ON EISENSTEIN SERIES AND THE COHOMOLOGY OF ARITHMETIC GROUPS

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Abstract. The automorphic cohomology of a reductive \( \mathbb{Q} \)-group \( G \) captures essential analytic aspects of the arithmetic subgroups of \( G \). The subspace spanned by all possible residues and principal values of derivatives of Eisenstein series, attached to cuspidal automorphic forms \( \pi \) on the Levi factor of proper parabolic \( \mathbb{Q} \)-subgroups of \( G \), forms the Eisenstein cohomology. We show that non–trivial classes can only arise if the point of evaluation features a “half–integral” property. Consequently, only the analytic behavior of the automorphic \( L \)–functions at half–integral arguments matters whether an Eisenstein series attached to a globally generic \( \pi \) gives rise to a residual class or not.

Résumé Sur les séries d’Eisenstein et la cohomologie des groupes arithmétiques. La cohomologie automorphe d’un \( \mathbb{Q} \)–groupe réductif \( G \) détecte des propriétés analytiques essentielles des sous–groupes arithmétiques de \( G \). La cohomologie d’Eisenstein est le sous–espace engendré par tous les résidus ainsi que par les valeurs principales des dérivées des séries d’Eisenstein, attachées aux formes automorphes cuspidales \( \pi \) sur les facteurs de Levi des \( \mathbb{Q} \)–sous–groupes paraboliques propres de \( G \). Nous montrons que les classes non triviales ne peuvent provenir que des évaluations aux points “demi–entiers”. Ainsi, savoir si une série d’Eisenstein attachée à une forme \( \pi \) générique donne lieu à une classe résiduelle ou non, ne dépend que du comportement analytique de fonctions \( L \) automorphes en des points demi–entiers.

1. Eisenstein Cohomology of Arithmetic Groups

Let \( G \) be a connected reductive algebraic group defined over \( \mathbb{Q} \). Let \( \mathbb{Q}_v \) be the completion of \( \mathbb{Q} \) at a place \( v \) of \( \mathbb{Q} \). Let \( \mathbb{A} \) be the ring of adèles of \( \mathbb{Q} \), and \( \mathbb{A}_f \) the finite adèles. We fix a choice of a minimal parabolic \( \mathbb{Q} \)–subgroup \( P_0 \) of \( G \) with Levi decomposition \( P_0 = L_0 N_0 \), and a choice of a maximal compact subgroup \( K = \prod_v K_v \) of \( G(\mathbb{A}) \) such that \( K \) is in good position with respect to \( P_0 \) (cf. [6, Sect. I.1.4]). Here \( K_v \) is a maximal compact subgroup of \( G(\mathbb{Q}_v) \), and we write \( K_\mathbb{R} \) for \( K_v \) at the archimedean place \( v = \infty \) of \( \mathbb{Q} \). Let \( M_G \) be the connected component of the intersection of the kernels of all \( \mathbb{Q} \)–rational characters of \( G \), and \( m_G \) its Lie algebra. Let \( A_G \) be a maximal \( \mathbb{Q} \)–split torus in the center of \( G \).

Let \( E \) be a finite–dimensional irreducible representation of \( G(\mathbb{C}) \) of highest weight \( \Lambda \). Let \( J_E \) be the annihilator of the dual representation of \( E \) in the center of the universal enveloping algebra of \( m_G \). Let \( A_E \) be the space of automorphic forms on \( A_G(\mathbb{R})^0 G(\mathbb{Q}) \backslash G(\mathbb{A}) \) (cf. [6], [1]) annihilated by a power of \( J_E \). It carries the structure of a \( (m_G, K_\mathbb{R}, G(\mathbb{A}_f)) \)–module. The automorphic cohomology of \( G \) with coefficients in \( E \) is defined as the Lie algebra cohomology

\[
H^\ast(G, E) = H^\ast(m_G, K_\mathbb{R}; A_E \otimes E).
\]

As proved in [2, Thm. 1.4, resp. 2.3], this cohomology decomposes according to the decomposition of the space of automorphic forms with respect to their cuspidal support. More precisely, let \( C \) be the set of associate classes of parabolic \( \mathbb{Q} \)–subgroups of \( G \), and, given a class \( \{P\} \in C \), represented by a parabolic \( \mathbb{Q} \)–subgroup \( P \) with Levi decomposition \( P = L_P N_P \), let \( \Phi_{E,\{P\}} \) be the set of associate

