# Inversion of the Pieri formula for Macdonald polynomials

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**Abstract**

We give the explicit analytic development of Macdonald polynomials in terms of "modified complete" and elementary symmetric functions. These expansions are obtained by inverting the Pieri formula. Specialization yields similar developments for monomial, Jack and Hall–Littlewood symmetric functions.

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# 1 Introduction

Fifty years ago, Hua [3] introduced a new family of polynomials defined on the space of complex symmetric matrices, and set the problem of finding their explicit analytic expansion in terms of Schur functions [3, p. 132, Eq. (6.2.5)].

These polynomials were further investigated by James [4], who named them "zonal polynomials", studied their connection with symmetric group algebra, and gave a method to compute them. A large literature followed, mostly due to statisticians, but no explicit analytic formula was found for the zonal polynomials.

Hua's problem is now better understood in the more general framework of Macdonald polynomials (of type $A_n$) [19]. Zonal polynomials are indeed a special case of Jack polynomials, which in turn are obtained from Macdonald polynomials by taking a particular limit.

Macdonald polynomials are indexed by partitions, i.e. finite decreasing sequences of positive integers. These polynomials form a basis of the algebra of symmetric functions with rational coefficients in two parameters $q, t$. They generalize many classical bases of this algebra, including monomial, elementary, Schur, Hall–Littlewood, and Jack symmetric functions. These particular cases correspond to various specializations of the indeterminates $q$ and $t$.

Two combinatorial formulas were known for Macdonald polynomials. The first one gives them as a sum of monomials associated with tableaux [19, p. 346, Eqs. (7.13)]. The second one writes their expansion in terms of Schur functions as a determinant [14]. However, in general both methods do not lead to an analytic formula, since they involve combinatorial quantities which cannot be written in analytic terms.

Thus Hua's problem kept open for Macdonald polynomials. Their analytic expansion was explicitly known only when the indexing partition is a hook [7], has length two [5] or three [15], and in the dual cases corresponding to parts at most equal to 3.

The aim of this paper is to present a general solution to this problem and to provide two explicit analytic developments for Macdonald polynomials. One of them is made in terms of elementary symmetric functions. The other one is made in terms of "modified complete" symmetric functions, which have themselves a known development in terms of any classical basis [16].

In the special case $q = t$, these two developments coincide with the classical Jacobi–Trudi formulas for Schur functions. Thus our results appear as generalized Jacobi–Trudi expansions for Macdonald polynomials.

Our method relies on two ingredients, firstly the Pieri formula for Macdonald polynomials, secondly a method developed by Krattenthaler [10, 11] for inverting infinite multidimensional matrices.

The Pieri formula has been computed by Macdonald [19]. Most of the time, it is stated in combinatorial terms. We formulate it in analytic terms, which defines an infinite multidimensional matrix. Then we derive the inverse of this "Pieri matrix", by adapting Krattenthaler's operator method to the multivariate case, as already done elsewhere [21] by the second author.

This article is organized as follows. Sections 2 and 3 are devoted to the inversion of infinite multidimensional matrices, and may be read independently of the rest of the paper. In Section 2 we recall the Krattenthaler method, which is used in Section 3 to get new multidimensional matrix inverses. We shall only need a particular case of these inversions, but we prefer to prove them in full generality for possible future reference.

In Sections 4 to 11 we apply these results to the theory of Macdonald polynomials. In Section 4 we introduce our notation and recall general facts about these polynomials. In particular we give the analytic form of the Pieri formula. The infinite multidimensional matrix thus defined is inverted in Section 5. The generalized Jacobi–Trudi expansions for Macdonald polynomials are derived in Section 6. Sections 7, 8 and 9 are devoted to various specializations of our results, in particular for Schur, monomial, Hall–Littlewood and Jack symmetric functions. Most of the expansions there obtained are new. The example of hook partitions, already studied by Kerov [7, 8], is then considered in Section 10. We conclude in Section 11 with a few remarks about the extension of Macdonald polynomials to multi-integers or sequences of complex numbers.

Our results were announced in [18]. An alternative proof of our main theorem has subsequently been given in [17] (but requires the explicit form of the result here obtained). It is an open question whether our method can be generalized to Macdonald polynomials associated with other root systems than $A_n$.

# 2 Krattenthaler's matrix inversion method

Let $\mathsf{Z}$ be the set of integers, $n$ some positive integer and $\mathsf{Z}^n$ the set of multi-integers $\mathbf{m}=(m_1,\ldots,m_n)$. We write $\mathbf{0}=(0,\ldots,0)$, $\mathbf{m}\geq\mathbf{k}$ for $m_i\geq k_i\,(1\leq i\leq n)$, and for any set of indeterminates $\mathbf{z}=(z_1,\ldots,z_n)$, we put $\mathbf{z}^{\mathbf{m}}=z_1^{m_1}z_2^{m_2}\cdots z_n^{m_n}$.

A formal Laurent series is a series of the form $a(\mathbf{z})=\sum_{\mathbf{m}\geq\mathbf{k}}a_{\mathbf{m}}\mathbf{z}^{\mathbf{m}}$, for some $\mathbf{k}\in\mathsf{Z}^n$. On the space $\mathcal{L}$ of formal Laurent series we introduce the bilinear form $\langle\,,\,\rangle$ defined by

$$\langle a(\mathbf{z}),b(\mathbf{z})\rangle=\langle\mathbf{z}^{\mathbf{0}}\rangle(a(\mathbf{z})\,b(\mathbf{z})),$$

where $\langle\mathbf{z}^{\mathbf{0}}\rangle(c(\mathbf{z}))$ denotes the coefficient of $\mathbf{z}^{\mathbf{0}}$ in $c(\mathbf{z})$. Given any linear operator $L$ on $\mathcal{L}$, we write $L\in\mathrm{End}(\mathcal{L})$ and denote $L^*$ its adjoint with respect to $\langle\,,\,\rangle$, i.e. $\langle La(\mathbf{z}),b(\mathbf{z})\rangle=\langle a(\mathbf{z}),L^*b(\mathbf{z})\rangle$.

Let $F=(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k}\in\mathsf{Z}^n}$ be an infinite lower-triangular $n$-dimensional matrix, i.e. $f_{\mathbf{mk}}=0$ unless $\mathbf{m}\geq\mathbf{k}$. The matrix $G=(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathsf{Z}^n}$ is said to be the inverse matrix of $F$ if and only if

$$\sum_{\mathbf{m}\geq\mathbf{k}\geq\mathbf{l}}f_{\mathbf{mk}}\,g_{\mathbf{kl}}=\delta_{\mathbf{ml}}$$

holds for all $\mathbf{m},\mathbf{l}\in\mathsf{Z}^n$, where $\delta_{\mathbf{ml}}$ is the usual Kronecker symbol. Since $F$ and $G$ are both lower-triangular, the above sum is finite. Moreover the dual relation $\sum_{\mathbf{m}\geq\mathbf{k}\geq\mathbf{l}}g_{\mathbf{mk}}\,f_{\mathbf{kl}}=\delta_{\mathbf{ml}}$ is also satisfied.

In [10] Krattenthaler gave a method for solving Lagrange inversion problems, which are closely connected with inversion of lower-triangular matrices. We shall need the following special case of [10, Theorem 1].

Let $F = (f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k}\in\mathsf{Z}^n}$ be an infinite lower-triangular matrix with all $f_{\mathbf{kk}} \neq 0$, and $G = (g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathsf{Z}^n}$ its uniquely determined inverse matrix. Define the formal Laurent series

$$f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{m}\geq\mathbf{k}} f_{\mathbf{mk}}\, \mathbf{z}^{\mathbf{m}}, \qquad g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\leq\mathbf{k}} g_{\mathbf{kl}}\, \mathbf{z}^{-\mathbf{l}}.$$

Assume that

(i) there exist operators $U_j, V \in \mathrm{End}(\mathcal{L})$, $V$ being bijective, such that for all $\mathbf{k} \in \mathsf{Z}^n$, one has

$$U_j f_{\mathbf{k}}(\mathbf{z}) = c_j(\mathbf{k}) V f_{\mathbf{k}}(\mathbf{z}), \qquad 1 \leq j \leq n, \tag{2.1}$$

with $c_j(\mathbf{k})$ arbitrary sequences of constants;

(ii) for all $\mathbf{m} \neq \mathbf{k} \in \mathsf{Z}^n$, there exists some $j \in \{1,\dots,n\}$ with $c_j(\mathbf{m}) \neq c_j(\mathbf{k})$. (2.2)

**Lemma 2.1 (Krattenthaler).** *Suppose $h_{\mathbf{k}}(\mathbf{z})$ is a solution of the dual system*

$$U_j^* h_{\mathbf{k}}(\mathbf{z}) = c_j(\mathbf{k}) V^* h_{\mathbf{k}}(\mathbf{z}), \qquad 1 \leq j \leq n, \tag{2.3}$$

*with $h_{\mathbf{k}}(\mathbf{z}) \not\equiv 0$ for all $\mathbf{k} \in \mathsf{Z}^n$. Then we have*

$$g_{\mathbf{k}}(\mathbf{z}) = \frac{1}{\langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{k}}(\mathbf{z})\rangle} V^* h_{\mathbf{k}}(\mathbf{z}). \tag{2.4}$$

We shall use Lemma 2.1 as follows. For all $\mathbf{k} \in \mathsf{Z}^n$ and $1 \leq i,j \leq n$, let $W_i$, $V_{ij} \in \mathrm{End}(\mathcal{L})$ and $c_j(\mathbf{k})$ arbitrary constants. Assume that

(i) the operators $W_i, V_{ij}$ satisfy the commutation relations

$$V_{i_1 j} W_{i_2} = W_{i_2} V_{i_1 j}, \qquad i_1 \neq i_2, \quad 1 \leq i_1, i_2, j \leq n, \tag{2.5a}$$

$$V_{i_1 j_1} V_{i_2 j_2} = V_{i_2 j_2} V_{i_1 j_1}, \qquad i_1 \neq i_2, \quad 1 \leq i_1, i_2, j_1, j_2 \leq n, \tag{2.5b}$$

(ii) the constants $c_j(\mathbf{k})$ satisfy (2.2),

(iii) the operator $\det_{1\leq i,j\leq n}(V_{ij})$ is invertible.

**Corollary 2.2.** *Suppose that we have*

$$\sum_{j=1}^{n} c_j(\mathbf{k}) V_{ij} f_{\mathbf{k}}(\mathbf{z}) = W_i f_{\mathbf{k}}(\mathbf{z}), \qquad 1 \leq i \leq n, \tag{2.6}$$

*and that $h_{\mathbf{k}}(\mathbf{z})$ is a solution of*

$$\sum_{j=1}^{n} c_j(\mathbf{k}) V_{ij}^* h_{\mathbf{k}}(\mathbf{z}) = W_i^* h_{\mathbf{k}}(\mathbf{z}), \qquad 1 \leq i \leq n, \tag{2.7}$$

*with $h_{\mathbf{k}}(\mathbf{z}) \not\equiv 0$ for all $\mathbf{k} \in \mathsf{Z}^n$. Then we have*

$$g_{\mathbf{k}}(\mathbf{z}) = \frac{1}{\langle f_{\mathbf{k}}(\mathbf{z}), \det(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z})\rangle} \det(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}). \tag{2.8}$$

This corollary is a special case of [21, Cor. 2.14], already used in [12, Cor. 2.2]. For convenience we reproduce its short proof from Lemma 2.1.

*Proof of Corollary* 2.2. Due to (2.5b), we can apply Cramer's rule to (2.6) and obtain, for $1 \leq j \leq n$,

$$c_j(\mathbf{k}) \det_{1\leq i,l\leq n}(V_{il}) f_{\mathbf{k}}(\mathbf{z}) = \sum_{i=1}^{n} (-1)^{i+j} V^{(i,j)} W_i f_{\mathbf{k}}(\mathbf{z}),$$

$V^{(i,j)}$ being the minor of $(V_{st})_{1\leq s,t\leq n}$ with the $i$-th row and $j$-th column omitted. The dual system (in the sense of Lemma 2.1) writes as

$$\begin{aligned} c_j(\mathbf{k}) \det_{1\leq i,l\leq n}(V_{il}^*) h_{\mathbf{k}}(\mathbf{z}) &= \sum_{i=1}^{n} (-1)^{i+j} W_i^* V^{*(i,j)} h_{\mathbf{k}}(\mathbf{z}) \\ &= \sum_{i=1}^{n} (-1)^{i+j} V^{*(i,j)} W_i^* h_{\mathbf{k}}(\mathbf{z}), \end{aligned} \tag{2.9}$$

and is easily seen to be equivalent to (2.7). Note that condition (2.5b) justifies to write the dual of $\det(V_{il})$ as $\det(V_{il}^*)$, and similarly for $V^{(i,j)}$. Note also that, because of (2.5a), we may commute $W_i^*$ and $V^{*(i,j)}$ in (2.9). Now apply Lemma 2.1 with $V = \det(V_{ij})$ and $U_j = \sum_{i=1}^{n} (-1)^{i+j} V^{(i,j)} W_i$. □

In general, for any pair of inverse matrices $(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k}\in\mathsf{Z}^n}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathsf{Z}^n}$, and any sequence $(d_{\mathbf{k}})_{\mathbf{k}\in\mathsf{Z}^n}$ with $d_{\mathbf{k}} \neq 0$, a new pair of inverse matrices is obtained by multiplying the entries of $(f_{\mathbf{mk}})$ term-wise by $d_{\mathbf{m}}/d_{\mathbf{k}}$ and those of $(g_{\mathbf{kl}})$ term-wise by $d_{\mathbf{k}}/d_{\mathbf{l}}$. In such case, we shall say that we "transfer" the factor $d_{\mathbf{k}}$ from one matrix to the other. This procedure will be applied several times in Section 3.

# 3 New multidimensional matrix inversions

## 3.1 Extensions of Krattenthaler's matrix inverse

Let $a_k, c_k$ $(k \in \mathsf{Z})$ be arbitrary sequences of indeterminates. In [11] Krattenthaler proved that the two matrices

$$f_{mk} = \frac{\prod_{y=k}^{m-1}(a_y - c_k)}{\prod_{y=k+1}^{m}(c_y - c_k)}, \tag{3.1a}$$

$$g_{kl} = \frac{(a_l - c_l)}{(a_k - c_k)} \frac{\prod_{y=l+1}^{k}(a_y - c_k)}{\prod_{y=l}^{k-1}(c_y - c_k)}. \tag{3.1b}$$

are inverses of each other. By using the method developed in Section 2, we now derive two new multidimensional extensions of this result. We start with the following theorem.

