A NEW MULTIVARIABLE $_6\psi_6$ SUMMATION FORMULA

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Abstract. By multidimensional matrix inversion, combined with an A_r extension of Jackson's $_8\phi_7$ summation formula by Milne, a new multivariable $_8\phi_7$ summation is derived. By a polynomial argument this $_8\phi_7$ summation is transformed to another multivariable $_8\phi_7$ summation which, by taking a suitable limit, is reduced to a new multivariable extension of the nonterminating $_6\phi_5$ summation. The latter is then extended, by analytic continuation, to a new multivariable extension of Bailey's very-well-poised $_6\psi_6$ summation formula.

1. Introduction

Bailey's [4, Eq. (4.7)] very-well-poised $_6\psi_6$ summation formula,

$$
{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e \end{matrix}; q, \frac{a^2q}{bcde}\right]
= \frac{(q, aq, q/a, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, a^2q/bcde)_\infty}, \tag{1.1}
$$

(see Section 2 for the notation) where $|a^2q/bcde| < 1$ (cf. [12, Eq. (5.3.1)]), is one of the most important identities in special functions, with applications to orthogonal polynomials, number theory, and combinatorics.

Several *multivariable* extensions of Bailey's formula exist (all of them associated with various root systems), including a couple of summations by Gustafson [13, 14, 15], a summation by van Diejen [10], by the author [34], and by Ito [18].

In this paper we supply a new multivariable extension of (1.1) to this list. We also provide several other new summations. These include new multivariable very-well-poised $_8\phi_7$ and $_6\phi_5$ summations. The series obtained in this paper are of a slightly different type than those usually labelled as A_r series, but, being closely related to these, we still decided to refer to them as A_r series, see Remark 2.3.

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Our paper is organized as follows. In Section 2 we introduce the notation and review some common facts about basic hypergeometric series. We first do this for the classical univariate case and then for the multivariate case. In Section 3 we explain the concept of multidimensional matrix inversion and list an explicit result, see Corollary 3.2 which is needed in Section 4 to derive, via multidimensional inverse relations and an A_r ${}_8\phi_7$ summation theorem by Milne [25], a new multivariable terminating very-well-poised ${}_8\phi_7$ summation. A polynomial argument gives yet another multivariable ${}_8\phi_7$ summation. In Section 5 we suitably specialize this summation in order to obtain new multivariable very-well-poised ${}_6\phi_5$ summations. Finally, in Section 6 we apply analytic continuation (in particular, an iterated application of Ismail's [3, 17] argument) to deduce a new multivariable extension of Bailey's very-well-poised ${}_6\psi_6$ summation.

2. Preliminaries

2.1. **Notation and basic hypergeometric series.** Here we recall some standard notation for q-series, and basic hypergeometric series (cf. [12]).

Let q be a complex number such that $0 < |q| < 1$. We define the *q-shifted factorial* for all integers k by

$$(a)_k := \frac{(a)_\infty}{(aq^k)_\infty}, \tag{2.1}$$

where

$$(a)_\infty := \prod_{j=0}^{\infty}(1 - aq^j).$$

For brevity, we occasionally employ for products the notation

$$(a_1, \dots, a_m)_k := (a_1)_k \dots (a_m)_k$$

where k is an integer or infinity. Further, we utilize

$${}_s\phi_{s-1}\left[\begin{matrix}a_1, a_2, \dots, a_s\\ b_1, b_2, \dots, b_{s-1}\end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_s)_k}{(q, b_1, \dots, b_{s-1})_k} z^k, \tag{2.2}$$

and

$${}_s\psi_s\left[\begin{matrix}a_1, a_2, \dots, a_s\\ b_1, b_2, \dots, b_s\end{matrix}; q, z\right] := \sum_{k=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_s)_k}{(b_1, b_2, \dots, b_s)_k} z^k, \tag{2.3}$$

to denote the *basic hypergeometric ${}_s\phi_{s-1}$ series*, and the *bilateral basic hypergeometric ${}_s\psi_s$ series*, respectively. In (2.2) or (2.3), $a_1, \dots, a_s$ are called the *upper parameters*, $b_1, \dots, b_s$ the *lower parameters*, z is the *argument*, and q the *base* of the series. See [12, p. 5 and p. 137] for the criteria of when these series terminate, or, if not, when they converge.

The classical theory of basic hypergeometric series contains numerous summation and transformation formulae involving ${}_s\phi_{s-1}$ or ${}_s\psi_s$ series. Many of these summation theorems require that the parameters satisfy the condition of being either balanced and/or very-well-poised. An ${}_s\phi_{s-1}$ basic hypergeometric series is called *balanced* if $b_1 \cdots b_{s-1} = a_1 \cdots a_s q$ and $z = q$. An ${}_s\phi_{s-1}$ series is *well-poised* if $a_1 q = a_2 b_1 = \cdots = a_s b_{s-1}$. An ${}_s\phi_{s-1}$ basic hypergeometric series is called *very-well-poised* if it is well-poised and if $a_2 = -a_3 = q\sqrt{a_1}$. Note that the factor

$$\frac{1 - a_1 q^{2k}}{1 - a_1}$$

appears in a very-well-poised series. The parameter a_1 is usually referred to as the *special parameter* of such a series. Similarly, a bilateral ${}_s\psi_s$ basic hypergeometric series is well-poised if $a_1 b_1 = a_2 b_2 \cdots = a_s b_s$ and very-well-poised if, in addition, $a_1 = -a_2 = q b_1 = -q b_2$.

A standard reference for basic hypergeometric series is Gasper and Rahman's texts [12]. In our computations in the subsequent sections we frequently use some elementary identities of q-shifted factorials, listed in [12, Appendix I].

One of the most important theorems in the theory of basic hypergeometric series is Jackson's [20] terminating very-well-poised balanced ${}_8\phi_7$ summation (cf. [12, Eq. (2.6.2)]):

$$
{}_8\phi_7\left[\begin{matrix} a,\, q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2q^{1+n}/bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcdq^{-n}/a, aq^{1+n} \end{matrix}; q, q\right]
= \frac{(aq, aq/bc, aq/bd, aq/cd)_n}{(aq/b, aq/c, aq/d, aq/bcd)_n}. \tag{2.4}
$$

This identity stands on the top of the classical hierarchy of summations for basic hypergeometric series. Special cases include the terminating and nonterminating very-well-poised ${}_6\phi_5$ summations, the q-Pfaff–Saalschütz summation, the q-Gauß summation, the q-Chu–Vandermonde summation and the termininating and nonterminating q-binomial theorem, see [12].

