A NEW A_n EXTENSION OF RAMANUJAN'S ${}_1\psi_1$ SUMMATION WITH APPLICATIONS TO MULTILATERAL A_n SERIES

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ABSTRACT. In this article, we derive some identities for multilateral basic hypergeometric series associated to the root system A_n. First, we apply Ismail's [15] argument to an A_n q-binomial theorem of Milne [25, Th. 5.42] and derive a new A_n generalization of Ramanujan's ${}_1\psi_1$ summation theorem. From this new A_n ${}_1\psi_1$ summation and from an A_n ${}_1\psi_1$ summation of Gustafson [9] we deduce two lemmas for deriving simple A_n generalizations of bilateral basic hypergeometric series identities. These lemmas are closely related to the Macdonald identities for A_n. As samples for possible applications of these lemmas, we provide several A_n extensions of Bailey's ${}_2\psi_2$ transformations, and several A_n extensions of a particular ${}_2\psi_2$ summation.

1. INTRODUCTION

The theory of basic hypergeometric series (cf. [8]) consists of many known summation and transformation formulas. The most important of these is probably the q-binomial theorem, a summation first discovered by Cauchy [6]. Surpringly, the q-binomial theorem admits a bilateral generalization, the ${}_1\psi_1$ summation theorem, first discovered by Ramanujan [11]. Other important identities for basic hypergeometric series include the q-Gauß summation and Heine's ${}_2\phi_1$ transformations. These and many other basic hypergeometric series identities conspicuously appear in combinatorics and in related areas such as number theory, statistics, physics, and representation theory of Lie algebras, see Andrews [1].

Multiple basic hypergeometric series associated to the root system A_n (or equivalently, associated to the unitary group $U(n+1)$) have been investigated by various

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authors. Many different types of such series exist in the literature. The multivariable series we consider in this article have their origin in the work of the three mathematical physicists Biedenharn, Holman and Louck, see [12] and [13]. Their work was done in the context of the quantum theory of angular momentum, using methods relying on the representation theory of $U(n)$. In the sequel, substantial developments have taken place. Extensive investigations in the theory of multiple basic hypergeometric series associated to the root system A_n have been carried out by Gustafson, Milne, and their co-workers. As result, many of the classical formulas for basic hypergeometric series (cf. [8]) have already been generalized to the setting of A_n series. For some selected results on multiple basic hypergeometric series associated to A_n, see the references [5], [7], [9], [10], [18], [19], [20], [21], [22], [24], [25], [26], [27], [28], [29], and [30].

There are different methods for obtaining identities for A_n basic hypergeometric series. Partial fraction decompositions and q-difference equations are often involved in initially deriving such identities (see, e.g., [5], [10], and [18]). Further, where summations for multidimensional basic hypergeometric series are already known, multidimensional matrix inversions can often be utilized for obtaining new summation theorems for multidimensional basic hypergeometric series (see [5], [25], [26], [28], [29], and [30]). But there is also another, simpler, way of obtaining identities for A_n basic hypergeometric series. By utilizing Lemma 7.3 of Milne [25], see Lemma 4.1 in this article, and by using identities of the classical one-dimensional theory, simple identities for A_n series can be derived.

In this article, we find two multilateral generalizations of [25, Lemma 7.3], see Lemmas 4.3 and 4.9. These lemmas are closely related to the Macdonald [17] identities for the affine root system A_n. By using our lemmas combined with bilateral one-dimensional series identities we are able to derive simple multilateral identities for A_n series. We give some particular applications of this method. The A_n ${}_2\psi_2$ transformations and summations given in this article are just samples of the possible applications. It must be said that the identities obtained by this method concern A_n series of "simpler type" and are apparently not as deep as many of the A_n identities in the above mentioned references. Nevertheless, in spite of, or maybe even because of the "simplicity" of these A_n series our formulas could be useful in future applications.

Our article is organized as follows: In Section 2, we introduce some notation and give some background information. In Section 3, we apply Ismail's [15] analytic continuation argument to an A_n q-binomial theorem of Milne [25, Theorem 5.42] to derive a new A_n extension of Ramanujan's [11] ${}_1\psi_1$ summation theorem. In [19] a similar argument was used to find the first $U(n)$ generalization of the ${}_1\psi_1$ summation. More recently, motivated by [23], Kaneko [16] utilized this type of argument to derive a ${}_1\psi_1$ summation theorem for multiple basic hypergeometric series of Macdonald polynomial argument. In Section 4, we deduce from our new A_n ${}_1\psi_1$ summation and from Gustafson's [9, Theorem 1.17] $_1\psi_1$ summation two lemmas for deriving simple multilateral series identities in A_n. We discuss the connection of these lemmas with the Macdonald identities for A_n, partly following the similar analysis of [19]. Finally, in Section 5, we apply these lemmas to classical (one-dimensional) formulas for $_2\psi_2$ series. As result, we deduce several (different) A_n extensions of Bailey's [3] $_2\psi_2$ transformations, and moreover, deduce several (different) A_n extensions of a particular summation for $_2\psi_2$ series.

2. Background and notation

Let us first recall some standard basic hypergeometric notation (cf. [8]). Let q be a complex number such that $0 < |q| < 1$. We define the *q-shifted factorial* for all integers k by

$$(a)_\infty \equiv (a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j) \qquad \text{and} \qquad (a)_k \equiv (a;q)_k := \frac{(a)_\infty}{(aq^k)_\infty}.$$

For brevity, we employ the usual notation

$$(a_1,\dots,a_m)_k \equiv (a_1)_k \dots (a_m)_k$$

where k is an integer or infinity. Further, we utilize the notations

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_r)_k}{(q, b_1, \dots, b_s)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k, \tag{2.1}$$

and

$$ {}_r\psi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z\right] := \sum_{k=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_r)_k}{(b_1, b_2, \dots, b_s)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{s-r} z^k, \tag{2.2}$$

for *basic hypergeometric $_r\phi_s$ series*, and *bilateral basic hypergeometric $_r\psi_s$ series*, respectively. See [8, p. 25 and p. 125] for the criteria of when these series terminate, or, if not, when they converge. In this article, we make use of some of the elementary identities for q-shifted factorials, listed in [8, Appendix I].

Next, we note the convention for naming the multiple series in this article as A_n basic hypergeometric series. We consider multiple series of the form

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} S(\mathbf{k}),$$

where $\mathbf{k} = (k_1,\dots,k_n)$, which reduce to classical basic hypergeometric series when $n = 1$. Such a multiple series is called an A_n basic hypergeometric series if the summand $S(\mathbf{k})$ contains the factor

$$\prod_{1\le i<j\le n}\left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right). \tag{2.3}$$

A typical example is the left hand side of (3.3). A reason for naming these series as A_n series is that (2.3) is closely associated with the product side of the Weyl denominator formula for the root system A_n, see [4] and [31].

For multidimensional series, we also employ the notation $|\mathbf{k}|$ for $(k_1+\cdots+k_n)$ where $\mathbf{k}=(k_1,\dots,k_n)$. The convergence of multiple series can be checked by application of the multiple power series ratio test [14]. For explicit examples of how to use the multiple power series ratio test, see [25, Sec. 5].

3. An A_n extension of Ramanujan's ${}_1\psi_1$ summation

One of the most important summation theorems for basic hypergeometric series is the classical q-binomial theorem (cf. [8, Eq. (II.3)]),

$$ {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix}; q, z\right] = \frac{(az)_\infty}{(z)_\infty}, \tag{3.1} $$

where $|z|<1$.

A bilateral extension of (3.1) is Ramanujan's [11] ${}_1\psi_1$ summation theorem (cf. [8, Eq. (5.2.1)]),

$$ {}_1\psi_1\left[\begin{matrix} a \\ b \end{matrix}; q, z\right] = \frac{(q, b/a, az, q/az)_\infty}{(b, q/a, z, b/az)_\infty}, \tag{3.2} $$

where $|b/a|<|z|<1$. Clearly, the $b=q$ case of (3.2) is (3.1).

Theorem 5.42 of [25] is one of the many multivariable generalizations of (3.1). It can be stated as follows:

Theorem 3.1 (An A_n q-binomial theorem)**.** *Let a, $x_1,\dots,x_n$, and z be indeterminate, let $n\ge 1$, and suppose that none of the denominators in* (3.3) *vanishes. Then*

$$ \sum_{k_1,\dots,k_n=0}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} q\right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} \\ \times (a)_{|\mathbf{k}|} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n}\binom{k_i}{2}} z^{|\mathbf{k}|} \Bigg) = \frac{(az)_\infty}{(z)_\infty}, \tag{3.3} $$

provided $|z| < \left| q^{\frac{n-1}{2}} x_j^{-n} \prod_{i=1}^{n} x_i \right|$ *for* $j=1,\dots,n$.

We now apply Ismail's [15] argument and extend Theorem 3.1 to

Theorem 3.2 (An A_n ${}_1\psi_1$ summation)**.** *Let a, $b_1,\dots,b_n$, $x_1,\dots,x_n$, and z be indeterminate, let $n\ge 1$, and suppose that none of the denominators in* (3.4) *vanishes. Then*

$$ \sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} b_j\right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} $$

$$\times (a)_{|\mathbf{k}|}(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n\binom{k_i}{2}}z^{|\mathbf{k}|}\Bigg)$$

$$= \frac{(az, q/az, b_1\dots b_n q^{1-n}/a)_\infty}{(z, b_1\dots b_n q^{1-n}/az, q/a)_\infty}\prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j}q\right)_\infty}{\left(\frac{x_i}{x_j}b_j\right)_\infty}, \quad (3.4)$$

provided $|b_1\dots b_n q^{1-n}/a| < |z| < \left|q^{\frac{n-1}{2}}x_j^{-n}\prod_{i=1}^n x_i\right|$ *for* $j = 1,\dots,n$.

Proof. We apply Ismail's argument successively to the parameters $b_1,\dots,b_n$ using (3.3). The multiple series identity in (3.4) is analytic in each of the parameters $b_1,\dots,b_n$ in a domain around the origin. Now, the identity is true for $b_1 = q^{1+m_1}, b_2 = q^{1+m_2},\dots,$ and $b_n = q^{1+m_n}$, by the A_n q-binomial theorem in Theorem 3.1 (see below for the details). This holds for all $m_1,\dots,m_n \geq 0$. Since $\lim_{m_1\to\infty} q^{1+m_1} = 0$ is an interior point in the domain of analycity of b_1, by analytic continuation, we obtain an identity for b_1. By iterating this argument for $b_2,\dots,b_n$, we establish (3.4) for general $b_1,\dots,b_n$.