**Theorem 3.1.** *Let $b$ be an indeterminate and $a_i(k), c_i(k)$ ($k \in \mathsf{Z}$, $1 \le i \le n$) be arbitrary sequences of indeterminates. Define*

$$f_{\mathbf{mk}} = \prod_{i=1}^{n} \frac{\prod_{y_i=k_i}^{m_i-1} \Big[ \big(a_i(y_i) - b/\prod_{j=1}^{n} c_j(k_j)\big) \prod_{j=1}^{n} \big(a_i(y_i) - c_j(k_j)\big) \Big]}{\prod_{y_i=k_i+1}^{m_i} \Big[ \big(c_i(y_i) - b/\prod_{j=1}^{n} c_j(k_j)\big) \prod_{j=1}^{n} \big(c_i(y_i) - c_j(k_j)\big) \Big]}, \tag{3.2a}$$

*and*

$$\begin{aligned} g_{\mathbf{kl}} = {} & \prod_{i=1}^{n} c_i(k_i)^{-1} \prod_{1 \le i < j \le n} \big(c_i(k_i) - c_j(k_j)\big)^{-1} \\ & \times \prod_{i=1}^{n} \prod_{y_i=l_i}^{k_i-1} \left[ \frac{\big(a_i(y_i) - b/\prod_{j=1}^{n} c_j(k_j)\big)}{\big(c_i(y_i) - b/\prod_{j=1}^{n} c_j(k_j)\big)} \prod_{j=1}^{n} \frac{\big(a_i(y_i) - c_j(k_j)\big)}{\big(c_i(y_i) - c_j(k_j)\big)} \right] \\ \times \det_{1 \le i,j \le n} & \left[ c_i(l_i)^{n-j+1} - a_i(l_i)^{n-j+1} \frac{\big(c_i(l_i) - b/\prod_{s=1}^{n} c_s(k_s)\big)}{\big(a_i(l_i) - b/\prod_{s=1}^{n} c_s(k_s)\big)} \prod_{s=1}^{n} \frac{\big(c_i(l_i) - c_s(k_s)\big)}{\big(a_i(l_i) - c_s(k_s)\big)} \right]. \end{aligned} \tag{3.2b}$$

*Then the infinite lower-triangular $n$-dimensional matrices $(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k} \in \mathsf{Z}^n}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l} \in \mathsf{Z}^n}$ are inverses of each other.*

*Remark* 3.2. This generalizes Krattenthaler's result [11] which is obtained for $n = 1$. Indeed for $n = 1$ the determinant in (3.2b) reduces (after relabeling) to

$$c_l - a_l \frac{(c_l - b/c_k)}{(a_l - b/c_k)} \frac{(c_l - c_k)}{(a_l - c_k)} = c_l \frac{(a_l - b/c_l)}{(a_l - b/c_k)} \frac{(a_l - c_l)}{(a_l - c_k)},$$

and the matrices in (3.2) (after relabeling) become

$$f_{mk} = \frac{\prod_{y=k}^{m-1} (a_y - b/c_k)(a_y - c_k)}{\prod_{y=k+1}^{m} (c_y - b/c_k)(c_y - c_k)}, \tag{3.3a}$$

$$g_{kl} = \frac{(b - a_l c_l)(a_l - c_l)}{(b - a_k c_k)(a_k - c_k)} \frac{\prod_{y=l+1}^{k} (a_y - b/c_k)(a_y - c_k)}{\prod_{y=l}^{k-1} (c_y - b/c_k)(c_y - c_k)}. \tag{3.3b}$$

It is not difficult to see that this matrix inverse is actually equivalent to its $b \to \infty$ special case, which is (3.1). To recover (3.3) from (3.1), do the substitutions $a_y \mapsto a_y + b/a_y$, $c_y \mapsto c_y + b/c_y$, transfer some factors from one matrix to the other, and simplify.

Other multidimensional extentions of Krattenthaler matrix inverse (3.1) have been obtained in [21, Th. 3.1], [12] and [22].

*Remark* 3.3. In case $a_i(k) = a$ for some constant $a$ (for all $k \in \mathsf{Z}$, $1 \le i \le n$), the determinant appearing in (3.2b) factors, due to the evaluation

$$\begin{aligned} & \det_{1 \le i,j \le n} \left[ c_i(l_i)^{n-j+1} - a^{n-j+1} \frac{\big(c_i(l_i) - b/\prod_{s=1}^{n} c_s(k_s)\big)}{\big(a - b/\prod_{s=1}^{n} c_s(k_s)\big)} \prod_{s=1}^{n} \frac{\big(c_i(l_i) - c_s(k_s)\big)}{\big(a - c_s(k_s)\big)} \right] \\ & \qquad = \frac{\big(a - b/\prod_{j=1}^{n} c_j(l_j)\big)}{\big(a - b/\prod_{j=1}^{n} c_j(k_j)\big)} \prod_{i=1}^{n} c_i(l_i) \frac{\big(a - c_i(l_i)\big)}{\big(a - c_i(k_i)\big)} \prod_{1 \le i < j \le n} \big(c_i(l_i) - c_j(l_j)\big), \end{aligned}$$

which was first proved in [21, Lemma A.1]. A slightly more general evaluation and a much quicker proof can be found in [22, Lemma A.1]. However, the resulting multidimensional matrix inversion is only the special case $a_t = a$ (for all $t \in \mathsf{Z}$) of [21, Th. 3.1].

*Proof of Theorem 3.1.* We apply the operator method of Section 2. From (3.2a), for all $\mathbf{m} \geq \mathbf{k}$ we deduce the recurrence

$$\begin{aligned}\big(c_i(m_i) - b/{\textstyle\prod_{j=1}^n} c_j(k_j)\big) \prod_{s=1}^n \big(c_i(m_i) - c_s(k_s)\big) f_{\mathbf{m}-\mathbf{e}_i,\mathbf{k}} \\ = \big(a_i(m_i-1) - b/{\textstyle\prod_{j=1}^n} c_j(k_j)\big) \prod_{s=1}^n \big(a_i(m_i-1) - c_s(k_s)\big) f_{\mathbf{m},\mathbf{k}}, \qquad 1 \leq i \leq n, \quad (3.4)\end{aligned}$$

where $\mathbf{e}_i \in \mathsf{Z}^n$ has all components zero except its $i$-th component equal to 1. We write

$$\begin{aligned} f_{\mathbf{k}}(\mathbf{z}) &= \sum_{\mathbf{m} \geq \mathbf{k}} f_{\mathbf{mk}} \mathbf{z}^{\mathbf{m}} \\ &= \sum_{\mathbf{m}\geq\mathbf{k}} \prod_{i=1}^n \frac{\prod_{y_i=k_i}^{m_i-1} \Big[\big(a_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big) \prod_{j=1}^n \big(a_i(y_i) - c_j(k_j)\big)\Big]}{\prod_{y_i=k_i+1}^{m_i} \Big[\big(c_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big) \prod_{j=1}^n \big(c_i(y_i) - c_j(k_j)\big)\Big]} \mathbf{z}^{\mathbf{m}}. \end{aligned}$$

We define linear operators $\mathcal{A}_i$ and $\mathcal{C}_i$ by $\mathcal{A}_i \mathbf{z}^{\mathbf{m}} = a_i(m_i)\mathbf{z}^{\mathbf{m}}$ and $\mathcal{C}_i \mathbf{z}^{\mathbf{m}} = c_i(m_i)\mathbf{z}^{\mathbf{m}}$ ($1 \leq i \leq n$). Then we may write (3.4) in the form

$$\begin{aligned}\big(\mathcal{C}_i - b/{\textstyle\prod_{j=1}^n} c_j(k_j)\big) \prod_{s=1}^n \big(\mathcal{C}_i - c_s(k_s)\big) f_{\mathbf{k}}(\mathbf{z}) \\ = z_i\big(\mathcal{A}_i - b/{\textstyle\prod_{j=1}^n} c_j(k_j)\big) \prod_{s=1}^n \big(\mathcal{A}_i - c_s(k_s)\big) f_{\mathbf{k}}(\mathbf{z}), \quad (3.5)\end{aligned}$$

valid for all $1 \leq i \leq n$ and $\mathbf{k} \in \mathsf{Z}^n$.

In order to write this system of equations in a way such that Corollary 2.2 may be applied, we expand the products on both sides in terms of the elementary symmetric functions of order $j$,

$$e_j\big(c_1(k_1), c_2(k_2), \dots, c_n(k_n), b/{\textstyle\prod_{s=1}^n} c_s(k_s)\big),$$

which we denote $e_j(\mathbf{c}(\mathbf{k}))$ for short. The recurrence system (3.5) then reads, using $e_{n+1}(\mathbf{c}(\mathbf{k})) = b$,

$$\begin{aligned}\sum_{j=1}^n e_j(\mathbf{c}(\mathbf{k}))\big[(-\mathcal{C}_i)^{n-j+1} - z_i(-\mathcal{A}_i)^{n-j+1}\big] f_{\mathbf{k}}(\mathbf{z}) \\ = \big[z_i(-\mathcal{A}_i)^{n+1} + bz_i - (-\mathcal{C}_i)^{n+1} - b\big] f_{\mathbf{k}}(\mathbf{z}). \quad (3.6)\end{aligned}$$

Now (3.6) is a system of type (2.6) with

$$\begin{aligned} W_i &= [z_i(-\mathcal{A}_i)^{n+1} + bz_i - (-\mathcal{C}_i)^{n+1} - b], \\ V_{ij} &= [(-\mathcal{C}_i)^{n-j+1} - z_i(-\mathcal{A}_i)^{n-j+1}], \\ c_j(\mathbf{k}) &= e_j(\mathbf{c}(\mathbf{k})). \end{aligned}$$

Conditions (2.2) and (2.5) are satisfied. Hence we may apply Corollary 2.2. In this case the dual system (2.7) for the auxiliary formal Laurent series $h_{\mathbf{k}}(\mathbf{z})$ writes as

$$\sum_{j=1}^{n} e_j(\mathbf{c}(\mathbf{k}))\big[(-\mathcal{C}_i^*)^{n-j+1} - (-\mathcal{A}_i^*)^{n-j+1}z_i\big]h_{\mathbf{k}}(\mathbf{z}) \\ = \big[(-\mathcal{A}_i^*)^{n+1}z_i + bz_i - (-\mathcal{C}_i^*)^{n+1} - b\big]h_{\mathbf{k}}(\mathbf{z}), \quad 1 \le i \le n.$$

Equivalently, we have

$$\big(\mathcal{C}_i^* - b/\textstyle\prod_{j=1}^n c_j(k_j)\big)\displaystyle\prod_{s=1}^{n}\big(\mathcal{C}_i^* - c_s(k_s)\big)h_{\mathbf{k}}(\mathbf{z}) \\ = \big(\mathcal{A}_i^* - b/\textstyle\prod_{j=1}^n c_j(k_j)\big)\displaystyle\prod_{s=1}^{n}\big(\mathcal{A}_i^* - c_s(k_s)\big)z_i h_{\mathbf{k}}(\mathbf{z}), \quad (3.7)$$

valid for all $1 \le i \le n$ and $\mathbf{k} \in \mathsf{Z}^n$. As is easily seen, we have $\mathcal{A}_i^*\mathbf{z}^{-\mathbf{l}} = a_i(l_i)\mathbf{z}^{-\mathbf{l}}$ and $\mathcal{C}_i^*\mathbf{z}^{-\mathbf{l}} = c_i(l_i)\mathbf{z}^{-\mathbf{l}}$. Thus, writing $h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} h_{\mathbf{kl}}\ \mathbf{z}^{-\mathbf{l}}$ and comparing coefficients of $\mathbf{z}^{-\mathbf{l}}$ in (3.7), we obtain

$$\big(c_i(l_i) - b/\textstyle\prod_{j=1}^n c_j(k_j)\big)\displaystyle\prod_{s=1}^{n}\big(c_i(l_i) - c_s(k_s)\big)h_{\mathbf{kl}} \\ = \big(a_i(l_i) - b/\textstyle\prod_{j=1}^n c_j(k_j)\big)\displaystyle\prod_{s=1}^{n}\big(a_i(l_i) - c_s(k_s)\big)h_{\mathbf{k},\mathbf{l}+\mathbf{e}_i}.$$

If we set $h_{\mathbf{kk}} = 1$, we get

$$h_{\mathbf{kl}} = \prod_{i=1}^{n}\prod_{y_i=l_i}^{k_i-1}\left[\frac{\big(a_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)}{\big(c_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)}\prod_{j=1}^{n}\frac{\big(a_i(y_i) - c_j(k_j)\big)}{\big(c_i(y_i) - c_j(k_j)\big)}\right].$$

Now taking into account (2.8), we have to compute the action of

$$\det_{1\le i,j\le n}(V_{ij}^*) = \det_{1\le i,j\le n}\big[(-\mathcal{C}_i^*)^{n-j+1} - (-\mathcal{A}_i^*)^{n-j+1}z_i\big]$$

when applied to

$$h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}}\prod_{i=1}^{n}\prod_{y_i=l_i}^{k_i-1}\left[\frac{\big(a_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)}{\big(c_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)}\prod_{j=1}^{n}\frac{\big(a_i(y_i) - c_j(k_j)\big)}{\big(c_i(y_i) - c_j(k_j)\big)}\right]\mathbf{z}^{-\mathbf{l}}.$$

Since

$$z_i h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} h_{\mathbf{kl}}\ \mathbf{z}^{-\mathbf{l}} \frac{\big(c_i(l_i) - b/\prod_{j=1}^n c_j(k_j)\big)}{\big(a_i(l_i) - b/\prod_{j=1}^n c_j(k_j)\big)} \prod_{j=1}^n \frac{\big(c_i(l_i) - c_j(k_j)\big)}{\big(a_i(l_i) - c_j(k_j)\big)},$$

we obtain

$$\det_{1\le i,j\le n}(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} h_{\mathbf{kl}}\ \mathbf{z}^{-\mathbf{l}}$$
$$\times \det_{1\le i,j\le n}\left[(-c_i(l_i))^{n-j+1} - (-a_i(l_i))^{n-j+1} \frac{\big(c_i(l_i) - b/\prod_{j=1}^n c_j(k_j)\big)}{\big(a_i(l_i) - b/\prod_{j=1}^n c_j(k_j)\big)} \prod_{j=1}^n \frac{\big(c_i(l_i) - c_j(k_j)\big)}{\big(a_i(l_i) - c_j(k_j)\big)}\right].$$

Note that since $f_{\mathbf{kk}} = 1$, the pairing $\langle f_{\mathbf{k}}(\mathbf{z}), \det(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z})\rangle$ is simply the coefficient of $\mathbf{z}^{-\mathbf{k}}$ in the above expression, i.e. $\det_{1\le i,j\le n}[(-c_i(k_i))^{n-j+1}]$. Thus equation (2.8) writes as

$$g_{\mathbf{k}}(\mathbf{z}) = \prod_{1\le i<j\le n} \big(c_j(k_j) - c_i(k_i)\big)^{-1} \prod_{i=1}^n \big(-c_i(k_i)\big)^{-1} \det_{1\le i,j\le n}(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}).$$

Since $g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}}\ \mathbf{z}^{-\mathbf{l}}$, we conclude by extracting the coefficient of $\mathbf{z}^{-\mathbf{l}}$ in $g_{\mathbf{k}}(\mathbf{z})$. $\square$

Surprisingly, although the determinant appearing in (3.2b) depends on both $\mathbf{k}$ and $\mathbf{l}$, one can virtually "transfer" this determinant from $g_{\mathbf{kl}}$ to $f_{\mathbf{mk}}$. Of course, this requires a proof in the particular situation. The following corresponding theorem is another multidimensional generalization of Krattenthaler's result [11].

