty}\sum_{k\ge0}\frac{(1-bcq^{2k-1})}{(1-bcq^{-1})}\frac{(bcq^{-1},d,e,c^2/deq,bz,b/z;q)_k}{(q,bc/d,bc/e,bdeq/c,cz,c/z;q)_k}q^k\\
+\frac{(bz,b/z;q)_\infty}{(cz,c/z;q)_\infty}\sum_{k\ge0}\frac{(1-c^3q^{2k-1}/bd^2e^2)}{(1-c^3/bd^2e^2q)}\\
\times\frac{(c^3/bd^2e^2q,c/bd,c/be,c^2/deq,cz/de,c/dez;q)_k}{(q,c^2/de^2,c^2/d^2e,cq/bde,c^2z/bde,c^2/bdez;q)_k}q^k.$$

To give another example, by expanding the "quadratic" infinite product

$$f(z)=\frac{(azq,aq/z,b^2z/a,b^2/az;q^2)_\infty}{(bz,b/z;q)_\infty}$$

in terms of the "well-poised monomials"

$$\frac{(az,a/z;q)_k}{(bz,b/z;q)_k},$$

we recover a particular nonterminating ${}_8\phi_7$ summation, namely Bailey's q-analogue of Watson's ${}_3F_2$ summation (cf. [5, Ex. 2.17(i)]), in the form

$$\frac{(azq,aq/z,b^2z/a,b^2/az;q^2)_\infty}{(bz,b/z;q)_\infty}=\frac{(q,a^2q,b^2,b^2/a^2;q^2)_\infty}{(-ab,-b/a;q)_\infty}\\
\times\sum_{k\ge0}\frac{(1+abq^{2k-1})}{(1+abq^{-1})}\frac{(-abq^{-1},bq^{-\frac12},-bq^{-\frac12},-aq/b,az,a/z;q)_k}{(q,-aq^{\frac12},aq^{\frac12},b^2q^{-1},bz,b/z;q)_k}\left(\frac{b}{a}\right)^k.$$

A paper featuring these well-poised basic expansions is under preparation.

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