The details are displayed as follows. Setting $b_i = q^{1+m_i}$, for $i = 1,\dots,n$, the left side of (3.4) becomes

$$\sum_{\substack{-m_i\leq k_i\leq\infty\\ i=1,\dots,n}}\Bigg(\prod_{1\leq i<j\leq n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^n\left(\frac{x_i}{x_j}q^{1+m_j}\right)_{k_i}^{-1}\prod_{i=1}^n x_i^{nk_i-|\mathbf{k}|}$$
$$\times (a)_{|\mathbf{k}|}(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n\binom{k_i}{2}}z^{|\mathbf{k}|}\Bigg). \quad (3.5)$$

We shift the summation indices in (3.5) by $k_i \mapsto k_i - m_i$, for $i = 1,\dots,n$ and obtain

$$q^{-\binom{|\mathbf{m}|+1}{2}+n\sum_{i=1}^n\binom{m_i+1}{2}}(-1)^{(n-1)|\mathbf{m}|}(a)_{-|\mathbf{m}|}z^{-|\mathbf{m}|}\prod_{i=1}^n x_i^{|\mathbf{m}|-nm_i}\prod_{i,j=1}^n\left(\frac{x_i}{x_j}q^{1+m_j}\right)_{-m_i}^{-1}$$
$$\times\sum_{k_1,\dots,k_n=0}^{\infty}\Bigg(\prod_{1\leq i<j\leq n}\left(\frac{x_iq^{-m_i+k_i}-x_jq^{-m_j+k_j}}{x_i-x_j}\right)\prod_{i,j=1}^n\left(\frac{x_i}{x_j}q^{1+m_j-m_i}\right)_{k_i}^{-1}$$
$$\times (aq^{-|\mathbf{m}|})_{|\mathbf{k}|}(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n\binom{k_i}{2}}z^{|\mathbf{k}|}\prod_{i=1}^n\left(x_iq^{-m_i}\right)^{nk_i-|\mathbf{k}|}\Bigg)$$
$$= q^{n\sum_{i=1}^n\binom{m_i+1}{2}}(-1)^{n|\mathbf{m}|}(az)^{-|\mathbf{m}|}(q/a)_{|\mathbf{m}|}^{-1}$$
$$\times\prod_{i=1}^n x_i^{|\mathbf{m}|-nm_i}\prod_{i,j=1}^n\frac{\left(\frac{x_i}{x_j}q\right)_{m_j}}{\left(\frac{x_i}{x_j}q\right)_{m_j-m_i}}\prod_{1\leq i<j\leq n}\left(\frac{x_iq^{-m_i}-x_jq^{-m_j}}{x_i-x_j}\right)$$

$$\times \sum_{k_1,\dots,k_n=0}^{\infty} \left(\prod_{1\le i<j\le n} \left(\frac{x_i q^{-m_i+k_i} - x_j q^{-m_j+k_j}}{x_i q^{-m_i} - x_j q^{-m_j}} \right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} q^{1+m_j-m_i} \right)_{k_i}^{-1} \right.$$
$$\left. \times\, (aq^{-|\mathbf{m}|})_{|\mathbf{k}|} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^{n}\binom{k_i}{2}} z^{|\mathbf{k}|} \prod_{i=1}^{n} \left(x_i q^{-m_i} \right)^{nk_i-|\mathbf{k}|} \right). \quad (3.6)$$

Next, we apply the $y_i \mapsto -m_i$, $i = 1, \dots, n$, case of [25, Lemma 3.12], specifically

$$\prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} q \right)_{m_j-m_i} = (-1)^{(n-1)|\mathbf{m}|} q^{-\binom{|\mathbf{m}|+1}{2}+n\sum_{i=1}^{n}\binom{m_i+1}{2}}$$
$$\times \prod_{i=1}^{n} x_i^{|\mathbf{m}|-nm_i} \prod_{1\le i<j\le n} \left(\frac{x_i q^{-m_i} - x_j q^{-m_j}}{x_i - x_j} \right), \quad (3.7)$$

and the $a \mapsto aq^{-|\mathbf{m}|}$, $x_i \mapsto x_i q^{-m_i}$, $i = 1, \dots, n$, case of the multidimensional summation theorem in (3.3) to simplify the expression obtained in (3.6) to

$$q^{\binom{|\mathbf{m}|+1}{2}} (-az)^{-|\mathbf{m}|} \frac{(azq^{-|\mathbf{m}|})_\infty}{(q/a)_{|\mathbf{m}|}(z)_\infty} \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} q \right)_{m_j}.$$

Now, this can easily be further transformed into

$$\frac{(q^{1+|\mathbf{m}|}/a, az, q/az)_\infty}{(q/a, z, q^{1+|\mathbf{m}|}/az)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q \right)_\infty}{\left(\frac{x_i}{x_j} q^{1+m_j} \right)_\infty},$$

which is exactly the $b_i = q^{1+m_i}$, $i = 1, \dots, n$, case of the right side of (3.4). ☐

If we set $z \mapsto -z/a$, and $b_i = 0$, $i = 1, \dots, n$ in (3.4), and then let $a \to \infty$, we obtain an A_n generalization of Jacobi's triple product identity, equivalent to Theorem 3.7 of [19].

4. Two lemmas for deriving multilateral A_n series identities

As an immediate consequence of a fundamental theorem for A_n series [18, Theorem 1.49], the first author [25, Lemma 7.3] of this article derived the following lemma, which is

Lemma 4.1 (Milne). *Let $a_1, \dots, a_n$ and $x_1, \dots, x_n$ be indeterminate, let N be a nonnegative integer, let $n \ge 1$, and suppose that none of the denominators in* (4.1) *vanishes. Then, if $f(m)$ is an arbitrary function of nonnegative integers m, we have*

$$\sum_{m=0}^{N} \frac{(a_1 a_2 \dots a_n)_m}{(q)_m} f(m)$$

$$= \sum_{\substack{k_1,\dots,k_n\ge 0\\ 0\le|\mathbf{k}|\le N}} \prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^n \frac{(\frac{x_i}{x_j}a_j)_{k_i}}{(\frac{x_i}{x_j}q)_{k_i}}\cdot f(|\mathbf{k}|). \tag{4.1}$$

With Lemma 4.1 and one-dimensional basic hypergeometric series identities, (simple) identities for A_n series can be derived. Some examples are given in [25, Sec. 7].

In this section, we provide two new lemmas, see Lemmas 4.3 and 4.9, which similarly can be used for deriving simple A_n generalizations of *bilateral* basic hypergeometric series identities. We make use of our A_n extension of Ramanujan's ${}_1\psi_1$ summation in Theorem 3.2 and of an A_n ${}_1\psi_1$ summation by Gustafson [9, Theorem 1.17], see Theorem 4.5.

Since, for $|b_1\dots b_nq^{1-n}/a|<|z|<1$,

$${}_1\psi_1\left[\begin{matrix}a\\ b_1\dots b_nq^{1-n}\end{matrix};q,z\right]=\frac{(q,b_1\dots b_nq^{1-n}/a,az,q/az)_\infty}{(b_1\dots b_nq^{1-n},q/a,z,b_1\dots b_nq^{1-n}/az)_\infty},$$

by Ramanujan's ${}_1\psi_1$ summation (3.2), we immediately see from (3.4) that

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty}\left(\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^n\left(\frac{x_i}{x_j}b_j\right)_{k_i}^{-1}\prod_{i=1}^n x_i^{nk_i-|\mathbf{k}|}\right.$$
$$\left.\times (a)_{|\mathbf{k}|}(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n\binom{k_i}{2}}z^{|\mathbf{k}|}\right)$$
$$=\frac{(b_1\dots b_nq^{1-n})_\infty}{(q)_\infty}\prod_{i,j=1}^n\frac{\left(\frac{x_i}{x_j}q\right)_\infty}{\left(\frac{x_i}{x_j}b_j\right)_\infty}\sum_{k=-\infty}^{\infty}\frac{(a)_k}{(b_1\dots b_nq^{1-n})_k}z^k \tag{4.2}$$

(provided $|z|<1$ and $|b_1\dots b_nq^{1-n}/a|<|z|<\left|q^{\frac{n-1}{2}}x_j^{-n}\prod_{i=1}^n x_i\right|$ for $j=1,\dots,n$).

In (4.2), we equate coefficients of $(a)_mz^m$ and extract

Proposition 4.2. *Let $b_1,\dots,b_n$ and $x_1,\dots,x_n$ be indeterminate, let m be an integer, let $n\ge 1$, and suppose that none of the denominators in (4.3) vanishes. Then*

$$\sum_{\substack{-\infty\le k_1,\dots,k_n\le\infty\\ |\mathbf{k}|=m}}\left(\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^n\left(\frac{x_i}{x_j}b_j\right)_{k_i}^{-1}\prod_{i=1}^n x_i^{nk_i-|\mathbf{k}|}\right.$$
$$\left.\times(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n\binom{k_i}{2}}\right)$$
$$=\frac{(b_1\dots b_nq^{1-n})_\infty}{(q)_\infty}\prod_{i,j=1}^n\frac{\left(\frac{x_i}{x_j}q\right)_\infty}{\left(\frac{x_i}{x_j}b_j\right)_\infty}\cdot\frac{1}{(b_1\dots b_nq^{1-n})_m}. \tag{4.3}$$

We state Proposition 4.2 although it is just a special case of Proposition 4.6. We utilize the $m=0$ case of Proposition 4.2 in the proof of Theorem 5.7.

Now, if we multiply both sides of (4.3) by

$$\frac{(q)_\infty}{(b_1\dots b_n q^{1-n})_\infty}\prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j}b_j\right)_\infty}{\left(\frac{x_i}{x_j}q\right)_\infty}\cdot f(m),$$

for suitable $f(m)$, and sum over all integers m, we obtain

Lemma 4.3. *Let $b_1,\dots,b_n$, $x_1,\dots,x_n$ be indeterminate, let $n\ge 1$, and suppose that none of the denominators in (4.4) vanishes. Then, if $f(m)$ is an arbitrary function of integers m, we have*

$$\sum_{m=-\infty}^{\infty}\frac{f(m)}{(b_1\dots b_n q^{1-n})_m} = \frac{(q)_\infty}{(b_1\dots b_n q^{1-n})_\infty}\prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j}b_j\right)_\infty}{\left(\frac{x_i}{x_j}q\right)_\infty}$$
$$\times\sum_{k_1,\dots,k_n=-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^n\left(\frac{x_i}{x_j}b_j\right)_{k_i}^{-1}\prod_{i=1}^n x_i^{nk_i-|\mathbf{k}|}$$
$$\times(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n\binom{k_i}{2}}\cdot f(|\mathbf{k}|)\Bigg), \quad (4.4)$$

provided the series converge.