**Theorem 3.4.** *Let $b$ be an indeterminate and $a_i(k), c_i(k)$ ($k \in \mathsf{Z}$, $1 \le i \le n$) be arbitrary sequences of indeterminates. Define*

$$f_{\mathbf{mk}} = \prod_{i=1}^n c_i(k_i)^{-1} \prod_{1\le i<j\le n} \big(c_i(k_i) - c_j(k_j)\big)^{-1}$$
$$\times \prod_{i=1}^n \prod_{y_i=k_i+1}^{m_i} \left[\frac{\big(a_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)}{\big(c_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)} \prod_{j=1}^n \frac{\big(a_i(y_i) - c_j(k_j)\big)}{\big(c_i(y_i) - c_j(k_j)\big)}\right]$$
$$\times \det_{1\le i,j\le n}\left[c_i(m_i)^{n-j+1} - a_i(m_i)^{n-j+1} \frac{\big(c_i(m_i) - b/\prod_{s=1}^n c_s(k_s)\big)}{\big(a_i(m_i) - b/\prod_{s=1}^n c_s(k_s)\big)} \prod_{s=1}^n \frac{\big(c_i(m_i) - c_s(k_s)\big)}{\big(a_i(m_i) - c_s(k_s)\big)}\right],$$

*and*

$$g_{\mathbf{kl}} = \prod_{i=1}^n \frac{\prod_{y_i=l_i+1}^{k_i}\Big[\big(a_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)\prod_{j=1}^n \big(a_i(y_i) - c_j(k_j)\big)\Big]}{\prod_{y_i=l_i}^{k_i-1}\Big[\big(c_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)\prod_{j=1}^n \big(c_i(y_i) - c_j(k_j)\big)\Big]}.$$

*Then the infinite lower-triangular $n$-dimensional matrices $(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k}\in\mathsf{Z}^n}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathsf{Z}^n}$ are inverses of each other.*

*Proof.* For any multi-integer $\mathbf{k} = (k_1, \dots, k_n)$, denote $-\mathbf{k} = (-k_1, \dots, -k_n)$. Define two multidimensional matrices $\tilde{g}_{\mathbf{mk}}$ and $\tilde{f}_{\mathbf{kl}}$ by $\tilde{g}_{\mathbf{mk}} = f_{-\mathbf{k},-\mathbf{m}}$ and $\tilde{f}_{\mathbf{kl}} = g_{-\mathbf{l},-\mathbf{k}}$. For $1 \le i \le n$ write $\tilde{a}_i(y_i) = a_i(-y_i)$ and $\tilde{c}_i(y_i) = c_i(-y_i)$. Then the matrices $\tilde{g}_{\mathbf{mk}}$ and $\tilde{f}_{\mathbf{kl}}$ are those considered in Theorem 3.1, associated to the sequences $\tilde{a}_i(k), \tilde{c}_i(k)$. Thus for all $\mathbf{m}, \mathbf{l} \in \mathsf{Z}^n$, we have

$$\sum_{\mathbf{m} \ge \mathbf{k} \ge \mathbf{l}} f_{\mathbf{mk}}\, g_{\mathbf{kl}} = \sum_{\mathbf{m} \ge \mathbf{k} \ge \mathbf{l}} \tilde{f}_{-\mathbf{l},-\mathbf{k}}\, \tilde{g}_{-\mathbf{k},-\mathbf{m}} = \delta_{\mathbf{ml}}.$$

□

For possible future reference, we now give two special cases of Theorems 3.1 and 3.4. These two corollaries are derived by the method used to get (3.3) from (3.1). Both are themselves multidimensional generalizations of (3.3).

**Corollary 3.5.** *Let $b$ be an indeterminate and $a_i(k), c_i(k)$ ($k \in \mathsf{Z}$, $1 \le i \le n$) be arbitrary sequences of indeterminates. Define*

$$f_{\mathbf{mk}} = \prod_{i,j=1}^{n} \frac{\prod_{y_i=k_i}^{m_i-1} \left[ \big(a_i(y_i) - b/c_j(k_j)\big)\big(a_i(y_i) - c_j(k_j)\big) \right]}{\prod_{y_i=k_i+1}^{m_i} \left[ \big(c_i(y_i) - b/c_j(k_j)\big)\big(c_i(y_i) - c_j(k_j)\big) \right]},$$

*and*

$$\begin{aligned} g_{\mathbf{kl}} = {} & \prod_{i=1}^{n} \frac{c_i(l_i)}{c_i(k_i)} \big(c_i(k_i) + b/c_i(k_i)\big)^{-1} \prod_{1 \le i < j \le n} \left[ \big(1 - b/c_i(k_i)c_j(k_j)\big)\big(c_i(k_i) - c_j(k_j)\big) \right]^{-1} \\ & \times \prod_{i,j=1}^{n} \prod_{y_i=l_i}^{k_i-1} \left[ \frac{\big(a_i(y_i) - b/c_j(k_j)\big)\big(a_i(y_i) - c_j(k_j)\big)}{\big(c_i(y_i) - b/c_j(k_j)\big)\big(c_i(y_i) - c_j(k_j)\big)} \right] \\ & \times \det_{1 \le i,j \le n} \Bigg[ \big(c_i(l_i) + b/c_i(l_i)\big)^{n-j+1} - \big(a_i(l_i) + b/a_i(l_i)\big)^{n-j+1} \\ & \qquad \times \prod_{s=1}^{n} \frac{\big(1 - b/c_i(l_i)c_s(k_s)\big)\big(c_i(l_i) - c_s(k_s)\big)}{\big(1 - b/a_i(l_i)c_s(k_s)\big)\big(a_i(l_i) - c_s(k_s)\big)} \Bigg]. \end{aligned}$$

*Then the infinite lower-triangular $n$-dimensional matrices $(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k} \in \mathsf{Z}^n}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l} \in \mathsf{Z}^n}$ are inverses of each other.*

*Proof.* In Theorem 3.1, first let $b \to \infty$, then perform the substitutions $a_i(y_i) \mapsto a_i(y_i) + b/a_i(y_i)$ and $c_i(y_i) \mapsto c_i(y_i) + b/c_i(y_i)$, for $1 \le i \le n$. Finally, transfer some factors from one matrix to the other. □

Starting from Theorem 3.4, the following result is proved identically.

**Corollary 3.6.** *Let $b$ be an indeterminate and $a_i(k), c_i(k)$ ($k \in \mathsf{Z}$, $1 \le i \le n$) be arbitrary sequences of indeterminates. Define*

$$f_{\mathbf{mk}} = \prod_{i=1}^{n} \frac{c_i(m_i)}{c_i(k_i)} \big(c_i(k_i) + b/c_i(k_i)\big)^{-1} \prod_{1\le i<j\le n} \big[\big(1 - b/c_i(k_i)c_j(k_j)\big)\big(c_i(k_i) - c_j(k_j)\big)\big]^{-1}$$
$$\times \prod_{i,j=1}^{n} \prod_{y_i=k_i+1}^{m_i} \left[\frac{\big(a_i(y_i) - b/c_j(k_j)\big)\big(a_i(y_i) - c_j(k_j)\big)}{\big(c_i(y_i) - b/c_j(k_j)\big)\big(c_i(y_i) - c_j(k_j)\big)}\right]$$
$$\times \det_{1\le i,j\le n}\Bigg[\big(c_i(m_i) + b/c_i(m_i)\big)^{n-j+1} - \big(a_i(m_i) + b/a_i(m_i)\big)^{n-j+1}$$
$$\times \prod_{s=1}^{n} \frac{\big(1 - b/c_i(m_i)c_s(k_s)\big)\big(c_i(m_i) - c_s(k_s)\big)}{\big(1 - b/a_i(m_i)c_s(k_s)\big)\big(a_i(m_i) - c_s(k_s)\big)}\Bigg],$$

*and*

$$g_{\mathbf{kl}} = \prod_{i,j=1}^{n} \frac{\prod_{y_i=l_i+1}^{k_i} \big[\big(a_i(y_i) - b/c_j(k_j)\big)\big(a_i(y_i) - c_j(k_j)\big)\big]}{\prod_{y_i=l_i}^{k_i-1} \big[\big(c_i(y_i) - b/c_j(k_j)\big)\big(c_i(y_i) - c_j(k_j)\big)\big]}.$$

*Then the infinite lower-triangular $n$-dimensional matrices $(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k}\in\mathsf{Z}^n}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathsf{Z}^n}$ are inverses of each other.*

## 3.2 An extension of Bressoud's matrix inverse

Let $q$ be an indeterminate. For any integer $k$, the classical $q$-shifted factorial $(a;q)_k$ is defined by

$$(a;q)_\infty = \prod_{j\ge 0}(1 - aq^j), \qquad (a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

Then we have the following important special case of Theorem 3.4.

**Corollary 3.7.** *Let $t_0, t_1, \ldots, t_n$ and $u_1, \ldots, u_n$ be indeterminates. Define*

$$f_{\mathbf{mk}} = \prod_{1\le i<j\le n} \big(q^{m_i}u_i - q^{m_j}u_j\big)^{-1} \prod_{i=1}^{n} t_i^{m_i-k_i} \frac{(q/t_i;q)_{m_i-k_i}}{(q;q)_{m_i-k_i}} \frac{(q^{k_i+|\mathbf{k}|+1}t_0u_i/t_i;q)_{m_i-k_i}}{(q^{k_i+|\mathbf{k}|+1}t_0u_i;q)_{m_i-k_i}}$$
$$\times \prod_{1\le i<j\le n} \frac{(q^{k_i-k_j+1}u_i/t_iu_j;q)_{m_i-k_i}}{(q^{k_i-k_j+1}u_i/u_j;q)_{m_i-k_i}} \frac{(q^{k_i-m_j}t_ju_i/u_j;q)_{m_i-k_i}}{(q^{k_i-m_j}u_i/u_j;q)_{m_i-k_i}}$$
$$\times \det_{1\le i,j\le n}\left[\big(q^{m_i}u_i\big)^{n-j}\left(1 - t_i^{j-n-1}\frac{\big(1 - q^{m_i+|\mathbf{k}|}t_0u_i\big)}{\big(1 - q^{m_i+|\mathbf{k}|}t_0u_i/t_i\big)} \prod_{s=1}^{n} \frac{\big(q^{m_i}u_i - q^{k_s}u_s\big)}{\big(q^{m_i}u_i/t_i - q^{k_s}u_s\big)}\right)\right],$$

*and*

$$g_{\mathbf{kl}} = \prod_{i=1}^n \frac{(t_i;q)_{k_i-l_i}}{(q;q)_{k_i-l_i}} \frac{(q^{l_i+|\mathbf{k}|+1}t_0u_i/t_i;q)_{k_i-l_i}}{(q^{l_i+|\mathbf{k}|}t_0u_i;q)_{k_i-l_i}}$$
$$\times \prod_{1\le i<j\le n} \frac{(q^{l_i-l_j}t_ju_i/u_j;q)_{k_i-l_i}}{(q^{l_i-l_j+1}u_i/u_j;q)_{k_i-l_i}} \frac{(q^{l_i-k_j+1}u_i/t_iu_j;q)_{k_i-l_i}}{(q^{l_i-k_j}u_i/u_j;q)_{k_i-l_i}}.$$

*Then the infinite lower-triangular $n$-dimensional matrices $(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k}\in\mathsf{Z}^n}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathsf{Z}^n}$ are inverses of each other.*

*Remark* 3.8. For $n=1$ this matrix inversion reduces to Bressoud's result [1], which he derived from the terminating very-well-poised ${}_6\phi_5$ summation [2, Eq. (II.21)].

*Proof.* We specialize Theorem 3.4 by letting $b \mapsto t_0^{-1}\prod_{j=1}^n u_j$, $a_i(y_i) \mapsto q^{y_i}u_i/t_i$, and $c_i(y_i) \mapsto q^{y_i}u_i$, for $1\le i\le n$, and rewrite the expressions using $q$-shifted factorials. After this first step, we obtain the inverse pair

$$f_{\mathbf{mk}} = q^{-|\mathbf{k}|}\prod_{i=1}^n u_i^{-1} \prod_{1\le i<j\le n}\big(q^{k_i}u_i - q^{k_j}u_j\big)^{-1}$$
$$\times \prod_{i=1}^n \frac{(q^{k_i+|\mathbf{k}|+1}t_0u_i/t_i;q)_{m_i-k_i}}{(q^{k_i+|\mathbf{k}|+1}t_0u_i;q)_{m_i-k_i}} \prod_{i,j=1}^n \frac{(q^{k_i-k_j+1}u_i/t_iu_j;q)_{m_i-k_i}}{(q^{k_i-k_j+1}u_i/u_j;q)_{m_i-k_i}}$$
$$\times \det_{1\le i,j\le n}\left[\big(q^{m_i}u_i\big)^{n-j}\left(1 - t_i^{j-n-1}\frac{\big(1-q^{m_i+|\mathbf{k}|}t_0u_i\big)}{\big(1-q^{m_i+|\mathbf{k}|}t_0u_i/t_i\big)}\prod_{s=1}^n \frac{\big(q^{m_i}u_i - q^{k_s}u_s\big)}{\big(q^{m_i}u_i/t_i - q^{k_s}u_s\big)}\right)\right],$$
$$g_{\mathbf{kl}} = \prod_{i=1}^n \frac{(q^{l_i+|\mathbf{k}|+1}t_0u_i/t_i;q)_{k_i-l_i}}{(q^{l_i+|\mathbf{k}|}t_0u_i;q)_{k_i-l_i}} \prod_{i,j=1}^n \frac{(q^{l_i-k_j+1}u_i/t_iu_j;q)_{k_i-l_i}}{(q^{l_i-k_j}u_i/u_j;q)_{k_i-l_i}}.$$