2.2. **Multidimensional series.** A_r (or, equivalently, $U(r+1)$) hypergeometric series were motivated by the work of Biedenharn, Holman, and Louck [16] in theoretical physics. The theory of A_r basic hypergeometric series (or "multiple basic hypergeometric series associated with the root system A_r", or "associated with the unitary group $U(r+1)$"), analogous to the classical theory of one-dimensional series, has been developed originally by R. A. Gustafson, S. C. Milne, and their coworkers, and later others (see [6, 13, 14, 25, 26, 28, 30] for a very small selection of papers in this area, neglecting a bunch of other important references which already have grown vast in number). Notably, several higher-dimensional extensions have been derived (in each case) for the q-binomial theorem, q-Chu–Vandermonde summation, q-Pfaff–Saalschütz summation, Jackson's ${}_8\phi_7$ summation, Bailey's ${}_{10}\phi_9$ transformation, and other important summation and transformation theorems.

See [27] for a survey on some of the main results and techniques from the theory of A_r basic hypergeometric series.

A recent major advance was the development of the theory of elliptic hypergeometric series initiated by Frenkel and Turaev [11]. This ultimately lead to the study of elliptic hypergeometric series associated with the root system A_r (and other root systems), see [30]. Some of these developments are described in Chapter 11 of Gasper and Rahman's texts [12].

A characteristic feature of A_r series is that they contain (the A_r-type product)

$$\prod_{1\le i<j\le r}\frac{(1-q^{k_i-k_j}x_i/x_j)}{(1-x_i/x_j)}$$

as a factor in the summand (while they should not, at the same time, contain factors that are characteristic for other types of multiple series, such as for C_r series, see [14]). This characteristic feature is indeed shared by all the multiple series considered in this paper which we therefore choose to label as A_r series.

When dealing with multivariable series, we shall use the compact notations

$$|\mathbf{k}|:=k_1+\dots+k_r,\quad \text{where}\quad \mathbf{k}=(k_1,\dots,k_r),$$

and

$$C:=c_1\cdots c_r,\qquad E:=e_1\cdots e_r.$$

We will need the following fundamental summation theorem by Milne [25], originally obtained by specializing an A_r q-Whipple transformation derived by partial fraction decompositions and functional equations. For a simpler, more direct proof (of the elliptic extension of Proposition 2.1) see Rosengren [30], which uses partial fraction decompositions and induction.

Proposition 2.1 ((Milne) An A_r terminating very-well-poised balanced ${}_8\phi_7$ summation)**.** *Let a, b, c, d and $x_1,\dots,x_r$ be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers, let $r\ge 1$, and suppose that none of the denominators in* (2.5) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\prod_{i=1}^r\frac{(1-ax_iq^{k_i+|\mathbf{k}|})}{(1-ax_i)}\prod_{1\le i<j\le r}\frac{(1-q^{k_i-k_j}x_i/x_j)}{(1-x_i/x_j)}\prod_{i,j=1}^r\frac{(q^{-n_j}x_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}}$$
$$\times\prod_{i=1}^r\frac{(ax_i)_{|\mathbf{k}|}\,(dx_i,a^2x_iq^{1+|\mathbf{n}|}/bcd)_{k_i}}{(ax_iq^{1+|\mathbf{n}|})_{|\mathbf{k}|}\,(ax_iq/b,ax_iq/c)_{k_i}}\cdot\frac{(b,c)_{|\mathbf{k}|}}{(aq/d,bcdq^{-|\mathbf{n}|}/a)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^r ik_i}$$
$$=\frac{(aq/bd,aq/cd)_{|\mathbf{n}|}}{(aq/d,aq/bcd)_{|\mathbf{n}|}}\prod_{i=1}^r\frac{(ax_iq,ax_iq/bc)_{n_i}}{(ax_iq/b,ax_iq/c)_{n_i}}.\tag{2.5}$$

In Section 4 we apply inverse relations to (2.5) to obtain a multivariable ${}_8\phi_7$ summation of a different type by which we extend a theorem of Bhatnagar [5, Thm. 3.6], whose result is stated as follows:

Proposition 2.2 ((Bhatnagar) An A_r terminating very-well-poised $_6\phi_5$ summation)**.** *Let a, b, c and $x_1,\dots,x_r$ be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers, let $r\ge 1$, and suppose that none of the denominators in* (2.6) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\prod_{1\le i<j\le r}\frac{(1-q^{k_i-k_j}x_i/x_j)}{(1-x_i/x_j)}\prod_{i,j=1}^{r}\frac{(q^{-n_j}x_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}}\prod_{i=1}^{r}\frac{(c/x_i)_{|\mathbf{k}|}\,x_i^{k_i}}{(ax_iq/c)_{k_i}\,(c/x_i)_{|\mathbf{k}|-k_i}}$$
$$\times\frac{(1-aq^{2|\mathbf{k}|})}{(1-a)}\frac{(a,b)_{|\mathbf{k}|}}{(aq^{1+|\mathbf{n}|},aq/b)_{|\mathbf{k}|}}\left(\frac{aq^{1+|\mathbf{n}|}}{bc}\right)^{|\mathbf{k}|}q^{-e_2(\mathbf{k})+\sum_{i=1}^{r}(i-1)k_i}$$
$$=\frac{(aq)_{|\mathbf{n}|}}{(aq/b)_{|\mathbf{n}|}}\prod_{i=1}^{r}\frac{(ax_iq/bc)_{n_i}}{(ax_iq/c)_{n_i}},\qquad(2.6)$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$.

Remark 2.3. The identity in Proposition 2.2 was derived in [5] by inverting an A_r terminating balanced $_3\phi_2$ summation from [26]. Bhatnagar calls his result a $U(r+1)$ summation theorem (which in our terminology corresponds to A_r), since the series in (2.6) looks very much like the usual $U(r+1)$ (or A_r) series. We believe this classification to be slightly inaccurate (but do not change it). The series in (2.6) is apparently of a different type than A_r which (after $r\mapsto r+1$) can often be written in form of an $\tilde{A}_r$ (or $SU(r)$) series. It actually may be more accurate to call the series in (2.6) (and the others in this paper, apart from the one appearing in (2.5)) to be of type $A_1\times A_{r-1}$ (or similar), due to the combination of two evidently different types of factors. However, in lack of a more solid (say, representation-theoretic) explanation, we (at least for now) refrain from labelling these series as $A_1\times A_{r-1}$ and simply keep referring to them as A_r series.