Thus, with Lemma 4.3, we can use one-dimensional bilateral series identities to obtain identities for multilateral A_n series.

The special case of Lemma 4.3, where $b_i=q$, for $i=1,\dots,n$ is worth noting:

Corollary 4.4. *Let $x_1,\dots,x_n$ be indeterminate, let $n\ge 1$, and suppose that none of the denominators in (4.5) vanishes. Then, if $f(m)$ is an arbitrary function of nonnegative integers m, we have*

$$\sum_{m=0}^{\infty}\frac{f(m)}{(q)_m} = \sum_{k_1,\dots,k_n=0}^{\infty}\Bigg(\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^n\left(\frac{x_i}{x_j}q\right)_{k_i}^{-1}\prod_{i=1}^n x_i^{nk_i-|\mathbf{k}|}$$
$$\times(-1)^{(n-1)|\mathbf{k}|}q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n\binom{k_i}{2}}\cdot f(|\mathbf{k}|)\Bigg), \quad (4.5)$$

provided the series converge.

Corollary 4.4 can be also obtained by specializing Lemma 4.1. Namely, by setting

$$f(m)\mapsto(-1)^m q^{-\binom{m}{2}}(a_1a_2\dots a_n)^{-m}f(m)$$

in Lemma 4.1, and then letting $N \to \infty$ and $a_i \to \infty$, for $i = 1, \dots, n$, we also obtain Corollary 4.4.

Next, we put our attention towards the derivation of another lemma for deriving multilateral series identities. For this, we utilize Gustafson's [9, Theorem 1.17] multivariable generalization of Ramanujan's ${}_1\psi_1$ summation (3.2).

Theorem 4.5 ((Gustafson) An A_n ${}_1\psi_1$ summation)**.** *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$, z, and $x_1, \dots, x_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (4.6) *vanishes. Then*

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\leq i<j\leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} a_j\right)_{k_i}}{\left(\frac{x_i}{x_j} b_j\right)_{k_i}} z^{|\mathbf{k}|}$$

$$= \frac{(a_1 \dots a_n z, q/a_1 \dots a_n z)_\infty}{(z, b_1 \dots b_n q^{1-n}/a_1 \dots a_n z)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q, \frac{x_i b_j}{x_j a_i}\right)_\infty}{\left(\frac{x_i}{x_j} b_j, \frac{x_i q}{x_j a_i}\right)_\infty}, \tag{4.6}$$

provided $|b_1 \dots b_n q^{1-n}/a_1 \dots a_n| < |z| < 1$.

Since, for $|b_1 \dots b_n q^{1-n}/a_1 \dots a_n| < |z| < 1$,

$${}_1\psi_1 \left[\begin{matrix} a_1 \dots a_n \\ b_1 \dots b_n q^{1-n} \end{matrix} ; q, z \right] = \frac{(q, b_1 \dots b_n q^{1-n}/a_1 \dots a_n, a_1 \dots a_n z, q/a_1 \dots a_n z)_\infty}{(b_1 \dots b_n q^{1-n}, q/a_1 \dots a_n, z, b_1 \dots b_n q^{1-n}/a_1 \dots a_n z)_\infty},$$

by Ramanujan's ${}_1\psi_1$ summation (3.2), we immediately see from (4.6) that

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\leq i<j\leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} a_j\right)_{k_i}}{\left(\frac{x_i}{x_j} b_j\right)_{k_i}} z^{|\mathbf{k}|}$$

$$= \frac{(b_1 \dots b_n q^{1-n}, q/a_1 \dots a_n)_\infty}{(q, b_1 \dots b_n q^{1-n}/a_1 \dots a_n)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q, \frac{x_i b_j}{x_j a_i}\right)_\infty}{\left(\frac{x_i}{x_j} b_j, \frac{x_i q}{x_j a_i}\right)_\infty} \sum_{k=-\infty}^{\infty} \frac{(a_1 \dots a_n)_k}{(b_1 \dots b_n q^{1-n})_k} z^k \tag{4.7}$$

(provided $|b_1 \dots b_n q^{1-n}/a_1 \dots a_n| < |z| < 1$).

In (4.7), we equate coefficients of z^m and extract

Proposition 4.6. *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$, and $x_1, \dots, x_n$ be indeterminate, let m be an integer, let $n \geq 1$, and suppose that none of the denominators in* (4.8) *vanishes. Then*

$$\sum_{\substack{-\infty\leq k_1,\dots,k_n\leq\infty \\ |\mathbf{k}|=m}} \prod_{1\leq i<j\leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} a_j\right)_{k_i}}{\left(\frac{x_i}{x_j} b_j\right)_{k_i}}$$

$$= \frac{(b_1 \dots b_n q^{1-n}, q/a_1 \dots a_n)_\infty}{(q, b_1 \dots b_n q^{1-n}/a_1 \dots a_n)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q, \frac{x_i b_j}{x_j a_i}\right)_\infty}{\left(\frac{x_i}{x_j} b_j, \frac{x_i q}{x_j a_i}\right)_\infty} \cdot \frac{(a_1 \dots a_n)_m}{(b_1 \dots b_n q^{1-n})_m}, \tag{4.8}$$

provided $|b_1 \dots b_n q^{1-n}/a_1 \dots a_n| < 1$.

The $b_i = b$, $i = 1, \dots, n$, case of Proposition 4.6 was established in [19, Theorem 1.21].

A specialization of Proposition 4.6 gives Proposition 4.2. Namely, if we divide both sides of (4.8) by $(a_1 \dots a_n)_m$ and then let $a_i \to \infty$, $i = 1, \dots, n$, we obtain (4.3).

The $m = 0$ case of Proposition 4.6 was established by Gustafson in [9, Theorem 1.15]:

Theorem 4.7 ((Gustafson) An A_{n-1} ${}_6\psi_6$ summation)**.** *Let* $a_1, \dots, a_n$, $b_1, \dots, b_n$, *and* $x_1, \dots, x_n$ *be indeterminate, let* $n \geq 1$, *and suppose that none of the denominators in* (4.9) *vanishes. Then*

$$\sum_{\substack{-\infty \leq k_1, \dots, k_n \leq \infty \\ |\mathbf{k}| = 0}} \prod_{1 \leq i < j \leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} a_j\right)_{k_i}}{\left(\frac{x_i}{x_j} b_j\right)_{k_i}}$$

$$= \frac{(b_1 \dots b_n q^{1-n}, q/a_1 \dots a_n)_\infty}{(q, b_1 \dots b_n q^{1-n}/a_1 \dots a_n)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q, \frac{x_i b_j}{x_j a_i}\right)_\infty}{\left(\frac{x_i}{x_j} b_j, \frac{x_i q}{x_j a_i}\right)_\infty}, \tag{4.9}$$

provided $|b_1 \dots b_n q^{1-n}/a_1 \dots a_n| < 1$.

The $n = 2$ case of Theorem 4.7 is equivalent to Bailey's [2, Eq. (4.7)] very-well-poised ${}_6\psi_6$ summation (cf. [8, Eq. (5.3.1)]).

We utilize Theorem 4.7 in the proof of Theorem 5.9.

From Theorem 4.7 we immediately deduce a ${}_1\psi_1/{}_6\psi_6$ generalization of the Macdonald identities for A_n, generalizing Theorem 1.24 of [19]. The analysis is similar to that in [19] where the $b_i = b$, $i = 1, \dots, n$, case of Theorem 4.7 was utilized to obtain [19, Theorem 1.24]. The following result appears implicitly in [10, Sec. 7].

Theorem 4.8 ((Gustafson) A ${}_1\psi_1$ generalization of the Macdonald identities for A_n)**.** *Let* $a_1, \dots, a_n$, $b_1, \dots, b_n$, *and* $x_1, \dots, x_n$ *be indeterminate, let* $n \geq 1$, *and suppose that none of the denominators in* (4.10) *vanishes. Then*

$$\sum_{\sigma \in \mathcal{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} x_{\sigma(i)}^{i-\sigma(i)} \sum_{\substack{-\infty \leq k_1, \dots, k_n \leq \infty \\ |\mathbf{k}| = 0}} q^{\sum_{i=1}^{r} (i-1) k_{\sigma(i)}} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} a_j\right)_{k_i}}{\left(\frac{x_i}{x_j} b_j\right)_{k_i}}$$

$$= \frac{(b_1 \dots b_n q^{1-n}, q/a_1 \dots a_n)_\infty}{(q, b_1 \dots b_n q^{1-n}/a_1 \dots a_n)_\infty} \prod_{1 \le i < j \le n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q, \frac{x_i b_j}{x_j a_i}\right)_\infty}{\left(\frac{x_i}{x_j} b_j, \frac{x_i q}{x_j a_i}\right)_\infty}, \quad (4.10)$$

provided $|b_1 \dots b_n q^{1-n}/a_1 \dots a_n| < 1$, *where* $\mathcal{S}_n$ *is the symmetric group of order* n, *and* $\varepsilon(\sigma)$ *is the sign of the permutation* σ.

Replacing a_i and b_i by $-1/c$ and 0, respectively, for $i = 1, \dots, n$, in Theorem 4.8, simplifying and then letting $c \to 0$ yields Equation (4.3) of [18] which is equivalent to the Macdonald identities for A_n, see [18, Sec. 4]. Thus, Theorem 4.8 may be viewed as a generalization of the Macdonald identities for A_n with the extra parameters $a_1, \dots, a_n$, and $b_1, \dots, b_n$.