Now note that $f_{\mathbf{mk}}$ contains the factors

$$\prod_{i,j=1}^n \frac{(q^{k_i-k_j+1}u_i/t_iu_j;q)_{m_i-k_i}}{(q^{k_i-k_j+1}u_i/u_j;q)_{m_i-k_i}}$$
$$= \prod_{i=1}^n \frac{(q/t_i;q)_{m_i-k_i}}{(q;q)_{m_i-k_i}} \prod_{1\le i<j\le n} \frac{(q^{k_i-k_j+1}u_i/t_iu_j;q)_{m_i-k_i}}{(q^{k_i-k_j+1}u_i/u_j;q)_{m_i-k_i}} \frac{(q^{k_j-k_i+1}u_j/t_ju_i;q)_{m_j-k_j}}{(q^{k_j-k_i+1}u_j/u_i;q)_{m_j-k_j}}$$
$$= \prod_{i=1}^n \frac{(q/t_i;q)_{m_i-k_i}}{(q;q)_{m_i-k_i}} \prod_{1\le i<j\le n} t_j^{k_j-m_j} \frac{(q^{k_i-k_j+1}u_i/t_iu_j;q)_{m_i-k_i}}{(q^{k_i-k_j+1}u_i/u_j;q)_{m_i-k_i}} \frac{(q^{k_i-m_j}u_it_j/u_j;q)_{m_j-k_j}}{(q^{k_i-m_j}u_i/u_j;q)_{m_j-k_j}}.$$

Similarly $g_{\mathbf{kl}}$ contains

$$\prod_{i,j=1}^n \frac{(q^{l_i-k_j+1}u_i/t_iu_j;q)_{k_i-l_i}}{(q^{l_i-k_j}u_i/u_j;q)_{k_i-l_i}} = \prod_{i=1}^n \left(\frac{q}{t_i}\right)^{k_i-l_i} \frac{(t_i;q)_{k_i-l_i}}{(q;q)_{k_i-l_i}}$$
$$\times \prod_{1\le i<j\le n} \left(\frac{q}{t_j}\right)^{k_j-l_j} \frac{(q^{l_i-k_j+1}u_i/t_iu_j;q)_{k_i-l_i}}{(q^{l_i-k_j}u_i/u_j;q)_{k_i-l_i}} \frac{(q^{k_i-k_j}u_i/t_iu_j;q)_{k_j-l_j}}{(q^{k_i-k_j+1}u_i/u_j;q)_{k_j-l_j}}.$$

Finally, we transfer the factor

$$d_{\mathbf{k}} = q^{-|\mathbf{k}|} \prod_{i=1}^{n} t_i^{k_i} \prod_{1\le i<j\le n} \left(q^{k_i}u_i - q^{k_j}u_j\right)^{-1} t_j^{k_j} \frac{(t_ju_i/u_j;q)_{k_i-k_j}}{(u_i/u_j;q)_{k_i-k_j}},$$

from one matrix to the other, and simplify the resulting expressions. $\square$

# 4 Macdonald polynomials

The standard reference for Macdonald polynomials is Chapter 6 of [19].

## 4.1 Symmetric functions

Let $X = \{x_1, x_2, x_3, \ldots\}$ be an infinite set of indeterminates, and $\mathcal{S}$ the corresponding algebra of symmetric functions with coefficients in $\mathsf{Q}$. Let $\mathsf{Q}[q,t]$ be the field of rational functions in two indeterminates $q, t$, and $\mathsf{Sym} = \mathcal{S} \otimes \mathsf{Q}[q,t]$ the algebra of symmetric functions with coefficients in $\mathsf{Q}[q,t]$.

The power sum symmetric functions are defined by $p_k(X) = \sum_{i\ge1} x_i^k$. Elementary and complete symmetric functions $e_k(X)$ and $h_k(X)$ are defined by their generating functions

$$\prod_{i\ge1}(1+ux_i) = \sum_{k\ge0} u^k\, e_k(X), \qquad \prod_{i\ge1} \frac{1}{1-ux_i} = \sum_{k\ge0} u^k\, h_k(X).$$

Each of these three sets form an algebraic basis of $\mathsf{Sym}$, which can thus be viewed as an abstract algebra over $\mathsf{Q}[q,t]$ generated by functions $e_k$, $h_k$ or $p_k$.

A partition $\lambda = (\lambda_1, ..., \lambda_n)$ is a finite weakly decreasing sequence of nonnegative integers, called parts. The number $l(\lambda)$ of positive parts is called the length of $\lambda$, and $|\lambda| = \sum_{i=1}^n \lambda_i$ the weight of $\lambda$. For any integer $i \ge 1$, $m_i(\lambda) = \mathrm{card}\{j : \lambda_j = i\}$ is the multiplicity of the part $i$ in $\lambda$. Clearly $l(\lambda) = \sum_{i\ge1} m_i(\lambda)$ and $|\lambda| = \sum_{i\ge1} i m_i(\lambda)$. We shall also write $\lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots)$. We set

$$z_\lambda = \prod_{i\ge1} i^{m_i(\lambda)} m_i(\lambda)!.$$

We denote $\lambda'$ the partition conjugate to $\lambda$, whose parts are given by $m_i(\lambda') = \lambda_i - \lambda_{i+1}$. We have $\lambda'_i = \sum_{j\ge i} m_j(\lambda)$.

For any partition $\lambda$, the symmetric functions $e_\lambda$, $h_\lambda$ and $p_\lambda$ defined by

$$f_\lambda = \prod_{i=1}^{l(\lambda)} f_{\lambda_i} = \prod_{i\ge1} (f_i)^{m_i(\lambda)}, \tag{4.1}$$

where $f_i$ stands for $e_i$, $h_i$ or $p_i$ respectively, form a linear basis of $\mathsf{Sym}$. Another classical basis is formed by the monomial symmetric functions $m_\lambda$, defined as the sum of all distinct monomials whose exponent is a permutation of $\lambda$.

For all $k \geq 0$, the "modified complete" symmetric function $g_k(X;q,t)$ is defined by the generating series

$$\prod_{i\geq 1} \frac{(tux_i;q)_\infty}{(ux_i;q)_\infty} = \sum_{k\geq 0} u^k g_k(X;q,t).$$

It is often written in $\lambda$-ring notation [16, p. 223], that is

$$g_k(X;q,t) = h_k\left[\frac{1-t}{1-q}\,X\right].$$

The symmetric functions $g_k(q,t)$ form an algebraic basis of $\mathsf{Sym}$. They may be expanded in terms of any classical basis. This development is explicitly given in [19, pp. 311 and 314] in terms of power sums and monomial symmetric functions, and in [16, Sec. 10, p. 237] in terms of other classical bases. The functions $g_\lambda(q,t)$, defined as in (4.1), form a linear basis of $\mathsf{Sym}$.

## 4.2 Macdonald operators

We now restrict to the case of a finite set of indeterminates $X = \{x_1,\ldots,x_n\}$. Let $T_{q,x_i}$ denote the $q$-deformation operator defined by

$$T_{q,x_i} f(x_1,\ldots,x_n) = f(x_1,\ldots,qx_i,\ldots,x_n),$$

and for all $1 \leq i \leq n$,

$$A_i(X;t) = \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{tx_i - x_j}{x_i - x_j}.$$

Macdonald polynomials $P_\lambda(X;q,t)$, with $\lambda$ a partition such that $l(\lambda) \leq n$, are defined as the eigenvectors of the following difference operator

$$E(X;q,t) = \sum_{i=1}^{n} A_i(X;t)\, T_{q,x_i}.$$

One has

$$E(X;q,t)\, P_\lambda(X;q,t) = \left(\sum_{i=1}^{n} q^{\lambda_i}\, t^{n-i}\right) P_\lambda(X;q,t).$$

Let $\Delta(X)$ be the Vandermonde determinant $\prod_{1\leq i<j\leq n}(x_i - x_j)$. More generally Macdonald polynomials $P_\lambda(X;q,t)$ are eigenvectors of the difference operator

$$D(u;q,t) = \frac{1}{\Delta(X)} \det_{1\leq i,j\leq n}\left[x_i^{n-j}\left(1 + ut^{n-j}T_{q,x_i}\right)\right],$$

where $u$ is some indeterminate. One has

$$D(u;q,t)\, P_\lambda(X;q,t) = \prod_{i=1}^{n}\left(1 + u\, q^{\lambda_i}\, t^{n-i}\right) P_\lambda(X;q,t).$$

The polynomials $P_\lambda(X;q,t)$ define symmetric functions, which form an orthogonal basis of $\mathsf{Sym}$ with respect to the scalar product $<\,,\,>_{q,t}$ defined by

$$< p_\lambda, p_\mu >_{q,t} = \delta_{\lambda\mu}\, z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Equivalently if $Y = \{y_1, \ldots, y_m\}$ is another set of $m$ indeterminates, and

$$\Pi(X,Y;q,t) = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty},$$

we have

$$\Pi(X,Y;q,t) = \sum_\lambda P_\lambda(X;q,t) Q_\lambda(Y;q,t),$$

where $Q_\lambda(X;q,t)$ denotes the dual basis of $P_\lambda(X;q,t)$ for the scalar product $<\,,\,>_{q,t}$. One has

$$Q_\lambda(X;q,t) = b_\lambda(q,t)\, P_\lambda(X;q,t), \tag{4.2}$$

with $b_\lambda(q,t) = {< P_\lambda(q,t), P_\lambda(q,t) >_{q,t}}^{-1}$ given by

$$b_\lambda(q,t) = \prod_{1\le i\le j\le l(\lambda)} \frac{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_{\lambda_j-\lambda_{j+1}}}{(q^{\lambda_i-\lambda_j+1}t^{j-i};q)_{\lambda_j-\lambda_{j+1}}}.$$

As shown in [19, p. 315], we have

$$D(u;q,t) = \sum_{K\subset\{1,\ldots,n\}} u^{|K|} t^{\binom{|K|}{2}} \prod_{\substack{k\in K\\ j\notin K}} \frac{tx_k - x_j}{x_k - x_j} \prod_{k\in K} T_{q,x_k}.$$

This yields

$$\frac{1}{\Delta(X)} \det_{1\le i,j\le n}\left[ x_i^{n-j}\left(1 + ut^{n-j}\prod_{k=1}^{m}\frac{1-x_iy_k}{1-tx_iy_k}\right)\right]$$
$$= \sum_{K\subset\{1,\ldots,n\}} u^{|K|} t^{\binom{|K|}{2}} \prod_{\substack{k\in K\\ j\notin K}} \frac{tx_k - x_j}{x_k - x_j} \prod_{i=1}^{m}\prod_{k\in K}\frac{1-x_ky_i}{1-tx_ky_i}. \tag{4.3}$$

Indeed since

$$\Pi^{-1}\, T_{q,x_i}\, \Pi = \prod_{k=1}^{m} \frac{1-x_iy_k}{1-tx_iy_k}$$

both terms are obviously $\Pi^{-1}\, D(u;q,t)_{(X)}\, \Pi$, where the suffix $(X)$ indicates operation on the $X$ variables.

There exists an automorphism $\omega_{q,t} = {\omega_{t,q}}^{-1}$ of $\mathsf{Sym}$ such that

$$\omega_{q,t}(Q_\lambda(q,t)) = P_{\lambda'}(t,q), \qquad \omega_{q,t}(g_k(q,t)) = e_k. \tag{4.4}$$

In particular the Macdonald symmetric functions associated with a row or a column partition are given by

$$P_{1^k}(q,t) = e_k, \qquad Q_{1^k}(q,t) = \frac{(t;t)_k}{(q;t)_k}\, e_k$$
$$P_{(k)}(q,t) = \frac{(q;q)_k}{(t;q)_k}\, g_k(q,t), \qquad Q_{(k)}(q,t) = g_k(q,t).$$

The parameters $q,t$ being kept fixed, we shall often write $P_\mu$ or $Q_\mu$ for $P_\mu(q,t)$ or $Q_\mu(q,t)$.

## 4.3 Pieri formula

Let $u_1,\ldots,u_n$ be $n$ indeterminates and $\mathsf{N}$ the set of nonnegative integers. For $\theta = (\theta_1,\ldots,\theta_n) \in \mathsf{N}^n$, let $|\theta| = \sum_{i=1}^n \theta_i$ and define

$$d_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n) = \prod_{k=1}^n \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(q^{|\theta|+1}u_k;q)_{\theta_k}}{(q^{|\theta|}tu_k;q)_{\theta_k}} \prod_{1\le i<j\le n} \frac{(tu_i/u_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}} \frac{(q^{-\theta_j+1}u_i/tu_j;q)_{\theta_i}}{(q^{-\theta_j}u_i/u_j;q)_{\theta_i}}.$$

If we set $u_{n+1} = 1/t$, $\theta_{n+1} = -|\theta|$, and $v_k = q^{\theta_k}u_k$ $(1 \le k \le n+1)$, we may write

$$d_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n) = \prod_{1\le i\le j\le n} \frac{(tu_i/u_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}} \prod_{1\le i<j\le n+1} \frac{(qu_i/tv_j;q)_{\theta_i}}{(u_i/v_j;q)_{\theta_i}}.$$

Macdonald symmetric functions satisfy a Pieri formula generalizing the classical Pieri formula for Schur functions. This generalization was obtained by Macdonald [19, p. 331], and independently by Koornwinder [9].

Most of the time this Pieri formula is stated in combinatorial terms. Its analytic form is less popular, but will be crucial for our purposes.

**Theorem 4.1.** *Let $\lambda = (\lambda_1, ..., \lambda_n)$ be an arbitrary partition with length $n$ and $\lambda_{n+1} \in \mathsf{N}$. For any $1 \le k \le n+1$ define $u_k = q^{\lambda_k-\lambda_{n+1}}t^{n-k}$. We have*

$$Q_{(\lambda_1,\ldots,\lambda_n)}\, Q_{(\lambda_{n+1})} = \sum_{\theta\in\mathsf{N}^n} d_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n)\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n,\lambda_{n+1}-|\theta|)}.$$

*Proof.* We make use of the expressions given in [19, p. 340, Eq. (6.24)(ii)] and [19, p. 342, Example 2(b)]. Specifically, we write

$$Q_\lambda\, Q_{(\lambda_{n+1})} = \sum_{\kappa\supset\lambda} \psi_{\kappa/\lambda}\, Q_\kappa,$$

where the skew-diagram $\kappa - \lambda$ is a horizontal $\lambda_{n+1}$-strip, i.e. has at most one square in each column, and $\psi_{\kappa/\lambda}$ is given by

$$\psi_{\kappa/\lambda} = \prod_{1\le i\le j\le n} \frac{f(q^{\lambda_i-\lambda_j}t^{j-i})}{f(q^{\kappa_i-\lambda_j}t^{j-i})} \frac{f(q^{\kappa_i-\kappa_{j+1}}t^{j-i})}{f(q^{\lambda_i-\kappa_{j+1}}t^{j-i})} = \prod_{1\le i\le j\le n} \frac{w_{\kappa_i-\lambda_i}(q^{\lambda_i-\lambda_j}t^{j-i})}{w_{\kappa_i-\lambda_i}(q^{\lambda_i-\kappa_{j+1}}t^{j-i})},$$

with $f(u) = (tu;q)_\infty/(qu;q)_\infty$ and $w_r(u) = (tu;q)_r/(qu;q)_r$. Since $\kappa - \lambda$ is a horizontal strip, the length of $\kappa$ is at most equal to $n+1$, so we can write $\kappa = (\lambda_1+\theta_1, \dots, \lambda_n+\theta_n, \lambda_{n+1}-|\theta|)$. Then

$$\psi_{\kappa/\lambda} = \prod_{1\le i\le j\le n} w_{\theta_i}(q^{\lambda_i-\lambda_j}t^{j-i}) \prod_{1\le i<j\le n+1} \left(w_{\theta_i}(q^{\lambda_i-\kappa_j}t^{j-i-1})\right)^{-1},$$

which is the statement. ☐

*Remark* 4.2. Theorem 4.1 translates into analytic terms the fact that $\kappa - \lambda$ must be a horizontal strip: the $q$-products in the numerator of $d_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)$ vanish if $\kappa-\lambda$ is not a horizontal strip. However, the fact that $\kappa$ must be a partition is not given any analytic translation: $d_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)$ does *not* vanish if $(\lambda_1+\theta_1,\dots,\lambda_n+\theta_n,\lambda_{n+1}-|\theta|)$ is not a partition. Thus Theorem 4.1 implicitly assumes that $Q_\kappa = 0$ if $\kappa$ is not a partition. This fact will be important in Subsection 11.1.