Remark 2.4. We note that for $r\ge 2$ it is not possible to extend the terminating summation in Proposition 2.2 by analytic continuation to a nonterminating identity. This is due to the appearance of the factor $q^{-e_2(\mathbf{k})}$ in the summand.

3. Multidimensional matrix inversions

Let $\mathbb{Z}$ denote the set of integers. In the following, we consider infinite lower-triangular r-dimensional matrices $F=(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $G=(g_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ (i.e., $f_{\mathbf{nk}}=0$ unless $\mathbf{n}\ge\mathbf{k}$, by which we mean $n_i\ge k_i$ for all $i=1,\dots,r$), and infinite sequences $(a_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^r}$ and $(b_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^r}$.

The matrix F is said to be the *inverse* of G, if and only if the following orthogonality relation holds:

$$\sum_{\mathbf{n}\ge\mathbf{k}\ge\mathbf{l}}f_{\mathbf{nk}}g_{\mathbf{kl}}=\delta_{\mathbf{nl}}\qquad\text{for all}\quad\mathbf{n},\mathbf{l}\in\mathbb{Z}^r,\qquad(3.1)$$

where $\delta_{\mathbf{nl}}$ is the usual Kronecker delta. Since F and G are lower-triangular, the sum in (3.1) is finite and also the dual relation, with the roles of F and G being interchanged, must hold at the same time.

It follows readily from the orthogonality relation (3.1) that

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} f_{\mathbf{nk}}a_{\mathbf{k}} = b_{\mathbf{n}} \qquad \text{for all } \mathbf{n}\in\mathbb{Z}^r, \tag{3.2a}$$

if and only if

$$\sum_{\mathbf{0}\le\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}}b_{\mathbf{l}} = a_{\mathbf{k}} \qquad \text{for all } \mathbf{k}\in\mathbb{Z}^r. \tag{3.2b}$$

Inverse relations are a powerful tool for proving or deriving identities. For instance, given an identity in the form (3.2a), we can immediately deduce (3.2b), which may possibly be a new identity. It is exactly this variant of multiple inverse relations which we apply in the derivation of Theorem 4.1.

One of the main results of [32] (see Thm. 3.1 therein) was the following explicit multidimensional matrix inverse which reduces to Krattenthaler's matrix inverse [21] for $r=1$.

Proposition 3.1 (A general A_r matrix inverse)**.** *Let $(a_t)_{t\in\mathbb{Z}}$ and $(c_j(t))_{t\in\mathbb{Z}}$, $1\le j\le r$, be arbitrary sequences of scalars. Then the lower-triangular r-dimensional matrices*

$$f_{\mathbf{nk}} = \frac{\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}\Big((1-a_tc_1(k_1)\cdots c_r(k_r))\prod_{j=1}^r(a_t-c_j(k_j))\Big)}{\prod_{i=1}^r\prod_{t=k_i+1}^{n_i}\Big((1-c_i(t)c_1(k_1)\cdots c_r(k_r))\prod_{j=1}^r(c_i(t)-c_j(k_j))\Big)}, \tag{3.3a}$$

and

$$g_{\mathbf{kl}} = \prod_{1\le i<j\le r}\frac{(c_i(l_i)-c_j(l_j))}{(c_i(k_i)-c_j(k_j))}\cdot\frac{(1-a_{|\mathbf{l}|}c_1(l_1)\cdots c_r(l_r))}{(1-a_{|\mathbf{k}|}c_1(k_1)\cdots c_r(k_r))}\prod_{j=1}^r\frac{(a_{|\mathbf{l}|}-c_j(l_j))}{(a_{|\mathbf{k}|}-c_j(k_j))}$$
$$\times\frac{\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}\Big((1-a_tc_1(k_1)\cdots c_r(k_r))\prod_{j=1}^r(a_t-c_j(k_j))\Big)}{\prod_{i=1}^r\prod_{t=l_i}^{k_i-1}\Big((1-c_i(t)c_1(k_1)\cdots c_r(k_r))\prod_{j=1}^r(c_i(t)-c_j(k_j))\Big)} \tag{3.3b}$$

are mutually inverse.

For an elliptic extension of the above, see [31]. Some important special cases of Proposition 3.1 include Bhatnagar and Milne's A_r matrix inverse [6, Thm. 3.48] (take $c_i(t)\mapsto x_iq^t$, for $i=1,\dots,r$), the author's D_r matrix inverse [32, Thm. 5.11] (take $a_t=0$ and $c_i(t)\mapsto x_iq^t+q^{-t}/x_i$, for $i=1,\dots,r$), and some other (non-hypergeometric) matrix inverses considered in [33, App. A].

The following corollary of Proposition 3.1 has not yet been explicitly stated, nor applied. (This is maybe surprising as it also can be derived from Bhatnagar and Milne's matrix inverse.) It is readily obtained from (3.3) by letting $a_t \mapsto (b/x_1 \dots x_r)^{\frac{1}{r+1}} aq^t$ and $c_i(t) \mapsto (b/x_1 \dots x_r)^{\frac{1}{r+1}} x_i q^t$, for $i = 1, \dots, r$, followed by some simplifications including

$$\prod_{1\le i<j\le r} \frac{(1-q^{k_i-k_j}x_i/x_j)}{(1-q^{l_i-l_j}x_i/x_j)} \prod_{i,j=1}^{r} \frac{(q^{l_i-k_j}x_i/x_j)_{k_i-l_i}}{(q^{1+l_i-l_j}x_i/x_j)_{k_i-l_i}} \\ = (-1)^{|\mathbf{k}|-|\mathbf{l}|} q^{-\binom{|\mathbf{k}|-|\mathbf{l}|}{2}-\sum_{i=1}^{r} i(k_i-l_i)}, \quad (3.4)$$

the latter of which is equivalent to Lemma 4.3 of [26] and is typical for dealing with A_r series.