For future reference, we write down the $b_i = a_i q$, $i = 1, \dots, n$, case of Theorems 4.8 and 4.6. Note that this case is valid since the convergence condition $|b_1 \dots b_n q^{1-n}/a_1 \dots a_n| < 1$ becomes $|q| < 1$. After a routine simplification, (4.10) becomes

$$\sum_{\sigma \in \mathcal{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} x_{\sigma(i)}^{i-\sigma(i)} \sum_{\substack{-\infty \le k_1, \dots, k_n \le \infty \\ |\mathbf{k}|=0}} q^{\sum_{i=1}^{r}(i-1)k_{\sigma(i)}} \prod_{i,j=1}^{n} \frac{\left(1 - \frac{x_i}{x_j} a_j\right)}{\left(1 - \frac{x_i}{x_j} a_j q^{k_i}\right)}$$

$$= \frac{(a_1 \dots a_n q, q/a_1 \dots a_n)_\infty}{(q, q)_\infty} \prod_{1 \le i < j \le n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q, \frac{x_i a_j}{x_j a_i} q\right)_\infty}{\left(\frac{x_i}{x_j} a_j q, \frac{x_i q}{x_j a_i}\right)_\infty}. \quad (4.11)$$

Similarly, (4.8) becomes

$$\sum_{\substack{-\infty \le k_1, \dots, k_n \le \infty \\ |\mathbf{k}|=m}} \prod_{1 \le i < j \le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \frac{\left(1 - \frac{x_i}{x_j} a_j\right)}{\left(1 - \frac{x_i}{x_j} a_j q^{k_i}\right)}$$

$$= \frac{(a_1 \dots a_n, q/a_1 \dots a_n)_\infty}{(1 - a_1 \dots a_n q^m)(q, q)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q, \frac{x_i a_j}{x_j a_i} q\right)_\infty}{\left(\frac{x_i}{x_j} a_j q, \frac{x_i q}{x_j a_i}\right)_\infty}. \quad (4.12)$$

Equations (4.11) and (4.12) extend (3.16) and (3.17) of [19], respectively, to which they reduce when $a_i = a$, for $i = 1, \dots, n$.

Now, let us return to our objective of finding a multilateral generalization of Lemma 4.1. If we multiply both sides of (4.8) by

$$\frac{(q, b_1 \dots b_n q^{1-n}/a_1 \dots a_n)_\infty}{(b_1 \dots b_n q^{1-n}, q/a_1 \dots a_n)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} b_j, \frac{x_i q}{x_j a_i}\right)_\infty}{\left(\frac{x_i}{x_j} q, \frac{x_i b_j}{x_j a_i}\right)_\infty} \cdot g(m),$$

for suitable $g(m)$, and sum over all integers m, we obtain

Lemma 4.9. *Let $a_1,\dots,a_n$, $b_1,\dots,b_n$, $x_1,\dots,x_n$ be indeterminate, let $n\ge 1$, and suppose that none of the denominators in (4.13) vanishes. Then, if $g(m)$ is an arbitrary function of integers m, we have*

$$\sum_{m=-\infty}^{\infty}\frac{(a_1\dots a_n)_m}{(b_1\dots b_nq^{1-n})_m}g(m)$$
$$=\frac{(q,b_1\dots b_nq^{1-n}/a_1\dots a_n)_\infty}{(b_1\dots b_nq^{1-n},q/a_1\dots a_n)_\infty}\prod_{i,j=1}^{n}\frac{\left(\frac{x_i}{x_j}b_j,\frac{x_iq}{x_ja_i}\right)_\infty}{\left(\frac{x_i}{x_j}q,\frac{x_ib_j}{x_ja_i}\right)_\infty}$$
$$\times\sum_{k_1,\dots,k_n=-\infty}^{\infty}\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^{n}\frac{\left(\frac{x_i}{x_j}a_j\right)_{k_i}}{\left(\frac{x_i}{x_j}b_j\right)_{k_i}}\cdot g(|\mathbf{k}|), \quad (4.13)$$

provided the series converge.

Hence, besides Lemma 4.3, we can also use Lemma 4.9 with one-dimensional bilateral series identities to obtain identities for multilateral A_n series. Lemma 4.9 generalizes the $N\to\infty$ case of Lemma 4.1 by additional parameters $b_1,\dots,b_n$, since the special case $b_i=q$, for $i=1,\dots,n$, of Lemma 4.9 boils down to the $N\to\infty$ case of Lemma 4.1.

5. Applications: Some ${}_2\psi_2$ formulas in A_n

In this section, we illustrate the usefulness of the lemmas of the preceding section and provide some multidimensional extensions of Bailey's [3] ${}_2\psi_2$ transformations, associated to the root system A_n. Further, as interesting special cases of these ${}_2\psi_2$ transformations in A_n, we provide some ${}_2\psi_2$ summations in A_n.

Using Ramanujan's ${}_1\psi_1$ summation (3.2) and elementary manipulations of series, Bailey [3, Eq. (2.3)] derived the transformation

$${}_2\psi_2\left[\begin{matrix}a,b\\c,d\end{matrix};q,z\right]=\frac{(az,d/a,c/b,dq/abz)_\infty}{(z,d,q/b,cd/abz)_\infty}\,{}_2\psi_2\left[\begin{matrix}a,abz/d\\az,c\end{matrix};q,\frac{d}{a}\right], \quad (5.1)$$

where $\max(|z|,|cd/abz|,|d/a|,|c/b|)<1$.

Bailey's ${}_2\psi_2$ transformation can be iterated. The result is [3, Eq. (2.4)]

$${}_2\psi_2\left[\begin{matrix}a,b\\c,d\end{matrix};q,z\right]=\frac{(az,bz,cq/abz,dq/abz)_\infty}{(q/a,q/b,c,d)_\infty}\,{}_2\psi_2\left[\begin{matrix}abz/c,abz/d\\az,bz\end{matrix};q,\frac{cd}{abz}\right], \quad (5.2)$$

where $\max(|z|,|cd/abz|)<1$.

We can specialize (5.1) (or (5.2)) to obtain a summation theorem for a particular ${}_2\psi_2$ series. If $d=bq$ and $z=q/a$ in (5.1), then the series on the right side reduces just to one term, 1, and we have the summation

$$ {}_2\psi_2\left[\begin{matrix} a, b \\ c, bq \end{matrix}; q, \frac{q}{a}\right] = \frac{(q, q, bq/a, c/b)_\infty}{(q/a, bq, q/b, c)_\infty}, \tag{5.3} $$

where $\max(|q/a|, |c|) < 1$.

In the following subsections, we combine our Lemmas 4.3 and 4.9 from Section 4 together with the above one-dimensional ${}_2\psi_2$ formulas. In Subsection 5.1, we derive several multivariable extensions of Bailey's ${}_2\psi_2$ transformation formulas (5.1) and (5.2). In Subsection 5.2 we derive multivariable extensions of the ${}_2\psi_2$ summation in (5.3).

5.1. **Some A_n extensions of Bailey's ${}_2\psi_2$ transformations.** We give several (but not all) of the possible A_n ${}_2\psi_2$ transformations which arise from Lemmas 4.3 and 4.9.

We start with two multivariable extensions of (5.1) which arise from Lemma 4.3.

Theorem 5.1 (An A_n ${}_2\psi_2$ transformation). *Let a, b, $c_1, \ldots, c_n$, d, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, and z be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (5.4) *vanishes. Then*

$$ \begin{aligned} &\sum_{k_1,\ldots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\leq i<j\leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} c_j\right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|} \\ &\qquad \times \frac{(a, b)_{|\mathbf{k}|}}{(d)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n}\binom{k_i}{2}} z^{|\mathbf{k}|} \Bigg) \\ &= \frac{(az, d/a, c_1 \ldots c_n q^{1-n}/b, dq/abz)_\infty}{(z, d, q/b, c_1 \ldots c_n dq^{1-n}/abz)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q, \frac{y_i}{y_j} c_j\right)_\infty}{\left(\frac{y_i}{y_j} q, \frac{x_i}{x_j} c_j\right)_\infty} \\ &\quad \times \sum_{k_1,\ldots,k_n=-\infty}^{\infty} \Bigg(\prod_{1\leq i<j\leq n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j}\right) \prod_{i,j=1}^{n} \left(\frac{y_i}{y_j} c_j\right)_{k_i}^{-1} \prod_{i=1}^{n} y_i^{nk_i - |\mathbf{k}|} \\ &\qquad \times \frac{(a, abz/d)_{|\mathbf{k}|}}{(az)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n}\binom{k_i}{2}} \left(\frac{d}{a}\right)^{|\mathbf{k}|} \Bigg), \end{aligned} \tag{5.4} $$

provided $|c_1 \ldots c_n dq^{1-n}/ab| < |z| < \left|q^{\frac{n-1}{2}} x_j^{-n} \prod_{i=1}^{n} x_i\right|$ *and* $|c_1 \ldots c_n dq^{1-n}/ab| < |d/a| < \left|q^{\frac{n-1}{2}} y_j^{-n} \prod_{i=1}^{n} y_i\right|$ *for* $j = 1, \ldots, n$.

Proof. We have, for $\max(|z|, |c_1 \ldots c_n dq^{1-n}/abz|, |d/a|, |c_1 \ldots c_n q^{1-n}/b|) < 1$,

$$ {}_2\psi_2\left[\begin{matrix} a, b \\ c_1 \ldots c_n q^{1-n}, d \end{matrix}; q, z\right] $$

$$= \frac{(az, d/a, c_1 \dots c_n q^{1-n}/b, dq/abz)_\infty}{(z, d, q/b, c_1 \dots c_n dq^{1-n}/abz)_\infty} \, {}_2\psi_2 \left[\begin{matrix} a, abz/d \\ az, c_1 \dots c_n q^{1-n} \end{matrix} ; q, \frac{d}{a} \right], \quad (5.5)$$

by Bailey's ${}_2\psi_2$ transformation in (5.1). Now we apply Lemma 4.3 to the ${}_2\psi_2$'s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on left side of (5.5) by the $b_i \mapsto c_i$, $i = 1, \dots, n$, and

$$f(m) = \frac{(a, b)_m}{(d)_m} z^m$$

case of Lemma 4.3. The ${}_2\psi_2$ on the right side of (5.5) is rewritten by the $b_i \mapsto c_i$, $x_i \mapsto y_i$, $i = 1, \dots, n$, and

$$f(m) = \frac{(a, abz/d)_m}{(az)_m} \left(\frac{d}{a} \right)^m$$

case of Lemma 4.3. Finally, we divide both sides of the resulting equation by

$$\frac{(q)_\infty}{(c_1 \dots c_n q^{1-n})_\infty} \prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j} c_j \right)_\infty}{\left(\frac{x_i}{x_j} q \right)_\infty} \quad (5.6)$$

and simplify to obtain (5.4). ☐

Theorem 5.2 (An A_n ${}_2\psi_2$ transformation). *Let $a_1, a_2, \dots, a_n$, b, $c_1, \dots, c_n$, d, $x_1, \dots, x_n$, $y_1, \dots, y_n$, and $z_1, \dots, z_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (5.7) *vanishes. Write $A \equiv a_1 \dots a_n$, $C \equiv c_1 \dots c_n$, and $Z \equiv z_1 \dots z_n$, for short. Then*