The Pieri formula defines an infinite transition matrix. Indeed, the Macdonald symmetric functions $\{Q_\lambda\}$ form a basis of $\mathsf{Sym}$, and so do the products $\{Q_\mu Q_{(r)}\}$. We shall now compute the inverse of this matrix explicitly.

# 5 Main result

Let $u = (u_1,\dots,u_n)$ be $n$ indeterminates and $\theta = (\theta_1,\dots,\theta_n) \in \mathsf{N}^n$. For clarity of notations, we introduce $n$ auxiliary variables $v = (v_1,\dots,v_n)$ defined by $v_k = q^{\theta_k}u_k$. We write

$$C^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n) = \prod_{k=1}^n t^{\theta_k} \frac{(q/t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(qu_k;q)_{\theta_k}}{(qtu_k;q)_{\theta_k}} \prod_{1\le i<j\le n} \frac{(qu_i/tu_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}} \frac{(tu_i/v_j;q)_{\theta_i}}{(u_i/v_j;q)_{\theta_i}}$$
$$\times \frac{1}{\Delta(v)} \det_{1\le i,j\le n}\left[v_i^{n-j}\left(1-t^{j-1}\frac{1-tv_i}{1-v_i}\prod_{k=1}^n \frac{u_k-v_i}{tu_k-v_i}\right)\right].$$

Setting $u_{n+1} = 1/t$ we have

$$C^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n) = \prod_{1\le i<j\le n+1} \frac{(qu_i/tu_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}} \prod_{1\le i\le j\le n} \frac{(tu_i/v_j;q)_{\theta_i}}{(u_i/v_j;q)_{\theta_i}}$$
$$\times \frac{1}{\Delta(v)} \det_{1\le i,j\le n}\left[v_i^{n-j}\left(1-t^j\prod_{k=1}^{n+1} \frac{u_k-v_i}{tu_k-v_i}\right)\right].$$

We are now in a position to prove our main result.

**Theorem 5.1.** *Let $\lambda=(\lambda_1,...,\lambda_{n+1})$ be an arbitrary partition with length $n+1$. For any $1\le k\le n+1$ define $u_k=q^{\lambda_k-\lambda_{n+1}}t^{n-k}$. We have*

$$Q_{(\lambda_1,\dots,\lambda_{n+1})}=\sum_{\theta\in\mathsf{N}^n} C^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)\,Q_{(\lambda_{n+1}-|\theta|)}\,Q_{(\lambda_1+\theta_1,\dots,\lambda_n+\theta_n)}.$$

*Proof.* Let $\beta=(\beta_1,\dots,\beta_n)$, $\kappa=(\kappa_1,\dots,\kappa_n)$, $\gamma=(\gamma_1,\dots,\gamma_n)\in\mathsf{Z}^n$. Defining

$$\begin{aligned}
f_{\beta\kappa}&=C^{(q,t)}_{\beta_1-\kappa_1,\dots,\beta_n-\kappa_n}\big(q^{\kappa_1+|\kappa|}u_1,\dots,q^{\kappa_n+|\kappa|}u_n\big),\\
g_{\kappa\gamma}&=d_{\kappa_1-\gamma_1,\dots,\kappa_n-\gamma_n}\big(q^{\gamma_1+|\gamma|}u_1,\dots,q^{\gamma_n+|\gamma|}u_n\big),
\end{aligned}$$

these infinite lower-triangular $n$-dimensional matrices are inverses of each other, by application of Corollary 3.7 written with $t_k=t$ for $0\le k\le n$. Now if in Theorem 4.1, we replace $\lambda_{n+1}$ by $\lambda_{n+1}-|\gamma|$ and (for $1\le i\le n$) $\lambda_i$ by $\lambda_i+\gamma_i$, $u_i$ by $q^{\gamma_i+|\gamma|}u_i$, we obtain (after shifting the summation indices)

$$\sum_{\kappa\in\mathsf{Z}^n} g_{\kappa\gamma}y_\kappa=w_\gamma \qquad (\gamma\in\mathsf{Z}^n),$$

with

$$\begin{aligned}
y_\kappa&=Q_{(\lambda_1+\kappa_1,\dots,\lambda_n+\kappa_n,\lambda_{n+1}-|\kappa|)},\\
w_\gamma&=Q_{(\lambda_1+\gamma_1,\dots,\lambda_n+\gamma_n)}Q_{(\lambda_{n+1}-|\gamma|)}.
\end{aligned}$$

This immediately yields

$$\sum_{\beta\in\mathsf{Z}^n} f_{\beta\kappa}w_\beta=y_\kappa \qquad (\kappa\in\mathsf{Z}^n).$$

We conclude by setting $\kappa_i=0$ for all $1\le i\le n$. □

In the case $n=1$, i.e. for partitions of length 2, Theorem 5.1 reads

$$Q_{(\lambda_1,\lambda_2)}=\sum_{\theta\in\mathsf{N}} C^{(q,t)}_\theta(u)\,Q_{(\lambda_2-\theta)}\,Q_{(\lambda_1+\theta)}, \tag{5.1}$$

with $u=q^{\lambda_1-\lambda_2}$ and

$$\begin{aligned}
C^{(q,t)}_\theta(u)&=t^\theta\,\frac{(q/t;q)_\theta}{(q;q)_\theta}\,\frac{(qu;q)_\theta}{(qtu;q)_\theta}\Big(1-\frac{1-q^\theta tu}{1-q^\theta u}\,\frac{u-q^\theta u}{tu-q^\theta u}\Big)\\
&=t^\theta\,\frac{(q/t;q)_\theta}{(q;q)_\theta}\,\frac{(qu;q)_\theta}{(qtu;q)_\theta}\,\frac{t-1}{t-q^\theta}\,\frac{1-q^{2\theta}u}{1-q^\theta u}\\
&=t^\theta\,\frac{(1/t;q)_\theta}{(q;q)_\theta}\,\frac{(u;q)_\theta}{(qtu;q)_\theta}\,\frac{1-q^{2\theta}u}{1-u}.
\end{aligned}$$

We thus recover Jing and Jósefiak's result [5], which appears as a consequence of Bressoud's matrix inverse [1].

The reader may also verify that for $n=2$, i.e. for partitions of length 3, our result gives the formula stated in an earlier note by the first author [15].

Applying the automorphism $\omega_{q,t}$ to Theorem 5.1, and taking into account (4.4), we obtain the following equivalent result.

**Theorem 5.2.** *Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots, (n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. For any $1 \le k \le n+1$ define $u_k = q^{n-k} t^{\sum_{j=k}^{n} m_j}$. We have*

$$P_\lambda = \sum_{\theta \in \mathsf{N}^n} C^{(t,q)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n)\, e_{m_{n+1}-|\theta|}\, P_{(1^{m_1+\theta_1-\theta_2},\ldots,(n-1)^{m_{n-1}+\theta_{n-1}-\theta_n}, n^{m_n+m_{n+1}+\theta_n})}.$$

*Remark* 5.3. Our proof of Theorem 5.1 looks somewhat external to Macdonald theory, and does not explain the particular form of $C^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n)$. Observe that its last factor may be written

$$\frac{1}{\Delta(v)} \det_{1\le i,j\le n}\left[ v_i^{n-j} \left( 1 - t^j \prod_{k=1}^{n+1} \frac{u_k - v_i}{t u_k - v_i} \right) \right]$$
$$= \sum_{K \subset \{1,\ldots,n\}} (-1)^{|K|} (1/t)^{\binom{|K|+1}{2}} \prod_{\substack{k\in K \\ j \notin K}} \frac{v_j - v_k/t}{v_j - v_k} \prod_{i=1}^{n+1} \prod_{k \in K} \frac{u_i - v_k}{u_i - v_k/t}. \quad (5.2)$$

This expression may be obtained from (4.3) by replacing $t$ by $1/t$, $u$ by $-1/t$, $X$ by $V = (v_1,\ldots,v_n)$, and $Y$ by $U = (1/u_1,\ldots,1/u_{n+1})$. If we write $\Pi$ for $\Pi(U,V,1/q,1/t)$, both sides of (5.2) are $\Pi^{-1}\, D(-1/t;1/q,1/t)_{(V)}\, \Pi$, where the suffix $(V)$ indicates operation on the $V$ variables. Unfortunately our proof of Theorem 5.1 does not provide any explanation for the mysterious occurrence of this Macdonald operator.

# 6 Analytic expansions

Theorems 5.1 and 5.2 immediately generate the analytic development of Macdonald polynomials in terms of the symmetric functions $g_k$ or $e_k$, which form two algebraic basis of $\mathsf{Sym}$.

Let $\mathsf{M}^{(n)}$ denote the set of upper triangular $n\times n$ matrices with nonnegative integers, and 0 on the diagonal. By a straightforward iteration of Theorem 5.1 we obtain the analytic development of Macdonald polynomials in terms of the symmetric functions $g_k$.

**Theorem 6.1.** *Let $\lambda = (\lambda_1, ..., \lambda_{n+1})$ be an arbitrary partition with length $n+1$. We have*

$$Q_\lambda(q,t) = \sum_{\theta \in \mathsf{M}^{(n+1)}} \prod_{k=1}^{n} C^{(q,t)}_{\theta_{1,k+1},\ldots,\theta_{k,k+1}}(\{u_i = q^{\lambda_i - \lambda_{k+1} + \sum_{j=k+2}^{n+1}(\theta_{i,j}-\theta_{k+1,j})} t^{k-i}; 1 \le i \le k\})$$
$$\times \prod_{k=1}^{n+1} g_{\lambda_k + \sum_{j=k+1}^{n+1}\theta_{kj} - \sum_{j=1}^{k-1}\theta_{jk}}.$$

*Proof.* By induction on $n$. The property is trivial for $n = 0$. Let us assume it is true when $n$ is replaced by $n-1$. We may write Theorem 5.1 in the form

$$Q_\lambda = \sum_{(\theta_{1,n+1},\ldots,\theta_{n,n+1}) \in \mathsf{N}^n} C^{(q,t)}_{\theta_{1,n+1},\ldots,\theta_{n,n+1}}(\{u_i = q^{\lambda_i - \lambda_{n+1}} t^{n-i}; 1 \le i \le n\})$$
$$\times\, g_{\lambda_{n+1} - \sum_{j=1}^{n}\theta_{j,n+1}}\, Q_{(\lambda_1+\theta_{1,n+1},\ldots,\lambda_n+\theta_{n,n+1})}.$$

Now each partition $\rho = (\lambda_1 + \theta_{1,n+1}, \ldots, \lambda_n + \theta_{n,n+1})$ has length $n$, and by the inductive hypothesis we have

$$Q_\rho = \sum_{\theta \in \mathsf{M}^{(n)}} \prod_{k=1}^{n-1} C^{(q,t)}_{\theta_{1,k+1},\ldots,\theta_{k,k+1}}(\{u_i = q^{\rho_i - \rho_{k+1} + \sum_{j=k+2}^{n}(\theta_{i,j}-\theta_{k+1,j})} t^{k-i}; 1 \le i \le k\})$$
$$\times \prod_{k=1}^{n} g_{\rho_k + \sum_{j=k+1}^{n} \theta_{kj} - \sum_{j=1}^{k-1}\theta_{jk}}.$$

Since

$$g_{\lambda_{n+1} - \sum_{j=1}^{n}\theta_{j,n+1}} \prod_{k=1}^{n} g_{\lambda_k + \theta_{k,n+1} + \sum_{j=k+1}^{n}\theta_{kj} - \sum_{j=1}^{k-1}\theta_{jk}} = \prod_{k=1}^{n+1} g_{\lambda_k + \sum_{j=k+1}^{n+1}\theta_{kj} - \sum_{j=1}^{k-1}\theta_{jk}},$$
$$\rho_i - \rho_{k+1} + \sum_{j=k+2}^{n}(\theta_{i,j} - \theta_{k+1,j}) = \lambda_i - \lambda_{k+1} + \sum_{j=k+2}^{n+1}(\theta_{i,j} - \theta_{k+1,j}),$$

the theorem follows immediately. □

This result may also be stated in terms of "raising operators" [19, p. 9]. For each pair of integers $1 \le i < j \le n+1$ define an operator $R_{ij}$ acting on multi-integers $a = (a_1, \ldots, a_{n+1})$ by $R_{ij}(a) = (a_1, \ldots, a_i + 1, \ldots, a_j - 1, \ldots, a_{n+1})$. Any product $R = \prod_{i<j} R_{ij}^{\theta_{ij}}$, with $\theta = (\theta_{ij})_{1 \le i<j \le n+1} \in \mathsf{M}^{(n+1)}$ is called a raising operator. Its action may be extended to any function $g_\mu = \prod_{k=1}^{n+1} g_{\mu_k}$, with $\mu$ a partition of length $n+1$, by setting $Rg_\mu = g_{R(\mu)}$. In particular $R_{ij} g_\mu = g_{\mu_1} \ldots g_{\mu_i+1} \ldots g_{\mu_j-1} \ldots g_{\mu_{n+1}}$. Then the last quantity appearing in the right-hand side of Theorem 6.1 may be written

$$\prod_{k=1}^{n+1} g_{\lambda_k + \sum_{j=k+1}^{n+1}\theta_{kj} - \sum_{j=1}^{k-1}\theta_{jk}} = \left(\prod_{1 \le i<j \le n+1} R_{ij}^{\theta_{ij}}\right) g_\lambda.$$

Applying $\omega_{q,t}$, we immediatly deduce the following analytic expansion of Macdonald polynomials in terms of elementary symmetric functions $e_k$.