Second author’s work supported in part by FWF Austrian Science Fund, grant number P 21090–N13.
denotes the group of cuspidal automorphic representations of the Levi factors of $Q \in \{P\}$ as in [2, Sect. 1.2]. Let $A_{E\{P\},\phi}$ be the subspace of $A_E$ consisting of automorphic forms whose constant term along a parabolic $Q$–subgroup $Q$ of $G$ is orthogonal to the space of cuspidal automorphic forms on $L_Q(A)$ if $Q \not\in \{P\}$, and belongs to the $\phi_Q$–isotypic component of that space if $Q \in \{P\}$. Then

$$H^*(G,E) = \bigoplus_{(P) \in C} \bigoplus_{\phi \in \Phi_{E\{P\}}} H^*(m_G, K_{\mathbb{R}}; A_{E\{P\},\phi} \otimes E).$$

For $\{P\} \neq \{G\}$, the cohomology classes in a summand $H^*(m_G, K_{\mathbb{R}}; A_{E\{P\},\phi} \otimes E)$ are constructed from the residues or principal values of the derivatives of Eisenstein series attached to a cuspidal representation $\pi$ of $L_P(A)$ belonging to an associate class $\phi \in \Phi_{E\{P\}}$. Thus, the family of these summands is called the Eisenstein cohomology. We assume, as we may, that $\pi$ is normalized in such a way that the poles of the Eisenstein series attached to $\pi$ are real.

As proved in [5, Sect. 3], from the representation theoretic point of view, the study of a summand in the above decomposition of the automorphic cohomology, reduces to the study of the induced representation

$$\text{Ind}_{P(A)}^{G(A)} H^*(\mathfrak{p}, K_{\mathbb{R}} \cap P(\mathbb{R}); V_\pi \otimes H^*(n_P, E) \otimes S(\hat{\mathfrak{a}}_P^G)),$$

where $\mathfrak{p}$, $n_P$ are the Lie algebras of $P$ and $N_P$, $V_\pi$ is the $\pi$–isotypic subspace of the space of cuspidal automorphic forms on $L_P(A)$, and $S(\hat{\mathfrak{a}}_P^G)$ is the symmetric algebra of $\hat{\mathfrak{a}}_P^G$ endowed with the $(\mathfrak{m}_G, K_{\mathbb{R}})$–module structure as in [1, p. 218]. Here $\hat{\mathfrak{a}}_P^G$ is the dual of $\mathfrak{a}_P \cap \mathfrak{m}_G$, where $\mathfrak{a}_P$ is the Lie algebra of the maximal split torus $A_P$ in the center of $L_P$.

### 2. Necessary Conditions for Non–vanishing

The necessary conditions for non–vanishing of cohomology classes are given in terms of the absolute root system of $G$. Hence, for simplicity of exposition, we assume from this point on that $G$ is $\mathbb{Q}$–split. Let $\Psi$ be the absolute root system of $G$ with respect to $L_0$, $\Psi^+$ and $\Delta$ the positive and simple roots determined by $P_0$. Let $\rho_{P_0}$ be the half–sum of positive roots. Let $W$ be the absolute Weyl group of $G$. Let $P$ be a standard (i.e. containing $P_0$) proper parabolic $\mathbb{Q}$–subgroup of $G$, with Levi decomposition $P = L_P N_P$. Let $W_P$ be the set of minimal coset representatives for $W_{L_P} \setminus W$ (cf. [3]), where $W_{L_P}$ is the absolute Weyl group of $L_P$. For $w \in W_P$, let $F_{\mu_w}$ be a representation of the Levi factor $L_P(\mathbb{C})$ of highest weight $\mu_w = w(\Lambda + \rho_{P_0}) - \rho_{P_0}$. Let $\hat{\mathfrak{a}}_P = X^*(P) \otimes \mathbb{R}$, where $X^*(P)$ denotes the group of $\mathbb{Q}$–rational characters of $P$. Representation theoretical arguments show

**Proposition 2.1.** The space $H^*(m_G, K_{\mathbb{R}}; A_{E\{P\},\phi} \otimes \mathbb{C} E)$ is trivial except possibly if there exists a representative $w \in W^P$ such that $F_{\mu_w}$ is isomorphic to its complex conjugate contragredient representation $F^*_{\mu_w}$, and so that for any $\pi \in \phi$ the infinitesimal characters of its infinite component $\pi_\infty$ and $F^*_{\mu_w}$ coincide.