Corollary 3.2 (An A_r matrix inverse)**.** *Let a, b and $x_1 \dots, x_r$ be indeterminates. Then the lower-triangular r-dimensional matrices*

$$f_{\mathbf{nk}} = \frac{(abq^{2|\mathbf{k}|})_{|\mathbf{n}|-|\mathbf{k}|} \prod_{i=1}^{r}(aq^{|\mathbf{k}|-k_i}/x_i)_{|\mathbf{n}|-|\mathbf{k}|}}{\prod_{i=1}^{r}(bx_iq^{1+k_i+|\mathbf{k}|})_{n_i-k_i} \prod_{i,j=1}^{r}(q^{1+k_i-k_j}x_i/x_j)_{n_i-k_i}}, \quad (3.5a)$$

$$g_{\mathbf{kl}} = (-1)^{|\mathbf{k}|-|\mathbf{l}|} q^{\binom{|\mathbf{k}|-|\mathbf{l}|}{2}} \frac{(1-abq^{2|\mathbf{l}|})}{(1-abq^{2|\mathbf{k}|})} \prod_{i=1}^{r} \frac{(1-aq^{|\mathbf{l}|-l_i}/x_i)}{(1-aq^{|\mathbf{k}|-k_i}/x_i)} \\ \times \frac{(abq^{1+|\mathbf{l}|+|\mathbf{k}|})_{|\mathbf{k}|-|\mathbf{l}|} \prod_{i=1}^{r}(aq^{1+|\mathbf{l}|-k_i}/x_i)_{|\mathbf{k}|-|\mathbf{l}|}}{\prod_{i=1}^{r}(bx_iq^{l_i+|\mathbf{k}|})_{k_i-l_i} \prod_{i,j=1}^{r}(q^{1+l_i-l_j}x_i/x_j)_{k_i-l_i}}. \quad (3.5b)$$

are mutually inverse.

Corollary 3.2 constitutes a multivariable extension of Bressoud's matrix inverse [7] which he extracted directly from the terminating very-well-poised ${}_6\phi_5$ summation. Bressoud's matrix inverse underlies the WP-Bailey lemma [2] which is a generalization of the classical Bailey lemma [1], both powerful tools for deriving (chains of) identities. Other multivariable extensions of Bressoud's matrix inverse have been derived (for type A) by Milne [26, Thm. 3.41], (for type C) by Lilly and Milne [24, 2nd Remark after Thm. 2.11], (for type D) by the author [32, Thm. 5.11], (of Carlitz type, 'twisted') by Krattenthaler and the author [22, Eqs. (6.4)/(6.5)], (related to A_{n-1} Macdonald polynomials) by Lassalle and the author [23, Thm. 2.7], and (related to an elliptic extension of BC_n Koornwinder–Macdonald polynomials) by Rains [29, Cor. 4.3] and by Coskun and Gustafson [9, Eq. (4.16)], and possibly others (of which the current author is not aware of).

4. New A_r terminating very-well-poised ${}_8\phi_7$ summations

We now combine the multidimensional matrix inverse in Corollary 3.2 with Milne's A_r ${}_8\phi_7$ summation in Proposition 2.1 to deduce a new multivariable ${}_8\phi_7$ summation theorem.

In particular, we have (3.2a) by the $(a,b,c)\mapsto(b,c,abq^{|\mathbf{n}|})$ case of Proposition 2.1, where

$$a_{\mathbf{k}}=\frac{(ab)_{2|\mathbf{k}|}\,(c)_{|\mathbf{k}|}}{(acd,bq/d)_{|\mathbf{k}|}}\prod_{i=1}^{r}\frac{(bx_i)_{|\mathbf{k}|}\,(a/x_i)_{|\mathbf{k}|-k_i}\,(dx_i,bx_iq/acd)_{k_i}}{(bx_i)_{k_i+|\mathbf{k}|}\,(bx_iq/c)_{k_i}}\\ \times q^{\binom{|\mathbf{k}|}{2}-\sum_{i=1}^{r}\binom{k_i}{2}}a^{|\mathbf{k}|}\prod_{i=1}^{r}x_i^{-k_i}\prod_{i,j=1}^{r}(qx_i/x_j)_{k_i}^{-1}$$

and

$$b_{\mathbf{n}}=\frac{(ab,ad,bq/cd)_{|\mathbf{n}|}}{(acd,bq/d)_{|\mathbf{n}|}}\prod_{i=1}^{r}\frac{(ac/x_i)_{|\mathbf{n}|}\,(a/x_i)_{|\mathbf{n}|-n_i}}{(bx_iq/c)_{n_i}\,(ac/x_i)_{|\mathbf{n}|-n_i}}\prod_{i,j=1}^{r}(qx_i/x_j)_{n_i}^{-1},$$

and $f_{\mathbf{nk}}$ as in (3.5a). Therefore we must have (3.2b) with the above sequences $b_{\mathbf{l}}$, $a_{\mathbf{k}}$, and $g_{\mathbf{kl}}$ as in (3.5b). In explicit terms this gives (after simplifications and the substitutions $(a,c,d,x_i,k_i,l_i)\mapsto(a/b,aq/bc,b^2/a,a^2qx_i/b^2cd,n_i,k_i)$, $i=1,\dots,r$) the following new multivariable extension of (2.4):

Theorem 4.1 (An A_r terminating very-well-poised balanced ${}_8\phi_7$ summation)**.** *Let a, b, c, d and $x_1,\dots,x_r$ be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers, let $r\ge 1$, and suppose that none of the denominators in* (4.1) *vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\prod_{1\le i<j\le r}\frac{(1-q^{k_i-k_j}x_i/x_j)}{(1-x_i/x_j)}\prod_{i,j=1}^{r}\frac{(q^{-n_j}x_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}}\\ \times\prod_{i=1}^{r}\frac{(bcd/ax_i)_{|\mathbf{k}|-k_i}\,(d/x_i)_{|\mathbf{k}|}\,(a^2x_iq^{1+|\mathbf{n}|}/bcd)_{k_i}}{(d/x_i)_{|\mathbf{k}|-k_i}\,(bcdq^{-n_i}/ax_i)_{|\mathbf{k}|}\,(ax_iq/d)_{k_i}}\\ \times\frac{(1-aq^{2|\mathbf{k}|})}{(1-a)}\frac{(a,b,c)_{|\mathbf{k}|}}{(aq^{1+|\mathbf{n}|},aq/b,aq/c)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^{r}ik_i}\\ =\frac{(aq,aq/bc)_{|\mathbf{n}|}}{(aq/b,aq/c)_{|\mathbf{n}|}}\prod_{i=1}^{r}\frac{(ax_iq/bd,ax_iq/cd)_{n_i}}{(ax_iq/d,ax_iq/bcd)_{n_i}}.\qquad(4.1)$$

By a polynomial argument, this is equivalent to the following result.