**Theorem 6.2.** *Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots, (n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. We have*

$$P_\lambda(q,t) = \sum_{\theta \in \mathsf{M}^{(n+1)}} \prod_{k=1}^{n} C^{(t,q)}_{\theta_{1,k+1},\ldots,\theta_{k,k+1}}(\{u_i = q^{k-i} t^{\sum_{j=i}^{k} m_j + \sum_{j=k+2}^{n+1}(\theta_{i,j}-\theta_{k+1,j})}; 1 \le i \le k\})$$
$$\times \prod_{k=1}^{n+1} e_{\sum_{j=k}^{n+1} m_j + \sum_{j=k+1}^{n+1}\theta_{kj} - \sum_{j=1}^{k-1}\theta_{jk}}.$$

It is clear that the analytic developments given by Theorems 6.1 and 6.2 are fully explicit. Two analogous formulas may be also obtained by using (4.2).

It seems that our method cannot provide a general analytic expansion for Macdonald polynomials in terms of monomial symmetric functions. However, this expansion may be easily derived from Theorem 6.2 any time the indexing partition is known explicitly. Indeed, the transition matrix from the basis $e_\mu$ to the monomial symmetric basis is well known [19, p. 102, Eq. (6.7)(i)].

# 7 Some special cases

It is worth considering our results in some particular cases [19, p. 324], for instance $q=t$ (Schur functions), or $q=1$ (elementary symmetric functions). Section 8 will be devoted to $q=0$ (Hall–Littlewood symmetric functions) and $t=1$ (monomial symmetric functions). Section 9 will de devoted to the $q=t^\alpha, t\to 1$ limit (Jack symmetric functions).

Let us first give a general property of the development (5.2). Since $v_k=q^{\theta_k}u_k$, we have

$$\frac{u_k-v_k}{u_k-v_k/t}=t\,\frac{1-q^{\theta_k}}{t-q^{\theta_k}}.$$

Obviously the summation on the right-hand side is therefore restricted to $K\subset T=\{k\in\{1,\dots,n\},\ \theta_k\neq 0\}$, and we have

$$\frac{1}{\Delta(v)}\det_{1\le i,j\le n}\left[v_i^{n-j}\left(1-t^j\prod_{k=1}^{n+1}\frac{u_k-v_i}{tu_k-v_i}\right)\right]=\prod_{k\in T}\Big(\frac{1-q^{\theta_k}}{t-q^{\theta_k}}\Big)F_\theta,$$

where $F_\theta$ may be easily written

$$\sum_{K\subset T}(-1)^{|K|}(1/t)^{\binom{|K|}{2}}\prod_{j\in T-K}\frac{t-q^{\theta_j}}{1-q^{\theta_j}}\prod_{\substack{k\in K\\ j\in T-K}}\frac{v_j-v_k/t}{v_j-v_k}\prod_{k\in K}\left(\frac{1-tv_k}{1-v_k}\prod_{\substack{i\in T\\ i\neq k}}\frac{u_i-v_k}{u_i-v_k/t}\right). \quad (7.1)$$

Since for $\theta_k\neq 0$ we have

$$t\,\frac{1-q^{\theta_k}}{t-q^{\theta_k}}\,\frac{(q/t;q)_{\theta_k}}{(q;q)_{\theta_k}}=\frac{(q/t;q)_{\theta_k-1}}{(q;q)_{\theta_k-1}},$$

we conclude that

$$C_\theta^{(q,t)}(u)=\prod_{k\in T}t^{\theta_k-1}\,\frac{(q/t;q)_{\theta_k-1}}{(q;q)_{\theta_k-1}}\,\frac{(qu_k;q)_{\theta_k}}{(qtu_k;q)_{\theta_k}}\prod_{1\le i<j\le n}\frac{(qu_i/tu_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}}\,\frac{(tu_i/v_j;q)_{\theta_i}}{(u_i/v_j;q)_{\theta_i}}\,F_\theta. \quad (7.2)$$

The specialization $q=t$ corresponds to the case of Schur functions. Then $g_k(t,t)=h_k$ and $P_\lambda(t,t)=Q_\lambda(t,t)=s_\lambda$.

**Lemma 7.1.** *For $q=t$, we have $C_\theta^{(t,t)}(u)=0$, except if $\theta_k\in\{0,1\}$ for $1\le k\le n$, in which case $C_\theta^{(t,t)}(u)$ is equal to $(-1)^{|\theta|}$.*

*Proof.* From (7.2) it is clear that $C_\theta^{(t,t)}(u) = 0$, except if all $\theta_k \in \{0,1\}$. It remains to compute the value of $C_\theta^{(t,t)}(u)$ in this case. Then $T = \{k \in \{1,\dots,n\},\ \theta_k = 1\}$, so that $v_k = u_k$ for $k \notin T$ and $v_k = tu_k$ for $k \in T$. We have only to prove

$$F_\theta = (-1)^{|T|} \prod_{k\in T} \frac{1-t^2u_k}{1-tu_k} \prod_{\substack{i,j\in T\\ i<j}} \frac{1-tu_i/u_j}{1-u_i/u_j}\, \frac{1-u_i/tu_j}{1-u_i/u_j}.$$

But in (7.1) we see that, when $q = t$ and $\theta_k = 1$ for $k \in T$, the only non zero contribution comes from $K = T$. Hence the result. ☐

Thus for $q = t$, Theorem 5.1 reads

$$s_{(\lambda_1,\dots,\lambda_{n+1})} = \sum_{\theta\in\{0,1\}^n} (-1)^{|\theta|}\, h_{\lambda_{n+1}-|\theta|}\, s_{(\lambda_1+\theta_1,\dots,\lambda_n+\theta_n)}, \tag{7.3}$$

The following lemma shows that this result is a variant of the classical Jacobi–Trudi formula [19, p. 41, Eq. (3.4)]

$$s_{(\lambda_1,\dots,\lambda_{n+1})} = \det_{1\le i,j\le n+1} [h_{\lambda_i-i+j}].$$

**Lemma 7.2.** *The right-hand side of* (7.3) *is the development of the Jacobi–Trudi determinant along its last row.*

*Proof.* For $0 \le j \le n$, let $M_j$ denote the minor obtained by deleting the $(n+1)$-th row and the $(n+1-j)$-th column of the Jacobi–Trudi determinant. We have

$$s_{(\lambda_1,\dots,\lambda_{n+1})} = \sum_{j=0}^{n} (-1)^j M_j\, h_{\lambda_{n+1}-j}.$$

Let $\Lambda = (\lambda_1+1,\dots,\lambda_n+1)$. Using the Jacobi–Trudi expansion for skew Schur functions [19, p. 70, Eq. (5.4)], it is clear that $M_j$ is exactly the skew Schur function $s_{\Lambda/(1^{n-j})}$. This skew Schur function can be expanded in terms of Schur functions by using [19, p. 70, Eq. (5.3)]. The classical Pieri rule yields

$$M_j = s_{\Lambda/(1^{n-j})} = \sum_\mu s_\mu,$$

with $\mu$ such that $\Lambda - \mu$ is a vertical $(n-j)$-strip. In other words, $\mu$ is obtained from $\Lambda$ by substracting $(n-j)$ nodes (at most one in each row), or alternatively from $(\lambda_1,\dots,\lambda_n)$ by adding $j$ nodes (at most one in each row). ☐

For $q = 1$ we readily obtain $C_\theta^{(t,1)}(u) = 0$ except if $\theta = (0,\dots,0)$. Theorem 5.2 thus reads

$$P_{(1^{m_1},\dots,(n+1)^{m_{n+1}})}(1,t) = e_{m_{n+1}}\, P_{(1^{m_1},\dots,(n-1)^{m_{n-1}},n^{m_n+m_{n+1}})}(1,t),$$

from which we deduce

$$P_\lambda(1,t) = \prod_{i=1}^{n+1} e_{\sum_{k=i}^{n+1} m_k(\lambda)} = e_{\lambda'}.$$

## 8 Hall–Littlewood polynomials

In this section we consider the case $q=0$, which is known [19, p. 324] to correspond to the Hall–Littlewood symmetric functions. We have $P_\lambda(0,t)=P_\lambda(t)$ and $Q_\lambda(0,t)=Q_\lambda(t)$, these functions being defined in [19, Ch. 3, pp. 208–210]. We shall follow the notation of [19], writing $q_k(t)$ for $g_k(0,t)=Q_{(k)}(t)$ and $q_\mu(t)$ for $g_\mu(0,t)$. The parameter $t$ being kept fixed, we shall also write $P_\lambda$, $Q_\lambda$, $q_k$ and $q_\mu$ for short.

The following expansion for Hall–Littlewood polynomials is well-known [19, p. 213]. If $\lambda$ is any partition with length $n+1$, one has

$$\begin{aligned} Q_\lambda &= \left(\prod_{1\le i<j\le n+1} \frac{1-R_{ij}}{1-tR_{ij}}\right) q_\lambda \\ &= \left(\prod_{1\le i<j\le n+1} \Big(1+(1-1/t)\sum_{\theta_{ij}\ge 1} t^{\theta_{ij}}\, R_{ij}^{\theta_{ij}}\Big)\right) q_\lambda . \end{aligned}$$

This property seems to be difficult to recover as the $q=0$ limit of Theorem 6.1. Already to take the $q=0$ limit of Theorem 5.1 does not seem to be an easy task (see however Subsection 11.1). We shall give the $q=0$ specialization of Theorem 5.2 instead.

Let $\left[\begin{smallmatrix} r\\ s\end{smallmatrix}\right]_t$ denote the $t$-binomial coefficient $(t^{r-s+1};t)_s/(t;t)_s$. The Pieri formula for Hall–Littlewood polynomials [19, p. 215, Eq. (3.2)] writes as

$$\begin{aligned} e_{m_{n+1}}\, P_{(1^{m_1},\ldots,n^{m_n})} = \sum_{\theta\in\mathsf{N}^n}\prod_{k=1}^n \begin{bmatrix} m_k+\theta_k-\theta_{k+1}\\ \theta_k\end{bmatrix}_t \\ \times P_{(1^{m_1+\theta_1-\theta_2},\ldots,(n-1)^{m_{n-1}+\theta_{n-1}-\theta_n},n^{m_n+\theta_n-\theta_{n+1}},(n+1)^{\theta_{n+1}})}, \end{aligned}$$

with $\theta_{n+1}=m_{n+1}-|\theta|$. This formula cannot be directly inverted by using the results of Section 3; if one applies the method of Section 2 to the matrix thus defined, the corresponding system of equations turns out to be *not* linear. We shall obtain the inverse relation as the $q=0$ limit of Theorem 5.2.

**Theorem 8.1.** *Let $\lambda=(1^{m_1},2^{m_2},\ldots,(n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. We have*

$$P_\lambda = \sum_{\theta\in\mathsf{N}^n} C^{(t)}_{\theta_1,\ldots,\theta_n}(m_1,\ldots,m_n)\; e_{m_{n+1}-|\theta|}\; P_{(1^{m_1+\theta_1-\theta_2},\ldots,(n-1)^{m_{n-1}+\theta_{n-1}-\theta_n},n^{m_n+m_{n+1}+\theta_n})},$$

*with $C^{(t)}_{\theta_1,\ldots,\theta_n}(m_1,\ldots,m_n)$ defined by*

$$C^{(t)}_{\theta_1,\ldots,\theta_n}(m_1,\ldots,m_n) = (-1)^{|\theta|}\prod_{k=1}^n t^{\binom{\theta_k}{2}} \begin{bmatrix} m_k+\theta_k\\ \theta_k\end{bmatrix}_t \left(1+\sum_{k=1}^n\prod_{j=k}^n \frac{t^{\theta_j}-1}{1-t^{-m_j-\theta_j}}\right). \qquad (8.1)$$

*Remark* 8.2. This result is new. It has no direct connection with Morris' recurrence formula [20], although in both cases, induction is done by removing the largest part of $\lambda$. Note that here all largest parts are simultaneously removed, whereas in [20] one part is removed at a time.

*Proof.* We define

$$C^{(t)}_{\theta_1,\ldots,\theta_n}(m_1,\ldots,m_n)=\lim_{q\to 0} C^{(t,q)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n),$$

with $u_k = q^{n-k}t^{M_k}$ and $M_k=\sum_{j=k}^n m_j$, $M_{n+1}=0$. Using the auxiliary variables $v_k = t^{\theta_k}u_k$, we first compute

$$\begin{aligned}\lim_{q\to 0}\prod_{1\le i<j\le n+1}\frac{(tu_i/qu_j;t)_{\theta_i}}{(tu_i/u_j;t)_{\theta_i}}&\prod_{1\le i\le j\le n}\frac{(qu_i/v_j;t)_{\theta_i}}{(u_i/v_j;t)_{\theta_i}}\\ &=\lim_{q\to 0}\prod_{1\le i<j\le n+1}\frac{(q^{j-i-1}t^{M_i-M_j+1};t)_{\theta_i}}{(q^{j-i}t^{M_i-M_j+1};t)_{\theta_i}}\prod_{1\le i\le j\le n}\frac{(q^{j-i+1}t^{M_i-M_j-\theta_j};t)_{\theta_i}}{(q^{j-i}t^{M_i-M_j-\theta_j};t)_{\theta_i}}.\end{aligned}$$

When $q\to 0$, all limits are 1 but

$$\prod_{i=1}^n\frac{(t^{M_i-M_{i+1}+1};t)_{\theta_i}}{(t^{-\theta_i};t)_{\theta_i}}=(-1)^{|\theta|}\prod_{i=1}^n t^{\binom{\theta_i+1}{2}}\begin{bmatrix} m_i+\theta_i\\ \theta_i\end{bmatrix}_t.$$

It remains to prove that

$$\begin{aligned}\lim_{q\to 0}\frac{1}{\Delta(v)}\det_{1\le i,j\le n}&\left[v_i^{n-j}\left(1-q^{j-1}\frac{1-qv_i}{1-v_i}\prod_{k=1}^n\frac{u_k-v_i}{qu_k-v_i}\right)\right]\\ &=t^{-|\theta|}\left(1+\sum_{k=1}^n\prod_{j=k}^n\frac{t^{\theta_j}-1}{1-t^{M_{j+1}-M_j-\theta_j}}\right).\end{aligned}$$

This is a direct consequence of the following more general result, applied for $a_k=t^{M_k}$, and $b_k=t^{M_k+\theta_k}$, i.e. $u_k=q^{n-k}a_k$, and $v_k=q^{n-k}b_k$. $\square$

**Lemma 8.3.** *Let* $a=(a_1,\ldots,a_n,a_{n+1})$ *and* $b=(b_1,\ldots,b_n)$ *be* $2n+1$ *indeterminates. Define*

$$F_n(q)=\prod_{1\le i<j\le n}(q^{n-i}b_i-q^{n-j}b_j)^{-1}\det_{1\le i,j\le n}\left[(q^{n-i}b_i)^{n-j}\left(1-q^j\prod_{k=1}^{n+1}\frac{b_i-q^{i-k}a_k}{b_i-q^{i-k+1}a_k}\right)\right]$$

*and*

$$G_n=\sum_{k=1}^{n+1}\prod_{j=1}^{k-1}\frac{a_j}{b_j}\prod_{j=k}^n\frac{a_j-b_j}{a_{j+1}-b_j}.$$

*Then we have* $\lim_{q\to 0}F_n(q)=G_n$.