**Proposition 2.2** (Thm. 4.11 in [7]). If the two necessary conditions in Proposition 2.1 are satisfied for certain $w \in W^P$, then the only possibly non–trivial cohomology classes are those obtained from the residues or the principal values of the derivatives of the Eisenstein series attached to $\pi$ as in [4] or [6, Sect. II.1.5] at the value $s_w = (w(\Lambda + \rho_{P_0})) |_{\hat{\mathfrak{a}}_P}$ of its complex parameter.

### 3. Evaluation Points and Automorphic $L$–functions at Half–integral Arguments

We retain the assumption that $G$ is $\mathbb{Q}$–split, and restrict our attention to classical groups. More precisely, $G$ is the $\mathbb{Q}$–split general linear group $GL_n$ ($n > 1$), the symplectic group $Sp_n$, the odd special orthogonal group $SO_{2n+1}$, or the even special orthogonal group $SO_{2n}$ ($n > 1$). Let $e_k \in \hat{\mathfrak{a}}_{P_0}$, for $k = 1, \ldots, n$, be the projection of $L_0$ to its $k^{th}$ component. The standard parabolic $\mathbb{Q}$–subgroups of $G$ are in bijection with the subsets of the set $\Delta$ of simple roots. Let $1 \leq R_1 < \ldots < R_d \leq n$ be
Theorem 3.1. Let \( s_w = -w(\Lambda + \rho_P)|_{\Phi_P} \) be the evaluation point written in the basis \( \{e_1, \ldots, e_n\} \) for \( \Phi_P \) as

\[
s_w = t_1 \sum_{l_1=1}^{R_1} e_{l_1} + t_2 \sum_{l_2=R_1+1}^{R_2} e_{l_2} + \cdots + t_d \sum_{l_d=R_{d-1}+1}^{R_d} e_{l_d},
\]

where \( t_1, \ldots, t_d \in \mathbb{R} \). Then the residue or a derivative of the Eisenstein series attached to \( \pi \), evaluated at \( s_w \), can possibly give rise to a non-trivial cohomology class in the space \( H^*(m_G, K_E; \mathcal{A}_{E,\{P\}} \otimes E) \) only if \( s_w \) has the property that \( t_l \in \frac{1}{2} \mathbb{Z} \) for \( l = 1, \ldots, d \), except in the case \( G = GL_n \) where we have \( t_k - t_l \in \frac{1}{2} \mathbb{Z} \) for \( 1 \leq k < l \leq d \).

The main technical tool in the proof is a combinatorial description of the sets \( W_P \) for classical groups which enables us to give explicit formulas for the action of \( w \in W_P \) on \( \Phi_P \). The divisibility properties of the coefficients of the evaluation point \( s_w = -w(\Lambda + \rho_P)|_{\Phi_P} \) can be controlled using the explicit formula for the action of \( w \in W_P \) and the necessary condition \( F_{\mu_w} \equiv F_{\mu_w} \) in Proposition 2.1.

This theorem shows that for computing Eisenstein cohomology one only needs to consider the Eisenstein series at evaluation points of a very special form. In particular, if \( \pi \) is globally generic, the Langlands–Shahidi method relates the poles of the Eisenstein series attached to \( \pi \), the analytic properties of certain automorphic \( L \)-functions. The point of evaluation \( s_w \) occurs in the arguments of those \( L \)-functions as \( k_\beta \langle s_w, \beta' \rangle \), where \( \beta \in \Psi_{\text{reg}}(G, A_P) \) ranges over the positive roots in the reduced root system of \( G \) with respect to \( A_P \), and either \( k_\beta = 1 \) or \( k_\beta \in \{1, 2\} \) depending on \( \beta \). Therefore, in all cases \( \langle s_w, \beta' \rangle \in \frac{1}{2} \mathbb{Z} \). Note that for different \( \beta \), different \( L \)-functions appear. Moreover, if the symmetric or exterior square \( L \)-function appears with \( k_\beta = 1 \), then \( R_d = n \), and either \( G = SO_{2n+1} \) with \( \beta \) of the form \( \beta = e_{R_k} \), or \( G = SO_{2n} \) with \( \beta \) of the form \( \beta = e_{R_k-1} + e_{R_k} \) and \( R_k - R_{k-1} \geq 2 \). Thus, in these two cases, in fact, \( \langle s_w, \beta' \rangle = 2t_k \in \mathbb{Z} \). Although the analytic properties of all the \( L \)-functions in the Langlands–Shahidi normalizing factors are not completely understood (e.g., the poles inside \( 0 < s < 1 \) for the symmetric and exterior square \( L \)-functions), it turns out, due to Theorem 3.1, that they are known at the evaluation points which are relevant for cohomology. We discuss an example in the next section.