$$\sum_{k_1, \dots, k_n = -\infty}^{\infty} \Bigg(\prod_{1 \leq i < j \leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^n \left(\frac{x_i}{x_j} c_j \right)_{k_i}^{-1} \prod_{i=1}^n x_i^{nk_i - |\mathbf{k}|}$$
$$\times \frac{(Aq^{1-n}, b)_{|\mathbf{k}|}}{(d)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n \sum_{i=1}^n \binom{k_i}{2}} Z^{|\mathbf{k}|} \Bigg)$$
$$= \frac{(Cq^{1-n}, dq^{n-1}/A, Cq^{1-n}/b, dq^n/AbZ)_\infty}{(Z, d, q/b, Cd/AbZ)_\infty} \prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j} q, \frac{y_i}{y_j} a_j z_j \right)_\infty}{\left(\frac{y_i}{y_j} q, \frac{x_i}{x_j} c_j \right)_\infty}$$
$$\times \sum_{k_1, \dots, k_n = -\infty}^{\infty} \Bigg(\prod_{1 \leq i < j \leq n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j} \right) \prod_{i,j=1}^n \left(\frac{y_i}{y_j} a_j z_j \right)_{k_i}^{-1} \prod_{i=1}^n y_i^{nk_i - |\mathbf{k}|}$$
$$\times \frac{(Aq^{1-n}, AbZq^{1-n}/d)_{|\mathbf{k}|}}{(Cq^{1-n})_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n \sum_{i=1}^n \binom{k_i}{2}} \left(\frac{dq^{n-1}}{A} \right)^{|\mathbf{k}|} \Bigg), \quad (5.7)$$

provided that $|Cd/Ab| < |Z| < \left|q^{\frac{n-1}{2}} x_j^{-n} \prod_{i=1}^n x_i\right|$ *and* $|Cd/Ab| < |dq^{n-1}/A| < \left|q^{\frac{n-1}{2}} y_j^{-n} \prod_{i=1}^n y_i\right|$ *for* $j = 1, \ldots, n$.

Proof. We have, for $\max(|Z|, |Cd/AbZ|, |dq^{n-1}/A|, |Cq^{1-n}/b|) < 1$,

$$
{}_2\psi_2\left[\begin{matrix} Aq^{1-n}, b \\ Cq^{1-n}, d \end{matrix}; q, Z\right] = \frac{(AZq^{1-n}, dq^{n-1}/A, Cq^{1-n}/b, dq^n/AbZ)_\infty}{(Z, d, q/b, Cd/AbZ)_\infty} \times {}_2\psi_2\left[\begin{matrix} Aq^{1-n}, AbZq^{1-n}/d \\ AZq^{1-n}, Cq^{1-n} \end{matrix}; q, \frac{dq^{n-1}}{A}\right], \quad (5.8)
$$

by Bailey's ${}_2\psi_2$ transformation in (5.1). Now we apply Lemma 4.3 to the ${}_2\psi_2$'s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on left side of (5.8) by the $b_i \mapsto c_i$, $i = 1, \ldots, n$, and

$$
f(m) = \frac{(Aq^{1-n}, b)_m}{(d)_m} Z^m
$$

case of Lemma 4.3. The ${}_2\psi_2$ on the right side of (5.8) is rewritten by the $b_i \mapsto a_i z_i$, $x_i \mapsto y_i$, $i = 1, \ldots, n$, and

$$
f(m) = \frac{(Aq^{1-n}, AbZq^{1-n}/d)_m}{(Cq^{1-n})_m} \left(\frac{dq^{n-1}}{A}\right)^m
$$

case of Lemma 4.3. Finally, we divide both sides of the resulting equation by (5.6) and simplify to obtain (5.7). □

Next, we give two multivariable extensions of (5.1) which arise from Lemma 4.9.

Theorem 5.3 (An A_n ${}_2\psi_2$ transformation)**.** *Let* $a_1, a_2, \ldots, a_n$, b, $c_1, \ldots, c_n$, d, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, *and* z *be indeterminate, let* $n \geq 1$, *and suppose that none of the denominators in* (5.9) *vanishes. Write* $A \equiv a_1 \ldots a_n$ *and* $C \equiv c_1 \ldots c_n$, *for short. Then*

$$
\sum_{k_1,\ldots,k_n=-\infty}^{\infty} \prod_{1\leq i<j\leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j} a_j\right)_{k_i}}{\left(\frac{x_i}{x_j} c_j\right)_{k_i}} \frac{(b)_{|\mathbf{k}|}}{(d)_{|\mathbf{k}|}} z^{|\mathbf{k}|}
$$

$$
= \frac{(Az, d/A, Cq^{1-n}/b, dq/Abz)_\infty}{(z, d, q/b, Cdq^{1-n}/Abz)_\infty} \prod_{i,j=1}^n \frac{\left(\frac{y_i}{y_j} c_j, \frac{y_i q}{y_j a_i}, \frac{x_i}{x_j} q, \frac{x_i c_j}{x_j a_i}\right)_\infty}{\left(\frac{x_i}{x_j} c_j, \frac{x_i q}{x_j a_i}, \frac{y_i}{y_j} q, \frac{y_i c_j}{y_j a_i}\right)_\infty}
$$

$$
\times \sum_{k_1,\ldots,k_n=-\infty}^{\infty} \prod_{1\leq i<j\leq n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j}\right) \prod_{i,j=1}^n \frac{\left(\frac{y_i}{y_j} a_j\right)_{k_i}}{\left(\frac{y_i}{y_j} c_j\right)_{k_i}} \frac{(Abz/d)_{|\mathbf{k}|}}{(Az)_{|\mathbf{k}|}} \left(\frac{d}{A}\right)^{|\mathbf{k}|}, \quad (5.9)
$$

provided $|Cdq^{1-n}/Ab| < |z| < 1$ *and* $|Cdq^{1-n}/Ab| < |d/A| < 1$.

Proof. We have, for $\max(|z|,|Cdq^{1-n}/Abz|,|d/A|,|Cq^{1-n}/b|)<1$,

$$
{}_2\psi_2\left[\begin{matrix}A,b\\ Cq^{1-n},d\end{matrix};q,z\right]=\frac{(Az,d/A,Cq^{1-n}/b,dq/Abz)_\infty}{(z,d,q/b,Cdq^{1-n}/Abz)_\infty}\,{}_2\psi_2\left[\begin{matrix}A,Abz/d\\ Az,Cq^{1-n}\end{matrix};q,\frac{d}{A}\right], \tag{5.10}
$$

by Bailey's ${}_2\psi_2$ transformation in (5.1). Now we apply Lemma 4.9 to the ${}_2\psi_2$'s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on left side of (5.10) by the $b_i\mapsto c_i$, $i=1,\dots,n$, and

$$
g(m)=\frac{(b)_m}{(d)_m}z^m
$$

case of Lemma 4.9. The ${}_2\psi_2$ on the right side of (5.10) is rewritten by the $b_i\mapsto c_i$, $x_i\mapsto y_i$, $i=1,\dots,n$, and

$$
g(m)=\frac{(Abz/d)_m}{(Az)_m}\left(\frac{d}{A}\right)^m
$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by

$$
\frac{(q,Cq^{1-n}/A)_\infty}{(Cq^{1-n},q/A)_\infty}\prod_{i,j=1}^n\frac{\left(\frac{x_i}{x_j}c_j,\frac{x_iq}{x_ja_i}\right)_\infty}{\left(\frac{x_i}{x_j}q,\frac{x_ic_j}{x_ja_i}\right)_\infty} \tag{5.11}
$$

and simplify to obtain (5.9). ∎

Theorem 5.4 (An A_n ${}_2\psi_2$ transformation)**.** *Let $a_1,\dots,a_n$, $b_1,\dots,b_n$, c, $d_1,\dots,d_n$, $x_1,\dots,x_n$, $y_1,\dots,y_n$, and $z_1,\dots,z_n$ be indeterminate, let $n\ge 1$, and suppose that none of the denominators in* (5.12) *vanishes. Write $A\equiv a_1\dots a_n$, $B\equiv b_1\dots b_n$, $D\equiv d_1\dots d_n$, and $Z\equiv z_1\dots z_n$, for short. Then*

$$
\begin{aligned}
&\sum_{k_1,\dots,k_n=-\infty}^{\infty}\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^n\frac{\left(\frac{x_i}{x_j}b_j\right)_{k_i}}{\left(\frac{x_i}{x_j}d_j\right)_{k_i}}\frac{(Aq^{1-n})_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}}Z^{|\mathbf{k}|}\\
&\quad=\frac{(D/A,c/B)_\infty}{(Z,cD/ABZ)_\infty}\prod_{i,j=1}^n\frac{\left(\frac{y_i}{y_j}a_jz_j,\frac{y_id_iq}{y_ja_ib_iz_i},\frac{x_i}{x_j}q,\frac{x_id_j}{x_jb_i}\right)_\infty}{\left(\frac{x_i}{x_j}d_j,\frac{x_iq}{x_jb_i},\frac{y_i}{y_j}q,\frac{y_id_ia_jz_j}{y_ja_ib_iz_i}\right)_\infty}\\
&\quad\times\sum_{k_1,\dots,k_n=-\infty}^{\infty}\prod_{1\le i<j\le n}\left(\frac{y_iq^{k_i}-y_jq^{k_j}}{y_i-y_j}\right)\prod_{i,j=1}^n\frac{\left(\frac{y_ia_jb_jz_j}{y_jd_j}\right)_{k_i}}{\left(\frac{y_i}{y_j}a_jz_j\right)_{k_i}}\frac{(Aq^{1-n})_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}}\left(\frac{D}{A}\right)^{|\mathbf{k}|},
\end{aligned} \tag{5.12}
$$

provided $|cD/AB|<|Z|<1$ and $|cD/AB|<|D/A|<1$.