*Proof.* Substituting $q$ for $t$, $q^{n-i}b_i$ for $v_i$, and $q^{n-i}a_i$ for $u_i$ in (5.2), we have

$$F_n(q) = \sum_{K\subset\{1,\dots,n\}} (-1)^{|K|}(1/q)^{\binom{|K|+1}{2}} \prod_{\substack{k\in K\\ j\notin K}} \frac{b_j - q^{j-k-1}b_k}{b_j - q^{j-k}b_k} \prod_{k\in K}\prod_{i=1}^{n+1} \frac{a_i - q^{i-k}b_k}{a_i - q^{i-k-1}b_k}.$$

The contribution of $K$ can be written as

$$(-1)^{|K|}(1/q)^{\binom{|K|+1}{2}} \prod_{\substack{k\in K\\ j\notin K}} \left( \frac{b_j - q^{j-k-1}b_k}{b_j - q^{j-k}b_k} \frac{a_j - q^{j-k}b_k}{a_j - q^{j-k-1}b_k} \right)$$
$$\times \prod_{k\in K} \left( \frac{a_{n+1} - q^{n-k+1}b_k}{a_{n+1} - q^{n-k}b_k} \right) \prod_{\substack{i\in K\\ k\in K}} \left( \frac{a_i - q^{i-k}b_k}{a_i - q^{i-k-1}b_k} \right).$$

When $q \to 0$, the limit of the various factors are

$$\lim_{q\to 0} \prod_{\substack{k\in K\\ j\notin K}} \left( \frac{b_j - q^{j-k-1}b_k}{b_j - q^{j-k}b_k} \frac{a_j - q^{j-k}b_k}{a_j - q^{j-k-1}b_k} \right) = \prod_{\substack{k\in K, k\neq n\\ k+1\notin K}} \left( \frac{a_{k+1}}{b_{k+1}} \frac{b_{k+1} - b_k}{a_{k+1} - b_k} \right),$$

$$\lim_{q\to 0} \prod_{k\in K} \frac{a_{n+1} - q^{n-k+1}b_k}{a_{n+1} - q^{n-k}b_k} = \frac{a_{n+1}}{a_{n+1} - b_n} \quad \text{if } n \in K,$$

$$\lim_{q\to 0} (-1/q)^{|K|} \prod_{k\in K} \frac{a_k - b_k}{a_k - b_k/q} = \prod_{i\in K} \frac{a_i - b_i}{b_i},$$

$$\lim_{q\to 0} (1/q)^{\binom{|K|}{2}} \prod_{\substack{i,j\in K\\ i<j}} \left( \frac{a_i - q^{i-j}b_j}{a_i - q^{i-j-1}b_j} \frac{a_j - q^{j-i}b_i}{a_j - q^{j-i-1}b_i} \right) = \prod_{\substack{i\in K, i\neq n\\ i+1\in K}} \frac{a_{i+1}}{a_{i+1} - b_i}.$$

Putting these limits together, we have

$$\lim_{q\to 0} F_n(q) = \sum_{K\subset\{1,\dots,n\}} \prod_{\substack{k\in K, k\neq n\\ k+1\notin K}} \frac{b_{k+1} - b_k}{b_{k+1}} \prod_{i\in K} \left( \frac{a_{i+1}}{b_i} \frac{a_i - b_i}{a_{i+1} - b_i} \right).$$

We are done once we have shown the following lemma. $\square$

**Lemma 8.4.** *Let $a = (a_1, \dots, a_n, a_{n+1})$ and $b = (b_1, \dots, b_n)$ be $2n+1$ indeterminates. Define*

$$F_n = \sum_{K\subset\{1,\dots,n\}} \prod_{\substack{k\in K, k\neq n\\ k+1\notin K}} \frac{b_{k+1} - b_k}{b_{k+1}} \prod_{i\in K} \left( \frac{a_{i+1}}{b_i} \frac{a_i - b_i}{a_{i+1} - b_i} \right).$$

*Then $F_n = G_n$.*

*Proof.* Obviously $G_n$ satisfies the recurrence relation

$$G_n = \prod_{i=1}^{n} \frac{a_i}{b_i} + \frac{a_n - b_n}{a_{n+1} - b_n}\, G_{n-1},$$

which yields

$$G_n = \left(\frac{a_n}{b_n} + \frac{a_n - b_n}{a_{n+1} - b_n}\right) G_{n-1} - \frac{a_n}{b_n}\,\frac{a_{n-1} - b_{n-1}}{a_n - b_{n-1}}\, G_{n-2}.$$

We have $F_0 = G_0 = 1$ and

$$F_1 = 1 + \frac{a_2}{b_1}\,\frac{a_1 - b_1}{a_2 - b_1} = \frac{a_1}{b_1} + \frac{a_1 - b_1}{a_2 - b_1} = G_1.$$

Thus we have only to prove that $F_n$ satisfies the second recurrence relation. Summing the contributions of sets $K = L \cup \{n\}$, with $L \subset \{1, \ldots, n-1\}$ possibly empty, we find

$$F_n = H_n + \frac{a_{n+1}}{b_n}\,\frac{a_n - b_n}{a_{n+1} - b_n} F_{n-1},$$

with

$$H_n = \sum_{L \subset \{1,\ldots,n-1\}} \prod_{\substack{k \in L \\ k+1 \notin L}} \frac{b_{k+1} - b_k}{b_{k+1}} \prod_{i \in L} \left(\frac{a_{i+1}}{b_i}\,\frac{a_i - b_i}{a_{i+1} - b_i}\right).$$

Summing separately sets with $n-1 \notin L$ and $n-1 \in L$, we have

$$H_n = H_{n-1} + \frac{b_n - b_{n-1}}{b_n}\,\frac{a_n}{b_{n-1}}\,\frac{a_{n-1} - b_{n-1}}{a_n - b_{n-1}}\, F_{n-2},$$

or equivalently

$$F_n - \frac{a_{n+1}}{b_n}\,\frac{a_n - b_n}{a_{n+1} - b_n} F_{n-1} = F_{n-1} + \frac{a_n}{b_{n-1}}\,\frac{a_{n-1} - b_{n-1}}{a_n - b_{n-1}} \left(\frac{b_n - b_{n-1}}{b_n} - 1\right) F_{n-2}.$$

Hence the result. □

From Theorem 6.2 we then deduce the following (new) expansion of Hall–Littlewood polynomials in terms of elementary symmetric functions.

**Theorem 8.5.** *Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots, (n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. We have*

$$P_\lambda(t) = \sum_{\theta \in \mathsf{M}^{(n+1)}} \prod_{k=1}^{n} C^{(t)}_{\theta_{1,k+1},\ldots,\theta_{k,k+1}}(\{m_i + \sum_{j=k+2}^{n+1} (\theta_{i,j} - \theta_{i+1,j}); 1 \le i \le k\})$$
$$\times \prod_{k=1}^{n+1} e_{\sum_{j=k}^{n+1} m_j + \sum_{j=k+1}^{n+1} \theta_{kj} - \sum_{j=1}^{k-1} \theta_{jk}},$$

*with $C^{(t)}_{\theta_1,\ldots,\theta_k}(m_1, \ldots, m_k)$ defined by equation* (8.1).

It is known [19, p. 208] that monomial symmetric functions are the specialization of Hall–Littlewood symmetric functions for $t = 1$. One has $P_\lambda(1) = m_\lambda$, and in this situation Theorem 8.1 reads as follows.

**Theorem 8.6.** *Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots, (n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. We have*

$$m_\lambda = \sum_{\theta \in \mathsf{N}^n} C_{\theta_1,\ldots,\theta_n}(m_1,\ldots,m_n)\; e_{m_{n+1}-|\theta|}\; m_{(1^{m_1+\theta_1-\theta_2},\ldots,(n-1)^{m_{n-1}+\theta_{n-1}-\theta_n},n^{m_n+m_{n+1}+\theta_n})},$$

*with $C_{\theta_1,\ldots,\theta_n}(m_1,\ldots,m_n)$ defined by*

$$C_{\theta_1,\ldots,\theta_n}(m_1,\ldots,m_n) = (-1)^{|\theta|} \prod_{k=1}^{n} \binom{m_k+\theta_k}{\theta_k} \left(1 + \sum_{k=1}^{n} \prod_{j=k}^{n} \frac{\theta_j}{m_j+\theta_j}\right). \tag{8.2}$$

This gives the expansion of monomial symmetric functions in terms of elementary symmetric functions, a problem which was studied by Waring [25] as early as 1762. Some years later, Vandermonde [24] computed tables up to weight 10 by a different approach[1].

**Theorem 8.7.** *Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots, (n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. We have*

$$m_\lambda = \sum_{\theta \in \mathsf{M}^{(n+1)}} \prod_{k=1}^{n} C_{\theta_{1,k+1},\ldots,\theta_{k,k+1}}(\{m_i + \sum_{j=k+2}^{n+1} (\theta_{i,j} - \theta_{i+1,j}); 1 \le i \le k\})$$
$$\times \prod_{k=1}^{n+1} e_{\sum_{j=k}^{n+1} m_j + \sum_{j=k+1}^{n+1} \theta_{kj} - \sum_{j=1}^{k-1} \theta_{jk}},$$

*with $C_{\theta_1,\ldots,\theta_k}(m_1,\ldots,m_k)$ defined by equation* (8.2).

# 9 Jack polynomials

Jack polynomials are the limit of Macdonald polynomials when $t \to 1$, with $q = t^\alpha$. The indeterminates $q, t$ are then considered as real variables, and $\alpha$ is some positive real number [19, p. 376]. We define

$$P_\lambda(\alpha) = \lim_{t\to 1} P_\lambda(t^\alpha, t), \qquad Q_\lambda(\alpha) = \lim_{t\to 1} Q_\lambda(t^\alpha, t).$$

The parameter $\alpha$ being kept fixed, we shall also write $P_\lambda, Q_\lambda$ for short.

These polynomials are normalized differently from their "integral form" $J_\lambda(\alpha)$ studied in [23]. We have $J_\lambda(\alpha) = c_\lambda(\alpha) P_\lambda(\alpha) = c'_\lambda(\alpha) Q_\lambda(\alpha)$, with $c_\lambda(\alpha)$ and $c'_\lambda(\alpha)$ given in [19, p. 381, Eq. (10.21)].

[1] Alain Lascoux [13, p. 12] mentions that these tables are free of any mistake.

The Jack polynomials $Q_{(k)}$ associated to row partitions $(k)$ have the generating series

$$\prod_{i\geq 1}(1-ux_i)^{-1/\alpha}=\sum_{k\geq 0}u^kQ_{(k)}(\alpha).$$

Their development in terms of any classical basis is given in [23, p. 80, Prop. 2.2].

We now fix some positive real number $a$. We denote by $(u)_k$ the classical rising factorial, defined by $(u)_0=1$ and $(u)_k=\prod_{i=1}^k(u+i-1)$ for $k\neq 0$.

Let $u=(u_1,\ldots,u_n)$ be $n$ indeterminates and $\theta=(\theta_1,\ldots,\theta_n)\in\mathsf{N}^n$. For clarity of notations, we introduce $n$ auxiliary variables $v=(v_1,\ldots,v_n)$ defined by $v_k=u_k+\theta_k$. We write

$$\begin{aligned}C^{(a)}_{\theta_1,\ldots,\theta_n}&(u_1,\ldots,u_n)\\&=\prod_{k=1}^n\frac{(1-a)_{\theta_k}}{\theta_k!}\,\frac{(u_k+1)_{\theta_k}}{(u_k+1+a)_{\theta_k}}\prod_{1\leq i<j\leq n}\frac{(u_i-u_j+1-a)_{\theta_i}}{(u_i-u_j+1)_{\theta_i}}\,\frac{(u_i-v_j+a)_{\theta_i}}{(u_i-v_j)_{\theta_i}}\\&\qquad\times\frac{1}{\Delta(v)}\det_{1\leq i,j\leq n}\left[v_i^{n-j}-(v_i-a)^{n-j}\,\frac{v_i+a}{v_i}\prod_{k=1}^n\frac{v_i-u_k}{v_i-u_k-a}\right].\end{aligned}$$

Setting $u_{n+1}=-a$, this may be written as

$$\begin{aligned}C^{(a)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n)=&\prod_{1\leq i<j\leq n+1}\frac{(u_i-u_j+1-a)_{\theta_i}}{(u_i-u_j+1)_{\theta_i}}\prod_{1\leq i\leq j\leq n}\frac{(u_i-v_j+a)_{\theta_i}}{(u_i-v_j)_{\theta_i}}\\&\times\frac{1}{\Delta(v)}\det_{1\leq i,j\leq n}\left[v_i^{n-j}-(v_i-a)^{n-j}\prod_{k=1}^{n+1}\frac{v_i-u_k}{v_i-u_k-a}\right].\end{aligned}$$

**Lemma 9.1.** *With $U=(q^{u_1},\ldots,q^{u_n})$, we have*

$$C^{(a)}_\theta(u)=\lim_{q\to 1}c^{(q,q^a)}_\theta(U).$$

*Proof.* Define $U_{n+1}=q^{u_{n+1}}$, so that the condition $U_{n+1}=1/t$ is satisfied for $t=q^a$. Introduce the auxiliary variables $V=(q^{v_1},\ldots,q^{v_n})$, so that $V_k=q^{\theta_k}U_k$. Then we only have to prove

$$\begin{aligned}\frac{1}{\Delta(v)}&\det_{1\leq i,j\leq n}\left[v_i^{n-j}-(v_i-a)^{n-j}\prod_{k=1}^{n+1}\frac{v_i-u_k}{v_i-u_k-a}\right]\\&=\lim_{\substack{t=q^a\\q\to 1}}\frac{1}{\Delta(V)}\det_{1\leq i,j\leq n}\left[V_i^{n-j}\left(1-t^j\prod_{k=1}^{n+1}\frac{U_k-V_i}{tU_k-V_i}\right)\right].\end{aligned}$$

Consider the following difference operator

$$D(z;a)=\frac{1}{\Delta(v)}\det_{1\leq i,j\leq n}\left[v_i^{n-j}+z\,(v_i-a)^{n-j}T_{a,v_i}\right],$$

acting on polynomials in $v$, where $z$ is some indeterminate and $T_{a,v_i}$ is the $a$-translation operator defined by

$$T_{a,v_i} f(v_1, \ldots, v_n) = f(v_1, \ldots, v_i + a, \ldots, v_n).$$

Then in a strictly parallel way to the proof given in [19, p. 315], we have

$$D(z;a) = \sum_{K \subset \{1,\ldots,n\}} z^{|K|} \prod_{\substack{k \in K \\ j \notin K}} \frac{v_k - v_j - a}{v_k - v_j} \prod_{k \in K} T_{a,v_k}.$$

Applying this result to $\prod_{i=1}^{n+1} \prod_{j=1}^{n} (v_j - u_i - a)$, with $z = -1$, we get

$$\frac{1}{\Delta(v)} \det_{1 \le i,j \le n} \left[ v_i^{n-j} - (v_i - a)^{n-j} \prod_{k=1}^{n+1} \frac{v_i - u_k}{v_i - u_k - a} \right]$$
$$= \sum_{K \subset \{1,\ldots,n\}} (-1)^{|K|} \prod_{\substack{k \in K \\ j \notin K}} \frac{v_k - v_j - a}{v_k - v_j} \prod_{i=1}^{n+1} \prod_{k \in K} \frac{v_k - u_i}{v_k - u_i - a}.$$

On the other hand (5.2), written for $t = q^a$, yields

$$\frac{1}{\Delta(V)} \det_{1 \le i,j \le n} \left[ V_i^{n-j} \left( 1 - t^j \prod_{k=1}^{n+1} \frac{U_k - V_i}{tU_k - V_i} \right) \right]$$
$$= \sum_{K \subset \{1,\ldots,n\}} (-1)^{|K|} q^{-a\binom{|K|+1}{2}} \prod_{\substack{k \in K \\ j \notin K}} \frac{V_j - q^{-a} V_k}{V_j - V_k} \prod_{i=1}^{n+1} \prod_{k \in K} \frac{U_i - V_k}{U_i - q^{-a} V_k}.$$

Hence the statement in the limit $q \to 1$. □

The two following results are straightforward consequences of Theorems 5.1 and 5.2.