Corollary 4.2 (An A_r terminating very-well-poised balanced ${}_8\phi_7$ summation)**.** *Let a, b, $c_1,\dots,c_r$, d and $x_1,\dots,x_r$ be indeterminate, let N be a nonnegative integer, let $r\ge 1$, and suppose that none of the denominators in* (4.2) *vanish. Then*

$$\sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le N}}\prod_{1\le i<j\le r}\frac{(1-q^{k_i-k_j}x_i/x_j)}{(1-x_i/x_j)}\prod_{i,j=1}^{r}\frac{(c_jx_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(bdq^{-N}/ax_i)_{|\mathbf{k}|-k_i}\,(d/x_i)_{|\mathbf{k}|}\,(a^2x_iq^{1+N}/bCd)_{k_i}}{(d/x_i)_{|\mathbf{k}|-k_i}\,(bc_idq^{-N}/ax_i)_{|\mathbf{k}|}\,(ax_iq/d)_{k_i}}$$
$$\times \frac{(1-aq^{2|\mathbf{k}|})}{(1-a)} \frac{(a,b,q^{-N})_{|\mathbf{k}|}}{(aq/C,aq/b,aq^{1+N})_{|\mathbf{k}|}}\, q^{\sum_{i=1}^r ik_i}$$
$$= \frac{(aq,aq/bC)_N}{(aq/b,aq/C)_N} \prod_{i=1}^{r} \frac{(ax_iq/bd,ax_iq/c_id)_N}{(ax_iq/d,ax_iq/bc_id)_N}, \quad (4.2)$$

where $C = c_1 \cdots c_r$.

Proof. First we write the right side of (4.2) as quotient of infinite products using (2.1). Then by the $c = q^{-N}$ case of Theorem 4.1 it follows that the identity (4.2) holds for $c_j = q^{-n_j}$, $j = 1, \ldots, r$. By clearing out denominators in (4.2), we get a polynomial equation in c_1, which is true for q^{-n_1}, $n_1 = 0, 1, \ldots$. Thus we obtain an identity in c_1. By carrying out this process for $c_2, c_3, \ldots, c_r$ also, we obtain Corollary 4.2. □

Remark 4.3. For $b \to \infty$, Theorem 4.1 and Corollary 4.2 reduce (after relabeling of parameters) to the two respective terminating ${}_6\phi_5$ summations in Bhatnagar [5, Thms. 3.6 and 3.7], of which the first is displayed in Proposition 2.2.

The (equivalent) $b \mapsto a^2q^{1+N}/bCd$ case of (4.2) appears to be particularly useful:

$$\sum_{\substack{k_1,\ldots,k_r\geq 0\\ 0\leq|\mathbf{k}|\leq N}} \prod_{1\leq i<j\leq r} \frac{(1-q^{k_i-k_j}x_i/x_j)}{(1-x_i/x_j)} \prod_{i,j=1}^{r} \frac{(c_jx_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}}$$
$$\times \prod_{i=1}^{r} \frac{(aq/bCx_i)_{|\mathbf{k}|-k_i}\,(d/x_i)_{|\mathbf{k}|}\,(bx_i)_{k_i}}{(d/x_i)_{|\mathbf{k}|-k_i}\,(ac_iq/bCx_i)_{|\mathbf{k}|}\,(ax_iq/d)_{k_i}}$$
$$\times \frac{(1-aq^{2|\mathbf{k}|})}{(1-a)} \frac{(a,a^2q^{1+N}/bCd,q^{-N})_{|\mathbf{k}|}}{(aq/C,bCdq^{-N}/a,aq^{1+N})_{|\mathbf{k}|}}\, q^{\sum_{i=1}^r ik_i}$$
$$= \frac{(aq,aq/bd)_N}{(aq/C,aq/bCd)_N} \prod_{i=1}^{r} \frac{(aq/bCx_i,ax_iq/c_id)_N}{(ax_iq/d,ac_iq/bCx_i)_N}, \quad (4.3)$$

where $C = c_1 \cdots c_r$.

Note that in the summand of the series on the left-hand side of (4.3) the terminating integer N appears only within factors depending on $|\mathbf{k}|$. This makes it particularly convenient to combine (4.3) with sums depending N to obtain further results such as a new multivariable ${}_{10}\phi_9$ transformation. (See [31] for details, in the more general setting of elliptic hypergeometric series.)

5. New A_r terminating and nonterminating very-well-poised ${}_6\phi_5$ summations

In (4.3) we now let $N \to \infty$ (while appealing to Tannery's theorem (cf. [8]) for justification of taking term-wise limits) and obtain the following result:

Corollary 5.1 (An A_r nonterminating very-well-poised ${}_6\phi_5$ summation)**.** *Let a, b, $c_1, \dots, c_r$, d and $x_1, \dots, x_r$ be indeterminate, let $r \geq 1$, and suppose that none of the denominators in* (5.1) *vanish. Then*

$$\sum_{k_1,\dots,k_r \geq 0} \prod_{1 \leq i<j \leq r} \frac{(1-q^{k_i-k_j}x_i/x_j)}{(1-x_i/x_j)} \prod_{i,j=1}^{r} \frac{(c_j x_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}} \\ \times \prod_{i=1}^{r} \frac{(aq/bCx_i)_{|\mathbf{k}|-k_i}\,(d/x_i)_{|\mathbf{k}|}\,(bx_i)_{k_i}}{(d/x_i)_{|\mathbf{k}|-k_i}\,(ac_iq/bCx_i)_{|\mathbf{k}|}\,(ax_iq/d)_{k_i}} \\ \times \frac{(1-aq^{2|\mathbf{k}|})}{(1-a)} \frac{(a)_{|\mathbf{k}|}}{(aq/C)_{|\mathbf{k}|}} \left(\frac{aq}{bCd}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r}(i-1)k_i} \\ = \frac{(aq, aq/bd)_\infty}{(aq/C, aq/bCd)_\infty} \prod_{i=1}^{r} \frac{(aq/bCx_i, ax_iq/c_id)_\infty}{(ax_iq/d, ac_iq/bCx_i)_\infty}, \quad (5.1)$$

provided $|aq/bCd| < 1$, where $C = c_1 \cdots c_r$.