Proof. We have, for $\max(|Z|,|cD/ABZ|,|D/A|,|c/B|)<1$,

$$
\begin{aligned}
&{}_2\psi_2\left[\begin{matrix}Aq^{1-n}, B\\ c, Dq^{1-n}\end{matrix}; q, Z\right]\\
&\qquad = \frac{(AZq^{1-n}, D/A, c/B, Dq/ABZ)_\infty}{(Z, Dq^{1-n}, q/B, cD/ABZ)_\infty}\, {}_2\psi_2\left[\begin{matrix}Aq^{1-n}, ABZ/D\\ AZq^{1-n}, c\end{matrix}; q, \frac{D}{A}\right], \qquad (5.13)
\end{aligned}
$$

by Bailey's ${}_2\psi_2$ transformation in (5.1). Now we apply Lemma 4.9 to the ${}_2\psi_2$'s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on left side of (5.13) by the $a_i \mapsto b_i$, $b_i \mapsto d_i$, $i = 1, \dots, n$, and

$$
g(m) = \frac{(Aq^{1-n})_m}{(c)_m} Z^m
$$

case of Lemma 4.9. The ${}_2\psi_2$ on the right side of (5.13) is rewritten by the $a_i \mapsto a_i b_i z_i/d_i$, $b_i \mapsto a_i z_i$, $x_i \mapsto y_i$, $i = 1, \dots, n$, and

$$
g(m) = \frac{(Aq^{1-n})_m}{(c)_m} \left(\frac{D}{A}\right)^m
$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by

$$
\frac{(q, Dq^{1-n}/B)_\infty}{(Dq^{1-n}, q/B)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} d_j, \frac{x_i q}{x_j b_i}\right)_\infty}{\left(\frac{x_i}{x_j} q, \frac{x_i d_j}{x_j b_i}\right)_\infty} \qquad (5.14)
$$

and simplify to obtain (5.12). □

Finally, we provide two multivariable extensions of (5.2) which arise from Lemmas 4.3 and 4.9, respectively.

Theorem 5.5 (An A_n ${}_2\psi_2$ transformation)**.** *Let $a_1, a_2, \dots, a_n$, b, $c_1, \dots, c_n$, d, $x_1, \dots, x_n$, $y_1, \dots, y_n$, and $z_1, \dots, z_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in (5.15) vanishes. Write $A \equiv a_1 \dots a_n$, $C \equiv c_1 \dots c_n$, and $Z \equiv z_1 \dots z_n$, for short. Then*

$$
\begin{aligned}
&\sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1 \leq i < j \leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} c_j\right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i - |\mathbf{k}|}\\
&\qquad \times \frac{(Aq^{1-n}, b)_{|\mathbf{k}|}}{(d)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n\sum_{i=1}^{n}\binom{k_i}{2}} Z^{|\mathbf{k}|} \Bigg)\\
&\qquad = \frac{(bZ, Cq/AbZ, dq^n/AbZ)_\infty}{(q^n/A, q/b, d)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q, \frac{y_i}{y_j} a_j z_j\right)_\infty}{\left(\frac{y_i}{y_j} q, \frac{x_i}{x_j} c_j\right)_\infty}\\
&\quad \times \sum_{k_1,\dots,k_n=-\infty}^{\infty} \Bigg(\prod_{1 \leq i < j \leq n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j}\right) \prod_{i,j=1}^{n} \left(\frac{y_i}{y_j} a_j z_j\right)_{k_i}^{-1} \prod_{i=1}^{n} y_i^{nk_i - |\mathbf{k}|}
\end{aligned}
$$

$$\times \frac{(AbZ/C, AbZq^{1-n}/d)_{|\mathbf{k}|}}{(bZ)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^n \binom{k_i}{2}} \left(\frac{Cd}{AbZ}\right)^{|\mathbf{k}|} \Bigg), \quad (5.15)$$

provided that $|Cd/Ab| < |Z| < \left|q^{\frac{n-1}{2}} x_j^{-n} \prod_{i=1}^n x_i\right|$ *and* $|Cd/Ab| < |Cd/AbZ| < \left|q^{\frac{n-1}{2}} y_j^{-n} \prod_{i=1}^n y_i\right|$ *for* $j = 1, \ldots, n$.

Proof. We have, for $\max(|Z|, |Cd/AbZ|) < 1$,

$$\begin{aligned} {}_2\psi_2\left[\begin{matrix} Aq^{1-n}, b \\ Cq^{1-n}, d \end{matrix}; q, Z\right] &= \frac{(AZq^{1-n}, bZ, Cq/AbZ, dq^n/AbZ)_\infty}{(q^n/A, q/b, Cq^{1-n}, d)_\infty} \\ &\quad \times {}_2\psi_2\left[\begin{matrix} AbZ/C, AbZq^{1-n}/d \\ AZq^{1-n}, bZ \end{matrix}; q, \frac{Cd}{AbZ}\right], \end{aligned} \quad (5.16)$$

by Bailey's ${}_2\psi_2$ transformation in (5.2). Now we apply Lemma 4.3 to the ${}_2\psi_2$'s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on left side of (5.16) by the $b_i \mapsto c_i$, $i = 1, \ldots, n$, and

$$f(m) = \frac{(Aq^{1-n}, b)_m}{(d)_m} Z^m$$

case of Lemma 4.3. The ${}_2\psi_2$ on the right side of (5.16) is rewritten by the $b_i \mapsto a_i z_i$, $x_i \mapsto y_i$, $i = 1, \ldots, n$, and

$$f(m) = \frac{(AbZ/C, AbZq^{1-n}/d)_m}{(bZ)_m} \left(\frac{Cd}{AbZ}\right)^m$$

case of Lemma 4.3. Finally, we divide both sides of the resulting equation by (5.6) and simplify to obtain (5.15). ☐

Theorem 5.6 (An A_n ${}_2\psi_2$ transformation)**.** *Let* $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, c, $d_1, \ldots, d_n$, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, *and* $z_1, \ldots, z_n$ *be indeterminate, let* $n \geq 1$, *and suppose that none of the denominators in* (5.17) *vanishes. Write* $A \equiv a_1 \ldots a_n$, $B \equiv b_1 \ldots b_n$, $D \equiv d_1 \ldots d_n$, *and* $Z \equiv z_1 \ldots z_n$, *for short. Then*

$$\begin{aligned} &\sum_{k_1,\ldots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j} b_j\right)_{k_i}}{\left(\frac{x_i}{x_j} d_j\right)_{k_i}} \frac{(Aq^{1-n})_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}} Z^{|\mathbf{k}|} \\ &= \frac{(BZ, cq^n/ABZ)_\infty}{(q^n/A, c)_\infty} \prod_{i,j=1}^n \frac{\left(\frac{y_i}{y_j} a_j z_j, \frac{y_i d_i q}{y_j a_i b_i z_i}, \frac{x_i}{x_j} q, \frac{x_i d_j}{x_j b_i}\right)_\infty}{\left(\frac{x_i}{x_j} d_j, \frac{x_i q}{x_j b_i}, \frac{y_i}{y_j} q, \frac{y_i d_i a_j z_j}{y_j a_i b_i z_i}\right)_\infty} \end{aligned}$$

$$\times \sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n} \left(\frac{y_iq^{k_i}-y_jq^{k_j}}{y_i-y_j}\right) \prod_{i,j=1}^{n} \frac{\left(\frac{y_ia_jb_jz_j}{y_jd_j}\right)_{k_i}}{\left(\frac{y_i}{y_j}a_jz_j\right)_{k_i}} \frac{(ABZq^{1-n}/c)_{|\mathbf{k}|}}{(BZ)_{|\mathbf{k}|}} \left(\frac{cD}{ABZ}\right)^{|\mathbf{k}|}, \tag{5.17}$$

provided $|cD/AB|<|Z|<1$.

Proof. We have, for $\max(|Z|,|cD/ABZ|)<1$,

$$ {}_2\psi_2\left[\begin{matrix}Aq^{1-n},B\\ c,Dq^{1-n}\end{matrix};q,Z\right] = \frac{(AZq^{1-n},BZ,cq^n/ABZ,Dq/ABZ)_\infty}{(q^n/A,q/B,c,Dq^{1-n})_\infty} \times {}_2\psi_2\left[\begin{matrix}ABZq^{1-n}/c,ABZ/D\\ AZq^{1-n},BZ\end{matrix};q,\frac{cD}{ABZ}\right], \tag{5.18}$$

by Bailey's ${}_2\psi_2$ transformation in (5.2). Now we apply Lemma 4.9 to the ${}_2\psi_2$'s on the left and on the right side of this transformation. Specifically, we rewrite the ${}_2\psi_2$ on left side of (5.18) by the $a_i\mapsto b_i$, $b_i\mapsto d_i$, $i=1,\dots,n$, and

$$g(m)=\frac{(Aq^{1-n})_m}{(c)_m}Z^m$$

case of Lemma 4.9. The ${}_2\psi_2$ on the right side of (5.18) is rewritten by the $a_i\mapsto a_ib_iz_i/d_i$, $b_i\mapsto a_iz_i$, $x_i\mapsto y_i$, $i=1,\dots,n$, and

$$g(m)=\frac{(ABZq^{1-n}/c)_m}{(BZ)_m}\left(\frac{cD}{ABZ}\right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by (5.14) and simplify to obtain (5.17). □

5.2. **Some A_n ${}_2\psi_2$ summations.** Here, we work out (all) the A_n extensions of the ${}_2\psi_2$ summation in (5.3) which arise from Lemmas 4.3 and 4.9, respectively.

First, we give two multivariable extensions of (5.3) which arise from Lemma 4.3.

Theorem 5.7 (An A_n ${}_2\psi_2$ summation)**.** *Let a, b, $c_1,\dots,c_n$, and $x_1,\dots,x_n$ be indeterminate, let $n\ge 1$, and suppose that none of the denominators in* (5.19) *vanishes. Then*

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \left(\prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right) \prod_{i,j=1}^{n}\left(\frac{x_i}{x_j}c_j\right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{nk_i-|\mathbf{k}|} \times \frac{(a,b)_{|\mathbf{k}|}}{(bq)_{|\mathbf{k}|}}(-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2}+n\sum_{i=1}^{n}\binom{k_i}{2}} \left(\frac{q}{a}\right)^{|\mathbf{k}|}\right)$$

$$= \frac{(q, bq/a, c_1 \dots c_n q^{1-n}/b)_\infty}{(q/a, bq, q/b)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q\right)_\infty}{\left(\frac{x_i}{x_j} c_j\right)_\infty}, \tag{5.19}$$

provided $|c_1 \dots c_n q^{2-n}/a| < |q/a| < \left| q^{\frac{n-1}{2}} x_j^{-n} \prod_{i=1}^n x_i \right|$ *for* $j = 1, \dots, n$.

Proof. We have, for $\max(|q/a|, |c_1 \dots c_n q^{1-n}|) < 1$,

$${}_2\psi_2 \left[\begin{matrix} a, b \\ c_1 \dots c_n q^{1-n}, bq \end{matrix}; q, \frac{q}{a} \right] = \frac{(q, q, bq/a, c_1 \dots c_n q^{1-n}/b)_\infty}{(q/a, bq, q/b, c_1 \dots c_n q^{1-n})_\infty}, \tag{5.20}$$

by the ${}_2\psi_2$ summation in (5.3). Now we apply Lemma 4.3 to the ${}_2\psi_2$ of this summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.20) by the $b_i \mapsto c_i$, $i = 1, \dots, n$, and

$$f(m) = \frac{(a, b)_m}{(bq)_m} \left(\frac{q}{a}\right)^m$$

case of Lemma 4.3. Finally, we divide both sides of the resulting equation by (5.6) and simplify to obtain (5.19).