**Theorem 9.2.** *Let $\lambda = (\lambda_1, ..., \lambda_{n+1})$ be an arbitrary partition with length $n+1$. For any $1 \le k \le n+1$ define $u_k = \lambda_k - \lambda_{n+1} + (n-k)/\alpha$. We have*

$$Q_{(\lambda_1,\ldots,\lambda_{n+1})} = \sum_{\theta \in \mathsf{N}^n} C^{(1/\alpha)}_{\theta_1,\ldots,\theta_n}(u_1, \ldots, u_n)\, Q_{(\lambda_{n+1}-|\theta|)}\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n)}.$$

**Theorem 9.3.** *Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots, (n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. For any $1 \le k \le n+1$ define $u_k = \sum_{j=k}^{n} m_j + (n-k)\alpha$. We have*

$$P_\lambda = \sum_{\theta \in \mathsf{N}^n} C^{(\alpha)}_{\theta_1,\ldots,\theta_n}(u_1, \ldots, u_n)\, e_{m_{n+1}-|\theta|}\, P_{(1^{m_1+\theta_1-\theta_2},\ldots,(n-1)^{m_{n-1}+\theta_{n-1}-\theta_n},n^{m_n+m_{n+1}+\theta_n})}.$$

As in Section 6 these formulas generate the explicit analytic developments of Jack polynomials in terms of the classical bases $Q_{(k)}$ and $e_k$. These expansions are easily written by replacing $C^{(q,t)}$ by $C^{(1/\alpha)}$, and $C^{(t,q)}$ by $C^{(\alpha)}$, in the corresponding statements for Macdonald polynomials. They are left to the reader.

# 10 The hook case

The explicit development of Macdonald polynomials in terms of the classical bases $g_k$ and $e_k$ was already known when the partition $\lambda$ is a hook. This result had been given by Kerov [7, Th. 6.3] (see also [8]). For $\lambda = (r, 1^s)$ Kerov's result writes elegantly as

$$Q_\lambda = \det_{1\leq i,j\leq s+1}\left[\frac{1-q^{\lambda_i-i+j}t^{s-j+1}}{1-q^{\lambda_i}t^{s-i+1}}\, g_{\lambda_i-i+j}\right].$$

It was derived by using the Pieri formula

$$Q_{1^s}\, Q_{(r)} = \frac{1-t^s}{1-qt^{s-1}}\,\frac{1-q^{r+1}t^{s-1}}{1-q^r t^s}\, Q_{(r+1,1^{s-1})} + Q_{(r,1^s)}, \tag{10.1}$$

which is readily obtained from Theorem 4.1, the two contributions on the right-hand side corresponding to $\theta_1 = r, \theta_2 = \ldots = \theta_s = 0$ and $\theta_1 = r-1, \theta_2 = \ldots = \theta_s = 0$, respectively.

Since the expansion of Theorem 5.1 involves the partition $(r, 2, 1^{s-2})$, it cannot provide a method to compute $Q_{(r,1^s)}$ through a recursion on $r$ and/or $s$. However we have obtained the following development, which may be worth giving here since its equivalence with Kerov's result is not trivial.

Let $n$ be a positive integer and $C(n)$ denote the set of positive multi-integers ("compositions") $c = (c_1, \ldots, c_l) \in \mathsf{N}^l$ with weight $|c| = \sum_{i=1}^l c_i = n$. The integer $l = l(c)$ is called the length of $c$. For any $c = (c_1, \ldots, c_l)$ we write $[c_i] = \sum_{1\leq k\leq i} c_k$ for the $i$-th partial sum.

In [15, p. 241] one of us has shown that the expansion of the column Macdonald polynomial $Q_{1^n}$ in terms of the modified complete symmetric functions $g_k$ may be written as

$$Q_{1^n} = (-1)^n \frac{(t;t)_n}{(q;t)_n} \sum_{c\in C(n)} \prod_{i=1}^{l(c)} \frac{q^{c_i}t^{[c_{i-1}]}-1}{1-t^{[c_i]}}\, g_{c_i}.$$

The following result gives the development of Kerov's determinant along its first row.

**Theorem 10.1.** *We have*

$$\begin{aligned} Q_{(r,1^s)}(q,t) &= (-1)^s \frac{(t;t)_s}{(q;t)_s} \\ &\quad\times \sum_{c\in C(s+1)} \left(\prod_{i=1}^{l(c)-1} \frac{q^{c_i}t^{[c_{i-1}]}-1}{1-t^{[c_i]}}\, g_{c_i}\right) \frac{1-q^{r+c_{l(c)}-1}t^{s-c_{l(c)}+1}}{1-q^r t^s}\, g_{r+c_{l(c)}-1}. \end{aligned}$$

*Proof.* Since $Q_{1^s}$ is known, the Pieri formula (10.1) defines $Q_{(r,1^s)}$ through induction on the integer $r$. We have $[c_{l(c)-1}] = |c| - c_{l(c)}$ and the property is true for $r = 1$. Assume that it is true for $Q_{(r,1^s)}$. In (10.1) we look for the compositions contributing both to $Q_{(r,1^s)}$ and $Q_{(r+1,1^{s-1})}$. Equivalently we substract from $Q_{(r,1^s)}$ the contributions coming from $Q_{1^s}\, Q_{(r)}$. These have the form

$$(-1)^s \frac{(t;t)_s}{(q;t)_s} \prod_{i=1}^{l-1} \frac{q^{c_i}t^{[c_{i-1}]}-1}{1-t^{[c_i]}}\, g_{c_i}\, g_r,$$

with $c = (c_1, \ldots, c_{l-1}) \in C(s)$. Such contributions can be rewritten as

$$(-1)^s \frac{(t;t)_s}{(q;t)_s} \prod_{i=1}^{l(c)-1} \left( \frac{q^{c_i} t^{[c_{i-1}]} - 1}{1 - t^{[c_i]}} \, g_{c_i} \right) \frac{1 - q^{r + c_{l(c)} - 1} t^{s - c_{l(c)} + 1}}{1 - q^r t^s} \, g_{r + c_{l(c)} - 1},$$

where $c \in C(s+1)$ is a composition having its last term $c_{l(c)} = 1$. Therefore the contributions to $Q_{(r+1,1^{s-1})}$ correspond to compositions $c \in C(s+1)$ having their last term $c_{l(c)} > 1$. Substracting 1 to the last component, we obtain a composition $c \in C(s)$ having the same length. Simplifying some factors, we are done. $\square$

In the case of hooks, the automorphism $\omega_{q,t}$ satisfies

$$\omega_{q,t}(Q_{(r,1^s)}(q,t)) = P_{(s+1,1^{r-1})}(t,q).$$

Applying this automorphism, we obtain the following equivalent result.

**Theorem 10.2.** *We have*

$$P_{(r,1^s)}(q,t) = (-1)^{r-1} \frac{(q;q)_{r-1}}{(t;q)_{r-1}} \sum_{c \in C(r)} \left( \prod_{i=1}^{l(c)-1} \frac{q^{[c_{i-1}]} t^{c_i} - 1}{1 - q^{[c_i]}} \, e_{c_i} \right) \frac{1 - q^{r - c_{l(c)}} t^{s + c_{l(c)}}}{1 - q^{r-1} t^{s+1}} \, e_{s + c_{l(c)}}.$$

# 11 Extension of Macdonald polynomials

## 11.1 Extension to multi-integers

In the Hall–Littlewood case, it is well known that the expansion

$$Q_\lambda = \left( \prod_{1 \le i < j \le n+1} \frac{1 - R_{ij}}{1 - tR_{ij}} \right) q_\lambda, \tag{11.1}$$

may be used to *define* Hall–Littlewood polynomials $Q_\lambda$ when $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ is *any* sequence of integers, positive or negative, not necessarily in descending order [19, p. 213, Example 2], see also [19, pp. 236–238, Example 8].

One may wonder whether Theorem 6.1 might be similarly used as a definition of Macdonald polynomials associated with any sequence of integers. Or equivalently, whether Theorem 5.1 might be inductively used to define $Q_\lambda$ in that case.

This can indeed be done but leads to a trivial result: one obtains $Q_\lambda = 0$ when $\lambda$ is not a partition. This fact shows a big difference between the general case (Macdonald) and its $q = 0$ limit (Hall–Littlewood).

Let us make this remark more precise through an elementary example. In the length 2 general case, as a consequence of (5.1), we have

$$\begin{aligned} Q_{(2,1)} &= Q_{(2)} Q_{(1)} + C_1^{(q,t)}(q) \; Q_3, \\ Q_{(1,2)} &= Q_{(1)} Q_{(2)} + C_1^{(q,t)}(1/q) \; Q_{(2)} Q_{(1)} + C_2^{(q,t)}(1/q) \; Q_3, \end{aligned}$$

the second equation being taken as a definition. Now

$$\begin{aligned} C_1^{(q,t)}(u) &= \frac{t-1}{1-q}\,\frac{1-q^2u}{1-qtu}, \\ C_2^{(q,t)}(u) &= \frac{t-1}{1-q}\,\frac{t-q}{1-q^2}\,\frac{1-qu}{1-qtu}\,\frac{1-q^4u}{1-q^2tu}, \end{aligned}$$

so that $Q_{(1,2)} = 0$.

However in the Hall–Littlewood case, (11.1) writes as

$$\begin{aligned} Q_{(2,1)} &= Q_{(2)}Q_{(1)} + (t-1)Q_3, \\ Q_{(1,2)} &= Q_{(1)}Q_{(2)} + (t-1)Q_{(2)}Q_{(1)} + t(t-1)Q_3, \end{aligned}$$

so that $Q_{(1,2)} = tQ_{(1,2)}$, as is well known.

In the Macdonald case, Theorem 5.1 always inductively gives $Q_\lambda = 0$ when $\lambda$ is not a partition. This fact may be easily explained as follows. Theorem 5.1 and Theorem 4.1 are equivalent by our matrix inversion. Thus Theorem 5.1 and Theorem 4.1 must yield the same value for any $Q_\lambda$. However, as already emphasized in Remark 4.2, Theorem 4.1 implicitly assumes that $Q_\lambda = 0$ when $\lambda$ is not a partition.

In the Hall–Littlewood situation, a specific structure does exist. Actually the definition (11.1) is equivalent to the following recurrence property

$$Q_{(\lambda_1,\ldots,\lambda_{n+1})} = \sum_{\theta\in\mathsf{N}^n} t^{|\theta|}(1-1/t)^{n(\theta)}\, Q_{(\lambda_{n+1}-|\theta|)}\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n)},$$

with $n(\theta) = \mathrm{card}\{j : \theta_j \neq 0\}$. We emphasize that the sum on the right-hand side is taken over *all* $\theta \in \mathsf{N}^n$, even over those $\theta$ for which $(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n)$ is *not* a partition.

It is easily shown that this relation may be inverted by writing the Pieri formula

$$Q_{(\lambda_1,\ldots,\lambda_n)}\, Q_{(\lambda_{n+1})} = \sum_{\theta\in\mathsf{N}^n} (1-t)^{n(\theta)}\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n,\lambda_{n+1}-|\theta|)}.$$

Here again we emphasize that the sum is taken over *all* $\theta \in \mathsf{N}^n$, even over those $\theta$ for which $(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n,\lambda_{n+1}-|\theta|)$, is *not* a partition.

Apparently this "analytic" Pieri formula had kept unnoticed. It is very different from the classical combinatorial one [19, p. 229, Eq. (5.7')]. Of course the latter may be recovered once all the $Q_\lambda$, where $\lambda$ is not a partition, are reduced to a linear combination of $Q_\mu$, where $\mu$ is a partition.

## 11.2 Extension to sequences of complex numbers

Kadell [6] defines Schur functions associated with any set of complex numbers by extending their classical definition as a ratio of alternants. Similarly it may be wondered whether Theorem 5.1 might be inductively used to extend Macdonald polynomials $Q_\lambda$ when $\lambda = (\lambda_1,\ldots,\lambda_{n+1})$ is any sequence of complex numbers.

Two difficulties are encountered here. The first one concerns the one row case, i.e. finding some reasonable extension of Macdonald polynomials $Q_{(k)}$ when $k$ is any complex number. Since $Q_{(k)}$ is not analytic in $k$ nor in $q^k$, such an extension is not unique. The second difficulty deals with convergence, the expansion of Theorem 5.1 being no longer terminating (and the extension thus defined being no longer a polynomial).

We have no clue that such $Q_\lambda$ would form a family of orthogonal functions, nor that they would be eigenfunctions of Macdonald operators (or some variant of them). These questions, among others, need investigation. Some results have been already obtained, about which we hope to report in a forthcoming paper.

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