An immediate consequence of Corollary 5.1 (obtained by letting $c_i = q^{-n_i}$, $i = 1, \dots, r$, and $d \mapsto c$, in (5.1)) is the following terminating summation:

Corollary 5.2 (An A_r terminating very-well-poised ${}_6\phi_5$ summation)**.** *Let a, b, c and $x_1, \dots, x_r$ be indeterminate, let $n_1, \dots, n_r$ be nonnegative integers, let $r \geq 1$, and suppose that none of the denominators in* (5.2) *vanish. Then*

$$\sum_{\substack{0 \leq k_i \leq n_i \\ i=1,\dots,r}} \prod_{1 \leq i<j \leq r} \frac{(1-q^{k_i-k_j}x_i/x_j)}{(1-x_i/x_j)} \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}} \\ \times \prod_{i=1}^{r} \frac{(aq^{1+|\mathbf{n}|}/bx_i)_{|\mathbf{k}|-k_i}\,(c/x_i)_{|\mathbf{k}|}\,(bx_i)_{k_i}}{(c/x_i)_{|\mathbf{k}|-k_i}\,(aq^{1+|\mathbf{n}|-n_i}/bx_i)_{|\mathbf{k}|}\,(ax_iq/c)_{k_i}} \\ \times \frac{(1-aq^{2|\mathbf{k}|})}{(1-a)} \frac{(a)_{|\mathbf{k}|}}{(aq^{1+|\mathbf{n}|})_{|\mathbf{k}|}} \left(\frac{aq^{1+|\mathbf{n}|}}{bc}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r}(i-1)k_i} \\ = (aq, aq/bc)_{|\mathbf{n}|} \prod_{i=1}^{r} \frac{(aq/bx_i)_{|\mathbf{n}|-n_i}}{(aq/bx_i)_{|\mathbf{n}|}(ax_iq/c)_{n_i}}. \quad (5.2)$$

Corollary 5.2 can also be obtained from Theorem 4.1 by first replacing b by $a^2q^{1+|\mathbf{n}|}/bcd$, then letting $c \to \infty$, followed by relabeling $d \mapsto c$.

On the other hand, we can also deduce the nonterminating ${}_6\phi_5$ summation in Corollary 5.1 from the terminating sum in Corollary 5.2 by analytic continuation (by which one can pretend to avoid the explicit application of Tannery's theorem in the derivation of Corollary 5.1, however, implicitly such an application is needed to show the analyticity of the series), in the form of a repeated application of a variant of Ismail's argument [17]. Indeed, both sides of the multiple series identity in (5.1) are analytic in each of the parameters $1/c_1, \dots, 1/c_r$ in a domain around the origin (see the following paragraph). Now, the identity is true for $1/c_1 = q^{n_1}, 1/c_2 = q^{n_2}, \dots,$ and $1/c_r = q^{n_r}$, by the A_r terminating ${}_6\phi_5$ summation in Corollary 5.2. This holds for all $n_1, \dots, n_r \geq 0$. Since $\lim_{n_1\to\infty} q^{n_1} = 0$ is an interior point in the domain of analyticity of $1/c_1$, by the identity theorem we obtain an identity for general $1/c_1$. By iterating this argument for $1/c_2, \dots, 1/c_r$, we establish (5.1) for general $1/c_1, \dots, 1/c_r$ (i.e., for general $c_1, \dots, c_r$).

What remains to be shown is the claim that both sides of (5.1) are analytic in the parameters $1/c_1, \dots, 1/c_r$. This is accomplished by multiple applications of the q-binomial theorem (cf. [12])

$$\sum_{l\geq 0} \frac{(a)_l}{(q)_l} z^j = \frac{(az)_\infty}{(z)_\infty}, \qquad \text{where} \quad |z| < 1, \tag{5.3}$$

to expand both sides of the identity as a convergent multiple power series in $1/c_1, \dots, 1/c_r$. We leave the details, similar in nature to those in the proof of Theorem 6.1, to the reader.

6. A new A_r very-well-poised ${}_6\psi_6$ summation

Having Corollary 5.1, we are ready to prove the following multivariable extension of Bailey's very-well-poised ${}_6\psi_6$ summation formula in (1.1):

Theorem 6.1 (An A_r very-well-poised ${}_6\psi_6$ summation)**.** *Let a, b, $c_1, \dots, c_r$, d, $e_1, \dots, e_r$ and $x_1, \dots, x_r$ be indeterminate, let $r \geq 1$, and suppose that none of the denominators in* (6.1) *vanish. Then*

$$\sum_{-\infty\leq k_1,\dots,k_r\leq\infty} \prod_{1\leq i<j\leq r} \frac{(1-q^{k_i-k_j}x_i/x_j)}{(1-x_i/x_j)} \prod_{i,j=1}^{r} \frac{(c_jx_i/x_j)_{k_i}}{(ax_iq/e_jx_j)_{k_i}}$$
$$\times \prod_{i=1}^{r} \frac{(aq/bCx_i)_{|\mathbf{k}|-k_i}\,(dE/a^{r-1}e_ix_i)_{|\mathbf{k}|}\,(bx_i)_{k_i}}{(dE/a^rx_i)_{|\mathbf{k}|-k_i}\,(ac_iq/bCx_i)_{|\mathbf{k}|}\,(ax_iq/d)_{k_i}}$$
$$\times \frac{(1-aq^{2|\mathbf{k}|})}{(1-a)} \frac{(E/a^{r-1})_{|\mathbf{k}|}}{(aq/C)_{|\mathbf{k}|}} \left(\frac{a^{r+1}q}{bCdE}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r}(i-1)k_i}$$
$$= \frac{(aq, q/a, aq/bd)_\infty}{(aq/C, a^{r+1}q/bCdE, a^{r-1}q/E)_\infty} \prod_{i,j=1}^{r} \frac{(qx_i/x_j, ax_iq/c_ie_jx_j)_\infty}{(qx_i/c_ix_j, ax_iq/e_jx_j)_\infty}$$

$$\times \prod_{i=1}^{r} \frac{(a^r x_i q/dE, aq/be_i x_i, aq/bCx_i, ax_i q/c_i d)_\infty}{(a^{r-1} e_i x_i q/dE, q/bx_i, ax_i q/d, ac_i q/bCx_i)_\infty}, \tag{6.1}$$

provided $|aq^{r+1}/bCdE| < 1$, *where* $C = c_1 \cdots c_r$ *and* $E = e_1 \cdots e_r$.

Clearly, Theorem 6.1 reduces to Corollary 5.1 for $e_1 = e_2 = \cdots = e_r = a$. A different very-well-poised ${}_6\psi_6$ summation for the root system A_r was given by Gustafson, see [13] and [14]. Other multivariable very-well-poised ${}_6\psi_6$ summations are listed by Ito [19].