For an alternative proof, set $z = q/a$ and $d = bq$ in Theorem 5.1. In this case the multilateral series on the right side of (5.4) reduces to

$$\sum_{\substack{-\infty \le k_1, \dots, k_n \le \infty \\ |\mathbf{k}| = 0}} \Bigg(\prod_{1 \le i < j \le n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j} \right) \prod_{i,j=1}^{n} \left(\frac{y_i}{y_j} c_j \right)_{k_i}^{-1} \prod_{i=1}^{n} y_i^{n k_i - |\mathbf{k}|}$$
$$\times (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n \sum_{i=1}^n \binom{k_i}{2}} \Bigg) = \frac{(c_1 \dots c_n q^{1-n})_\infty}{(q)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{y_i}{y_j} q\right)_\infty}{\left(\frac{y_i}{y_j} c_j\right)_\infty},$$

the last evaluation by the $m = 0$ case of Proposition 4.2. ☐

Theorem 5.8 (An A_n ${}_2\psi_2$ summation)**.** *Let* a, $b_1, \dots, b_n$, c, *and* $x_1, \dots, x_n$ *be indeterminate, let* $n \ge 1$, *and suppose that none of the denominators in* (5.21) *vanishes. Then*

$$\sum_{k_1, \dots, k_n = -\infty}^{\infty} \Bigg(\prod_{1 \le i < j \le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^{n} \left(\frac{x_i}{x_j} b_j q \right)_{k_i}^{-1} \prod_{i=1}^{n} x_i^{n k_i - |\mathbf{k}|}$$
$$\times \frac{(a, b_1 \dots b_n)_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}} (-1)^{(n-1)|\mathbf{k}|} q^{-\binom{|\mathbf{k}|}{2} + n \sum_{i=1}^n \binom{k_i}{2}} \left(\frac{q}{a}\right)^{|\mathbf{k}|} \Bigg)$$
$$= \frac{(q, b_1 \dots b_n q/a, c/b_1 \dots b_n)_\infty}{(q/a, q/b_1 \dots b_n, c)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q\right)_\infty}{\left(\frac{x_i}{x_j} b_j q\right)_\infty}, \tag{5.21}$$

provided $|cq/a| < |q/a| < \left| q^{\frac{n-1}{2}} x_j^{-n} \prod_{i=1}^n x_i \right|$ *for* $j = 1, \ldots, n$.

Proof. We have, for $\max(|q/a|, |c|) < 1$,

$$ {}_2\psi_2 \left[\begin{matrix} a, b_1 \ldots b_n \\ c, b_1 \ldots b_n q \end{matrix} ; q, \frac{q}{a} \right] = \frac{(q, q, b_1 \ldots b_n q/a, c/b_1 \ldots b_n)_\infty}{(q/a, b_1 \ldots b_n q, q/b_1 \ldots b_n, c)_\infty}, \tag{5.22} $$

by the ${}_2\psi_2$ summation in (5.3). Now we apply Lemma 4.3 to the ${}_2\psi_2$ of this summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.22) by the $b_i \mapsto b_i q$, $i = 1, \ldots, n$, and

$$ f(m) = \frac{(a, b_1 \ldots b_n)_m}{(c)_m} \left(\frac{q}{a} \right)^m $$

case of Lemma 4.3. Finally, we divide both sides of the resulting equation by

$$ \frac{(q)_\infty}{(b_1 \ldots b_n q^{1-n})_\infty} \prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j} b_j \right)_\infty}{\left(\frac{x_i}{x_j} q \right)_\infty} $$

and simplify to obtain (5.21).

For an alternative proof, set $c_i = b_i q$, $z_i = q/a_i$, $i = 1, \ldots, n$, and $b \mapsto b_1 \ldots b_n$ in Theorem 5.5. In this case the multilateral series on the right side of (5.15) is terminated from below and from above and reduces just to one term, 1. In the resulting equation, we replace A by aq^{n-1} and d by c. □

Finally, we give four multivariable extensions of (5.3) which arise from Lemma 4.9.

Theorem 5.9 (An A_n ${}_2\psi_2$ summation)**.** *Let* $a_1, \ldots, a_n$*,* b*,* $c_1, \ldots, c_n$*, and* $x_1, \ldots, x_n$ *be indeterminate, let* $n \geq 1$*, and suppose that none of the denominators in* (5.23) *vanishes. Then*

$$ \sum_{k_1, \ldots, k_n = -\infty}^{\infty} \prod_{1 \leq i < j \leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j} \right) \prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j} a_j \right)_{k_i}}{\left(\frac{x_i}{x_j} c_j \right)_{k_i}} \frac{(b)_{|\mathbf{k}|}}{(bq)_{|\mathbf{k}|}} \left(\frac{q}{a_1 \ldots a_n} \right)^{|\mathbf{k}|} $$
$$ = \frac{(q, bq/a_1 \ldots a_n, c_1 \ldots c_n q^{1-n}/b)_\infty}{(bq, q/b, c_1 \ldots c_n q^{1-n}/a_1 \ldots a_n)_\infty} \prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j} q, \frac{x_i c_j}{x_j a_i} \right)_\infty}{\left(\frac{x_i}{x_j} c_j, \frac{x_i q}{x_j a_i} \right)_\infty}, \tag{5.23} $$

provided $\max(|c_1 \ldots c_n q^{1-n}|, |q/a_1 \ldots a_n|) < 1$.

Proof. We have, for $\max(|q/a_1 \ldots a_n|, |c_1 \ldots c_n q^{1-n}|) < 1$,

$$ {}_2\psi_2 \left[\begin{matrix} a_1 \ldots a_n, b \\ c_1 \ldots c_n q^{1-n}, bq \end{matrix} ; q, \frac{q}{a_1 \ldots a_n} \right] = \frac{(q, q, bq/a_1 \ldots a_n, c_1 \ldots c_n q^{1-n}/b)_\infty}{(q/a_1 \ldots a_n, bq, q/b, c_1 \ldots c_n q^{1-n})_\infty}, \tag{5.24} $$

by the ${}_2\psi_2$ summation in (5.3). Now we apply Lemma 4.9 to the ${}_2\psi_2$ of this summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.24) by the $b_i \mapsto c_i$, $i = 1, \dots, n$, and

$$g(m) = \frac{(b)_m}{(bq)_m}\left(\frac{q}{a_1 \dots a_n}\right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by (5.11) and simplify to obtain (5.23).

For an alternative proof, set $z = q/a_1 \dots a_n$ and $d = bq$ in Theorem 5.3. In this case the multilateral series on the right side of (5.9) reduces to

$$\sum_{\substack{-\infty \leq k_1, \dots, k_n \leq \infty \\ |\mathbf{k}|=0}} \prod_{1 \leq i<j \leq n} \left(\frac{y_i q^{k_i} - y_j q^{k_j}}{y_i - y_j}\right) \prod_{i,j=1}^n \frac{\left(\frac{y_i}{y_j} a_j\right)_{k_i}}{\left(\frac{y_i}{y_j} c_j\right)_{k_i}}$$

$$= \frac{(c_1 \dots c_n q^{1-n}, q/a_1 \dots a_n)_\infty}{(q, c_1 \dots c_n q^{1-n}/a_1 \dots a_n)_\infty} \prod_{i,j=1}^n \frac{\left(\frac{y_i}{y_j} q, \frac{y_i c_j}{y_j a_i}\right)_\infty}{\left(\frac{y_i}{y_j} c_j, \frac{y_i q}{y_j a_i}\right)_\infty},$$

the last evaluation by Theorem 4.7. ☐

Theorem 5.10 (An A_n ${}_2\psi_2$ summation). *Let a, $b_1, \dots, b_n$, c, and $x_1, \dots, x_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (5.25) *vanishes. Then*

$$\sum_{k_1, \dots, k_n = -\infty}^{\infty} \prod_{1 \leq i<j \leq n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j} b_j\right)_{k_i}}{\left(\frac{x_i}{x_j} b_j q\right)_{k_i}} \frac{(a)_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}} \left(\frac{q}{a}\right)^{|\mathbf{k}|}$$

$$= \frac{(b_1 \dots b_n q/a, c/b_1 \dots b_n)_\infty}{(q/a, c)_\infty} \prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j} q, \frac{x_i b_j}{x_j b_i} q\right)_\infty}{\left(\frac{x_i}{x_j} b_j q, \frac{x_i q}{x_j b_i}\right)_\infty}, \tag{5.25}$$

provided $\max(|c|, |q/a|) < 1$.

Proof. We utilize the ${}_2\psi_2$ summation in (5.20) and apply Lemma 4.9 to the ${}_2\psi_2$ in that summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.20) by the $a_i \mapsto b_i$, $b_i \mapsto b_i q$, $i = 1, \dots, n$, and

$$g(m) = \frac{(a)_m}{(c)_m}\left(\frac{q}{a}\right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by

$$\frac{(q,q)_\infty}{(b_1 \dots b_n q, q/b_1 \dots b_n)_\infty} \prod_{i,j=1}^n \frac{\left(\frac{x_i}{x_j} b_j q, \frac{x_i q}{x_j b_i}\right)_\infty}{\left(\frac{x_i}{x_j} q, \frac{x_i b_j}{x_j b_i} q\right)_\infty}$$

and simplify to obtain (5.25).

For an alternative proof, set $z_i = q/a_i$, and $d_i = b_i q$, $i = 1, \dots, n$, in Theorem 5.4. In this case the multilateral series on the right side of (5.12) is terminated from below and from above and reduces just to one term, 1. In the resulting summation, replace A by aq^{n-1}. $\square$

Theorem 5.11 (An A_n ${}_2\psi_2$ summation)**.** *Let a, $b_1, \dots, b_n$, $c_1, \dots, c_n$, and $x_1, \dots, x_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (5.26) *vanishes. Then*

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n} \left(\frac{x_i q^{k_i} - x_j q^{k_j}}{x_i - x_j}\right) \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} b_j\right)_{k_i}}{\left(\frac{x_i}{x_j} c_j\right)_{k_i}} \frac{(a)_{|\mathbf{k}|}}{(b_1 \dots b_n q)_{|\mathbf{k}|}} \left(\frac{q}{a}\right)^{|\mathbf{k}|}$$
$$= \frac{(q, b_1 \dots b_n q/a)_\infty}{(q/a, b_1 \dots b_n q)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} q, \frac{x_i c_j}{x_j b_i}\right)_\infty}{\left(\frac{x_i}{x_j} c_j, \frac{x_i q}{x_j b_i}\right)_\infty}, \quad (5.26)$$

provided $\max(|c_1 \dots c_n q^{1-n}|, |q/a|) < 1$.