4. AN EXAMPLE: MAXIMAL PARABOLIC SUBGROUPS OF THE SYMPLECTIC GROUP

We consider the \( \mathbb{Q} \)-split symplectic group \( Sp_n \) of \( \mathbb{Q} \)-rank \( n \geq 2 \). The highest weight \( \Lambda \) of the representation \( E \) of \( Sp_n(\mathbb{C}) \) is of the form \( \Lambda = \sum_{k=1}^n \lambda_k e_k \), where all \( \lambda_k \in \mathbb{Z} \) and \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0 \). Let \( P_n = L_n N_n \) be the standard maximal proper parabolic \( \mathbb{Q} \)-subgroup of \( Sp_n \) with the Levi factor \( L_n \cong GL_n \). Let \( \pi \) be a cuspidal automorphic representation of \( L_n(\mathbb{A}) \) in an associate class \( \phi \in \Phi_{E,\{P_n\}} \).

Theorem 4.1. Let \( \mathcal{L}_{E,\{P_n\},\phi} \) be the subspace of \( \mathcal{A}_{E,\{P_n\},\phi} \) consisting of square–integrable automorphic forms. The cohomology space \( H^*(sp_n, K_{\mathbb{R}}; \mathcal{L}_{E,\{P_n\},\phi} \otimes E) \) is trivial except possibly in the case where the following conditions are satisfied:

1. the representation \( \pi \) is selfdual, \( L(s, \pi, \Lambda^2) \) has a pole at \( s = 1 \), and \( L(1/2, \pi) \neq 0 \),
2. the \( \mathbb{Q} \)-rank \( n \) of the algebraic group \( Sp_n/\mathbb{Q} \) is even,
3. the highest weight \( \Lambda \) of \( E \) satisfies \( \lambda_{2l-1} = \lambda_{2l} \) for all \( l = 1, 2, \ldots, n/2 \),
(4) the infinite component $\pi_\infty$ of $\pi$ is a tempered representation of $GL_n(\mathbb{R})$ fully induced from $n/2$ unitary discrete series representations of $GL_2(\mathbb{R})$ having the lowest $O(2)$–types $2\mu_l + 2n - 4l + 4$ for $l = 1, \ldots, n/2$, where $\mu_l = \lambda_{2l-1} = \lambda_{2l}$.

Square–integrable automorphic forms in $L^2(E, \{P_n\})$ are obtained as residues of Eisenstein series attached to $\pi$ at the poles inside the open positive Weyl chamber in $\hat{\alpha}_{P_n}$. Since all cuspidal automorphic representations of $GL_n(A)$ are globally generic, the Langlands–Shahidi method implies that those poles coincide with the poles of the normalizing factor

$$\frac{L(s, \pi)}{L(1+s, \pi)\varepsilon(s, \pi) L(1+2s, \pi, \lambda^2)\varepsilon(2s, \pi, \lambda^2)},$$

where $s > 0$ is identified with the character $\det^s \in \hat{\alpha}_{P_n}$. The poles of that ratio at $s > 0$ are among the poles of $L(2s, \pi, \lambda^2)$. However, this $L$–function has no poles for $2s > 1$, it has a simple pole at $2s = 1$ for $\pi$ as in Theorem 4.1.(i), but its analytic behavior inside the critical strip $0 < 2s < 1$ is not known. At this point the strength of Theorem 3.1 reveals, because it shows that possible poles inside $0 < 2s < 1$ play no role in understanding the cohomology space $H^*(\mathfrak{sp}_n, K_\mathbb{R}; A_{E, \{P_n\}}, \phi \otimes E)$. The rest of the theorem follows from the explicit formulas for the action of $w \in W_{P_n}$, Propositions 2.1 and 2.2, and $s_w = 1/2$.

A treatment of the other maximal proper parabolic subgroups $P_r$, with the Levi factor $L_r \cong GL_r \times Sp_{n-r}$ where $r < n$, is also carried through. In that case, given a globally generic $\pi \cong \tau \otimes \sigma$, the analytic behavior of the exterior square $L$–function $L(2s, \tau, \lambda^2)$ at $s = 1/2$, and the Rankin–Selberg $L$–function $L(s, \tau \times \sigma)$ at $s = 1$, plays a decisive role.

References


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