Proof of Theorem 6.1. We apply Ismail's argument [17] (see also [3]) successively to the parameters $1/e_1, \ldots, 1/e_r$ using the A_r nonterminating ${}_6\phi_5$ summation in Corollary 5.1. The multiple series identity on the left-hand side of (6.1) is analytic in each of the parameters $1/e_1, \ldots, 1/e_r$ in a domain around the origin (which is not difficult to verify, see further below). Now, the identity is true for $1/e_1 = q^{m_1}/a, 1/e_2 = q^{m_2}/a, \ldots,$ and $1/e_r = q^{m_r}/a$, by Corollary 5.1 (see the next paragraph for the details). This holds for all $m_1, \ldots, m_n \geq 0$. Since $\lim_{m_1 \to \infty} q^{m_1}/a = 0$ is an interior point in the domain of analyticity of $1/e_1$, by the identity theorem, we obtain an identity for general $1/e_1$. By iterating this argument for $1/e_2, \ldots, 1/e_r$, we establish (6.1) for general $1/e_1, \ldots, 1/e_r$ (i.e., for general $e_1, \ldots, e_r$).

The details are displayed as follows. Setting $1/e_i = q^{m_i}/a$, for $i = 1, \ldots, r$, the left-hand side of (6.1) becomes

$$\sum_{\substack{-m_i \leq k_i \leq \infty \\ i=1,\ldots,r}} \prod_{1 \leq i < j \leq r} \frac{(1 - q^{k_i - k_j} x_i/x_j)}{(1 - x_i/x_j)} \prod_{i,j=1}^{r} \frac{(c_j x_i/x_j)_{k_i}}{(q^{1+m_j} x_i/x_j)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(aq/bCx_i)_{|\mathbf{k}|-k_i} \, (dq^{m_i - |\mathbf{m}|}/x_i)_{|\mathbf{k}|} \, (bx_i)_{k_i}}{(dq^{-|\mathbf{m}|}/x_i)_{|\mathbf{k}|-k_i} \, (ac_i q/bCx_i)_{|\mathbf{k}|} \, (ax_i q/d)_{k_i}}$$

$$\times \frac{(1 - aq^{2|\mathbf{k}|})}{(1-a)} \frac{(aq^{-|\mathbf{m}|})_{|\mathbf{k}|}}{(aq/C)_{|\mathbf{k}|}} \left(\frac{aq^{1+|\mathbf{m}|}}{bCd} \right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r}(i-1)k_i}. \tag{6.2}$$

We shift the summation indices in (6.2) by $k_i \mapsto k_i - m_i$, for $i = 1, \ldots, r$ and obtain

$$\prod_{1 \leq i < j \leq r} \frac{(1 - q^{m_j - m_i} x_i/x_j)}{(1 - x_i/x_j)} \prod_{i,j=1}^{r} \frac{(c_j x_i/x_j)_{-m_i}}{(q^{1+m_j} x_i/x_j)_{-m_i}}$$

$$\times \prod_{i=1}^{r} \frac{(aq/bCx_i)_{m_i - |\mathbf{m}|} \, (dq^{m_i - |\mathbf{m}|}/x_i)_{-|\mathbf{m}|} \, (bx_i)_{-m_i}}{(dq^{-|\mathbf{m}|}/x_i)_{m_i - |\mathbf{m}|} \, (ac_i q/bCx_i)_{-|\mathbf{m}|} \, (ax_i q/d)_{-m_i}}$$

$$\times \frac{(1 - aq^{-2|\mathbf{m}|})}{(1-a)} \frac{(aq^{-|\mathbf{m}|})_{-|\mathbf{m}|}}{(aq/C)_{-|\mathbf{m}|}} \left(\frac{aq^{1+|\mathbf{m}|}}{bCd} \right)^{-|\mathbf{m}|} q^{-\sum_{i=1}^{r}(i-1)m_i}$$

$$\times \sum_{k_1,\dots k_r\geq 0} \prod_{1\leq i<j\leq r} \frac{(1-q^{m_j-m_i+k_i-k_j}x_i/x_j)}{(1-q^{m_j-m_i}x_i/x_j)} \prod_{i,j=1}^{r} \frac{(c_jq^{-m_i}x_i/x_j)_{k_i}}{(q^{1+m_j-m_i}x_i/x_j)_{k_i}}$$
$$\times \prod_{i=1}^{r} \frac{(aq^{1+m_i-|\mathbf{m}|}/bCx_i)_{|\mathbf{k}|-k_i}\,(dq^{m_i-2|\mathbf{m}|}/x_i)_{|\mathbf{k}|}\,(bx_iq^{-m_i})_{k_i}}{(dq^{m_i-2|\mathbf{m}|}/x_i)_{|\mathbf{k}|-k_i}\,(ac_iq^{1-|\mathbf{m}|}/bCx_i)_{|\mathbf{k}|}\,(ax_iq^{1-m_i}/d)_{k_i}}$$
$$\times \frac{(1-aq^{-2|\mathbf{m}|+2|\mathbf{k}|})}{(1-aq^{-2|\mathbf{m}|})} \frac{(aq^{-2|\mathbf{m}|})_{|\mathbf{k}|}}{(aq^{1-|\mathbf{m}|}/C)_{|\mathbf{k}|}} \left(\frac{aq^{1+|\mathbf{m}|}}{bCd}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^{r}(i-1)k_i}.$$

Next, we apply the $a\mapsto aq^{-2|\mathbf{m}|}$, $x_i\mapsto x_iq^{-m_i}$, $c_i\mapsto c_iq^{-m_i}$, $i=1,\dots,r$, $d\mapsto dq^{-2|\mathbf{m}|}$ case of Corollary 5.1, and obtain