Proof. Write $B \equiv b_1 \dots b_n$ and $C \equiv c_1 \dots c_n$. We have, for $\max(|q/a|, |Cq^{1-n}|) < 1$,

$${}_2\psi_2\left[\begin{matrix} a, B \\ Cq^{1-n}, Bq \end{matrix}; q, \frac{q}{a}\right] = \frac{(q, q, Bq/a, Cq^{1-n}/B)_\infty}{(q/a, Bq, q/B, Cq^{1-n})_\infty}, \quad (5.27)$$

by the ${}_2\psi_2$ summation in (5.3). Now we apply Lemma 4.9 to the ${}_2\psi_2$ of this summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.27) by the $a_i \mapsto b_i$, $b_i \mapsto c_i$, $i = 1, \dots, n$, and

$$g(m) = \frac{(a)_m}{(Bq)_m} \left(\frac{q}{a}\right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by

$$\frac{(q, Cq^{1-n}/B)_\infty}{(Cq^{1-n}, q/B)_\infty} \prod_{i,j=1}^{n} \frac{\left(\frac{x_i}{x_j} c_j, \frac{x_i q}{x_j b_i}\right)_\infty}{\left(\frac{x_i}{x_j} q, \frac{x_i c_j}{x_j b_i}\right)_\infty}$$

and simplify to obtain (5.26).

For an alternative proof, set $z_i = q/a_i$, $i = 1, \dots, n$, and $c = b_1 \dots b_n q$ in Theorem 5.6. In this case the multilateral series on the right side of (5.17) is terminated from below and from above and reduces just to one term, 1. In the resulting summation, replace d_i by c_i, $i = 1, \dots, n$, and A by aq^{n-1}. $\square$

Theorem 5.12 (An A_n ${}_2\psi_2$ summation)**.** *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$, c, and $x_1, \dots, x_n$ be indeterminate, let $n \geq 1$, and suppose that none of the denominators in* (5.28) *vanishes. Then*

$$\sum_{k_1,\dots,k_n=-\infty}^{\infty} \prod_{1\le i<j\le n}\left(\frac{x_iq^{k_i}-x_jq^{k_j}}{x_i-x_j}\right)\prod_{i,j=1}^{n}\frac{\left(\frac{x_i}{x_j}a_j\right)_{k_i}}{\left(\frac{x_i}{x_j}b_jq\right)_{k_i}}\frac{(b_1\dots b_n)_{|\mathbf{k}|}}{(c)_{|\mathbf{k}|}}\left(\frac{q}{a_1\dots a_n}\right)^{|\mathbf{k}|}$$

$$=\frac{(q,c/b_1\dots b_n)_\infty}{(q/b_1\dots b_n,c)_\infty}\prod_{i,j=1}^{n}\frac{\left(\frac{x_i}{x_j}q,\frac{x_ib_jq}{x_ja_i}\right)_\infty}{\left(\frac{x_i}{x_j}b_jq,\frac{x_iq}{x_ja_i}\right)_\infty}, \quad (5.28)$$

provided $\max(|c|,|q/a_1\dots a_n|)<1$.

Proof. Write $A\equiv a_1\dots a_n$ $B\equiv b_1\dots b_n$. We have, for $\max(|q/A|,|c|)<1$,

$$ {}_2\psi_2\left[\begin{matrix}A,B\\c,Bq\end{matrix};q,\frac{q}{A}\right]=\frac{(q,q,Bq/A,c/B)_\infty}{(q/A,Bq,q/B,c)_\infty}, \quad (5.29)$$

by the ${}_2\psi_2$ summation in (5.3). Now we apply Lemma 4.9 to the ${}_2\psi_2$ of this summation. Specifically, we rewrite the ${}_2\psi_2$ in (5.29) by the $b_i\mapsto b_iq$, $i=1,\dots,n$, and

$$g(m)=\frac{(B)_m}{(c)_m}\left(\frac{q}{A}\right)^m$$

case of Lemma 4.9. Finally, we divide both sides of the resulting equation by

$$\frac{(q,Bq/A)_\infty}{(Bq,q/A)_\infty}\prod_{i,j=1}^{n}\frac{\left(\frac{x_i}{x_j}b_jq,\frac{x_iq}{x_ja_i}\right)_\infty}{\left(\frac{x_i}{x_j}q,\frac{x_ib_jq}{x_ja_i}\right)_\infty}$$

and simplify to obtain (5.28).

For an alternative proof, set $z_i=q^{\frac{1}{n}}/b_i$, and $d_i=a_iq^{\frac{1}{n}}$, $i=1,\dots,n$, in Theorem 5.6. In this case the multilateral series on the right side of (5.17) is terminated from below and from above and reduces just to one term, 1. In the resulting summation, replace a_i by $b_iq^{1-\frac{1}{n}}$, and b_i by a_i, for $i=1,\dots,n$. □

References

[1] G. E. Andrews, *q-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra*, CBMS Regional Conference Lectures Series **66** (Amer. Math. Soc., Providence, RI, 1986).

[2] W. N. Bailey, "Series of hypergeometric type which are infinite in both directions", *Quart. J. Math.* (Oxford) **7** (1936), 105–115.

[3] W. N. Bailey, "On the basic bilateral basic hypergeometric series ${}_2\psi_2$", *Quart. J. Math.* (Oxford) (2) **1** (1950), 194–198.

[4] G. Bhatnagar, "D_n basic hypergeometric series", *The Ramanujan J.* **3** (1999), 175–203.

[5] G. Bhatnagar and S. C. Milne, "Generalized bibasic hypergeometric series and their $U(n)$ extensions", *Adv. Math.* **131** (1997), 188–252.

[6] A.-L. Cauchy, "Mémoire sur les fonctions dont plusieurs valeurs…", *C.R. Acad. Sci. Paris* **17** (1843), 523; reprinted in *Oeuvres de Cauchy*, Ser. 1 **8**, Gauthier-Villars, Paris (1893), 42–50.

[7] R. Y. Denis and R. A. Gustafson, "An $SU(n)$ q-beta integral transformation and multiple hypergeometric series identities", *SIAM J. Math. Anal.* **23** (1992), 552–561.

[8] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics And Its Applications 35, Cambridge University Press, Cambridge (1990).

[9] R. A. Gustafson, "Multilateral summation theorems for ordinary and basic hypergeometric series in $U(n)$", *SIAM J. Math. Anal.* **18** (1987), 1576–1596.

[10] R. A. Gustafson, "The Macdonald identities for affine root systems of classical type and hypergeometric series very well-poised on semi-simple Lie algebras", *in* Ramanujan International Symposium on Analysis (Dec. 26th to 28th, 1987, Pune, India), N. K. Thakare (ed.) (1989), 187–224.

[11] G. H. Hardy, *Ramanujan*, Cambridge Univ. Press, Cambridge (1940); Chelsea, New York (1978).

[12] W. J. Holman III, "Summation theorems for hypergeometric series in $U(n)$", *SIAM J. Math. Anal.* **II** (1980), 523–532.

[13] W. J. Holman III, L. C. Biedenharn, J. D. Louck, "On hypergeometric series well-poised in $SU(n)$", *SIAM J. Math. Anal.* **7** (1976), 529–541.

[14] J. Horn, "Ueber die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen", *Math. Ann.* **34** (1889), 544–600.

[15] M. E. H. Ismail, "A simple proof of Ramanujan's $_1\psi_1$ sum", *Proc. Amer. Math. Soc.* **63** (1977), 185–186.

[16] J. Kaneko, "A $_1\Psi_1$ summation theorem for Macdonald polynomials", *The Ramanujan J.* **2** (1996), 379-386.

[17] I. G. Macdonald, "Affine root systems and Dedekind's η-function", *Invent. Math.* **15** (1972), 91–143.

[18] S. C. Milne, "An elementary proof of the Macdonald identities for $A_\ell^{(1)}$", *Adv. Math.* **57** (1985), 34–70.

[19] S. C. Milne, "A $U(n)$ generalization of Ramanujan's $_1\Psi_1$ summation", *J. Math. Anal. Appl.* **118** (1986), 263–277.

[20] S. C. Milne, "Basic hypergeometric series very well-poised in $U(n)$", *J. Math. Anal. Appl.* **122** (1987), 223–256.

[21] S. C. Milne, "Multiple q-series and $U(n)$ generalizations of Ramanujan's $_1\psi_1$ sum", in *Ramanujan Revisited* (G. E. Andrews et al., eds.), Academic Press, New York (1988), 473–524.

[22] S. C. Milne, "The multidimensional $_1\Psi_1$ sum and Macdonald identities for $A_l^{(1)}$", in *Theta Functions Bowdoin* 1987 (L. Ehrenpreis and R. C. Gunning, eds.), *Proc. Sympos. Pure Math.* **49** (1989), 323–359.

[23] S. C. Milne, "Summation theorems for basic hypergeometric series of Schur function argument", in *Progress in Approximation Theory* (A. A. Gonchar and E. B. Saff, eds.), Springer-Verlag, New York (1992), 51–77.

[24] S. C. Milne, "A q-analog of a Whipple's transformation for hypergeometric series in $U(n)$", *Adv. Math.* **108** (1994), 1–76.

[25] S. C. Milne, "Balanced $_3\phi_2$ summation theorems for $U(n)$ basic hypergeometric series", *Adv. Math.* **131** (1997), 93–187.

[26] S. C. Milne and G. M. Lilly, "Consequences of the A_l and C_l Bailey transform and Bailey lemma", *Discrete Math.* **139** (1995), 319–346.

[27] S. C. Milne and J. W. Newcomb, "$U(n)$ very-well-poised $_{10}\phi_9$ transformations", *J. Comput. Appl. Math.* **68** (1996), 239–285.

[28] M. Schlosser, "Multidimensional matrix inversions and A_r and D_r basic hypergeometric series", *The Ramanujan J.* **1** (1997), 243–274.

[29] M. Schlosser, "Some new applications of matrix inversions in A_r", *The Ramanujan J.* **3** (1999), 405–461.
[30] M. Schlosser, "A new multidimensional matrix inversion in A_r", *Contemp. Math.* **254** (2000), 413–432.
[31] D. Stanton, "An elementary approach to the Macdonald identities", in *q-Series and partitions* (D. Stanton, ed.), IMA Vol. Math. Appl. **18** (1989), 139–150.

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