$$\prod_{1\leq i<j\leq r} \frac{(1-q^{m_j-m_i}x_i/x_j)}{(1-x_i/x_j)} \prod_{i,j=1}^{r} \frac{(c_jx_i/x_j)_{-m_i}}{(q^{1+m_j}x_i/x_j)_{-m_i}}$$
$$\times \prod_{i=1}^{r} \frac{(aq/bCx_i)_{m_i-|\mathbf{m}|}\,(dq^{m_i-|\mathbf{m}|}/x_i)_{-|\mathbf{m}|}\,(bx_i)_{-m_i}}{(dq^{-|\mathbf{m}|}/x_i)_{m_i-|\mathbf{m}|}\,(ac_iq/bCx_i)_{-|\mathbf{m}|}\,(ax_iq/d)_{-m_i}}$$
$$\times \frac{(1-aq^{-2|\mathbf{m}|})}{(1-a)} \frac{(aq^{-|\mathbf{m}|})_{-|\mathbf{m}|}}{(aq/C)_{-|\mathbf{m}|}} \left(\frac{aq^{1+|\mathbf{m}|}}{bCd}\right)^{-|\mathbf{m}|} q^{-\sum_{i=1}^{r}(i-1)m_i}$$
$$\times \frac{(aq^{1-2|\mathbf{m}|}, aq/bd)_\infty}{(aq^{1-|\mathbf{m}|}/C, aq^{1+|\mathbf{m}|}/bCd)_\infty} \prod_{i=1}^{r} \frac{(aq^{1+m_i-|\mathbf{m}|}/bCx_i, ax_iq/c_id)_\infty}{(ax_iq^{1-m_i}/d, ac_iq^{1-|\mathbf{m}|}/bCx_i)_\infty}. \tag{6.3}$$

Finally, we apply several elementary identities from [12, App. I], and the $n\mapsto r$, $y_i\mapsto -m_i$, $i=1,\dots,r$, case of [26, Lem. 3.12], specifically

$$\prod_{i,j=1}^{r} (qx_i/x_j)_{m_j-m_i} = (-1)^{(r-1)|\mathbf{m}|}\, q^{-\binom{|\mathbf{m}|+1}{2}+r\sum_{i=1}^{r}\binom{m_i+1}{2}}\, q^{-\sum_{i=1}^{r}(i-1)m_i}$$
$$\times \prod_{i=1}^{r} x_i^{|\mathbf{m}|-rm_i} \prod_{1\leq i<j\leq r} \frac{(1-q^{m_j-m_i}x_i/x_j)}{(1-x_i/x_j)}, \tag{6.4}$$

to transform the expression obtained in (6.3) to

$$\frac{(aq, q/a, aq/bd)_\infty}{(aq/C, aq^{1+|\mathbf{m}|}/bCd, q^{1+|\mathbf{m}|}/a)_\infty} \prod_{i,j=1}^{r} \frac{(qx_i/x_j, q^{1+m_j}x_i/c_ix_j)_\infty}{(qx_i/c_ix_j, q^{1+m_j}x_i/x_j)_\infty}$$
$$\times \prod_{i=1}^{r} \frac{(x_iq^{1+|\mathbf{m}|}/d, q^{1+m_i}/bx_i, aq/bCx_i, ax_iq/c_id)_\infty}{(x_iq^{1+|\mathbf{m}|-m_i}/d, q/bx_i, ax_iq/d, ac_iq/bCx_i)_\infty},$$

which is exactly the $1/e_i = q^{m_i}/a$, $i=1,\dots,r$, case of the right-hand side of (6.1).

We still need to show that both sides of (6.1) are analytic in $1/e_1,\dots,1/e_r$ around the origin, i.e., that both sides can be expanded as convergent multiple powers of the variables $1/e_1, \dots, 1/e_r$. This is easily achieved by multiple use of the q-binomial theorem (5.3). For the right-hand side the claim is immediate (because all we need is to multiply expressions of the form $(z)_\infty = \sum(-1)^l q^{\binom{l}{2}} z^l/(q)_l$ and $(z)_\infty^{-1} = \sum z^l/(q)_l$). At the left-hand side we manipulate those factors in the summand of the series which involve $1/e_1, \dots, 1/e_r$ (in order to obtain a convergent multiple power series). In particular, we use

$$\begin{aligned}
&\prod_{i,j=1}^r \frac{1}{(ax_iq/e_jx_j)_{k_i}} \prod_{i=1}^r \frac{(dE/a^{r-1}e_ix_i)_{|\mathbf{k}|}}{(dE/a^rx_i)_{|\mathbf{k}|-k_i}} \cdot (E/a^{r-1})_{|\mathbf{k}|} E^{-|\mathbf{k}|} \\
&\qquad = \prod_{i,j=1}^r \frac{(ax_iq^{1+k_i}/e_jx_j)_\infty}{(ax_iq/e_jx_j)_\infty} \prod_{i=1}^r \frac{(a^{r-1}e_ix_iq^{1-|\mathbf{k}|}/dE)_{|\mathbf{k}|}}{(a^rx_iq^{1+k_i-|\mathbf{k}|}/dE)_{|\mathbf{k}|-k_i}} x_i^{-k_i} \\
&\qquad\quad \times (a^{r-1}q^{1-|\mathbf{k}|}/E)_{|\mathbf{k}|} \left(\frac{d}{a^{r-1}}\right)^{|\mathbf{k}|} q^{(r+1)\binom{|\mathbf{k}|}{2} - \sum_{i=1}^r \binom{|\mathbf{k}|-k_i}{2}} \\
&\quad = \prod_{i,j=1}^r \frac{(ax_iq^{1+k_i}/e_jx_j)_\infty}{(ax_iq/e_jx_j)_\infty} \prod_{i=1}^r \frac{(a^{r-1}e_ix_iq^{1-|\mathbf{k}|}/dE)_\infty\,(a^rx_iq/dE)_\infty}{(a^{r-1}e_ix_iq/dE)_\infty\,(a^rx_iq^{1+k_i-|\mathbf{k}|}/dE)_\infty} x_i^{-k_i} \\
&\qquad\qquad \times \frac{(a^{r-1}q^{1-|\mathbf{k}|}/E)_\infty}{(a^{r-1}q/E)_\infty} \left(\frac{d}{a^{r-1}}\right)^{|\mathbf{k}|} q^{\binom{|\mathbf{k}|}{2} + |\mathbf{k}|^2 - \sum_{i=1}^r \binom{k_i+1}{2}}, \qquad (6.5)
\end{aligned}$$

followed by multiple applications of the q-binomial theorem. The last expression thus appears implicitly as a factor of the summand of the multilateral series over $k_1, \dots, k_r$. However, it is most important that the "quadratic" powers of q in the last line of (6.5), specifically

$$q^{\binom{|\mathbf{k}|}{2} + |\mathbf{k}|^2 - \sum_{i=1}^r \binom{k_i+1}{2}} = q^{e_2(\mathbf{k}) + 2\binom{|\mathbf{k}|}{2}}$$

(where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$), ensure absolut convergence of the multiple power series in $1/e_1, \dots, 1/e_r$. $\square$

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